

# Pacific Journal of Mathematics

**INTEGRAL CLOSURE OF DIFFERENTIAL RINGS**

EDWARD C. POSNER

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We prove that a commutative differentially simple ring of characteristic zero finitely generated over its field of constants is integrally closed in its field of quotients. (A ring is differentially simple if it has non-trivial multiplication and has no ideal invariant under a given family of derivations; i.e., has no differential ideals other than  $(0)$ . The field of constants is the subring of the ring annihilated by each derivation of the family of derivations.) The result of the first sentence is used to obtain a condition that the powers of an element of a function field in one variable form an integral basis. The following results from [1] will be used: A commutative differentially simple ring of characteristic zero is an integral domain whose ring of constants is a field. Crucial is the following lemma:

**LEMMA.** *Let  $F$  be a field of characteristic zero;  $x_1, \dots, x_n$  be  $n$  independent transcendentals over  $F$ ;  $y_1, \dots, y_q$  be integral over  $x_1, \dots, x_n$ ; and  $d$  an  $F$ -derivation of  $F[x, y]$  into itself. Then  $d$  (or rather its natural extension to  $F(x, y)$ ) sends  $O_x$  (the set of elements of  $F(x, y)$  integral over  $x_1, \dots, x_n$ ) into itself.*

*Proof.* In general any  $F$ -derivation of  $F(x, y)$  into itself can be written as

$$d = \sum_{i=1}^n A_i \frac{\partial}{\partial x_i},$$

$A_i$  elements of  $F(x, y)$ ,  $1 \leq i \leq n$ . Further,  $d$  maps  $F[x, y]$  into itself if and only if  $d(x_i)$  is in  $F[x, y]$  for each  $i$  and  $d(y_j)$  is in  $F[x, y]$  for each  $j$ . The first set of conditions is equivalent to the condition that  $A_i$  be in  $F[x, y]$  for each  $i$ .

In order to be able to use power series, we assume that  $F$  is algebraically closed. For if not, let  $\bar{F}$  be its algebraic closure. Let  $d$  also be the extension of  $d$  to  $\bar{F}(x, y)$ . Since  $d$  sends  $\bar{F}[x, y]$  into itself,  $d$  send  $\bar{O}_x$  into itself, where  $\bar{O}_x$  denotes the ring of integral functions of  $\bar{F}(x, y)$ . A fortiori,  $d$  sends  $O_x$  into  $\bar{O}_x$ . But  $\bar{O}_x \cap F[x, y] = O_x$  so actually  $d$  sends  $O_x$  into itself as required.

Let  $P$  be a place of  $F(x, y)$  over  $F$  which has residue field  $F$  and which is finite on  $x_1, \dots, x_n$ . We will prove that if  $g$ , in  $F(x, y)$ , is finite at  $P$ ,  $d(g)$  is finite at  $P$ . Let  $a_i$  denote the residue of  $x_i$  at  $P$ ; then there exist uniformizing parameters  $t_1, \dots, t_n$  at  $P$  such that

$x_i - \alpha_i$  is a positive integral power of  $t_i$ , say  $x_i - \alpha_i = t_i^{p_i}$ . Every element  $B$  of  $F(x, y)$  finite at  $P$  has a power series in  $t_1, \dots, t_n$  with coefficients in  $F$ . We call the smallest power of  $t_i$  occurring in this series the  $i$ -order of  $B$  at  $P$ , and denote it by  $\text{ord}_{P,i} B$ ; the definition of  $\text{ord}_{P,i} B$  extends to arbitrary elements  $B$  of  $F(x, y)$  in an obvious way. Fixing  $i$ , we see that if  $\text{ord}_{P,i} d(B) \geq \text{ord}_{P,i} B$  for every  $B$  finite at  $P$  then  $\text{ord}_{P,i} d(B) \geq 0$  for every such  $B$ . Suppose there exists some  $B$  finite at  $P$  with  $\text{ord}_{P,i} d(B) < \text{ord}_{P,i} B$ . Then  $\alpha_i - p_i < 0$ , where  $\alpha_i = \text{ord}_{P,i} A_i$ , so that  $r_i = p_i - \alpha_i > 0$ , and  $\text{ord}_{P,i} B = r_i + \text{ord}_{P,i} dB$  for every  $(B)$  in  $F(x, y)$  with  $\text{ord}_{P,i} B \neq 0$ . Since  $d$  maps  $F[x, y]$  into itself, the only values which  $\text{ord}_{P,i} B$  can have when  $B$  is in  $F[x, y]$  are integral multiples of  $r_i$ , for otherwise some element of  $F[x, y]$  would have negative  $i$ -order. Since  $t_1, \dots, t_n$  are uniformizing parameters, it follows that  $r_i = 1$ , for otherwise we could replace  $t_i$  by  $t_i^{r_i}$ . Thus,  $d$  drops positive  $i$ -orders by 1, so that  $\text{ord}_{P,i} d(B) \geq 0$  for every  $B$  finite at  $P$ . Since this holds for every  $i$ ,  $d(B)$  is finite at  $P$  whenever  $B$  is. Since this holds for every  $P$ , we conclude that  $d$  maps  $O_x$  into itself.

**THEOREM 1.** *Let  $F$  be a field of characteristic zero,  $A = F[z_1, z_2, \dots, z_k]$  a commutative finitely generated ring extension of  $F$ . Let  $D$  be a (finite or infinite) family of derivations of  $A$  into itself over  $F$ . Let  $A$  be differentially simple under  $D$ . Then  $A$  is integrally closed in its quotient field  $K$ .*

*Proof.*  $A$  is an integral domain by (1). By Noether's Normalization Lemma, we can write  $A = F[x_1, \dots, x_n; y_1, \dots, y_q]$ , with  $n$  the transcendence degree of  $K/F$  and  $y_1, \dots, y_q$  integral over  $x_1, \dots, x_n$ . To prove  $A = O_x$ , let  $I$  denote the conductor of  $O_x$ , that is, the set of elements  $u$  of  $F[x, y]$  such that  $u \cdot O_x \subset F[x, y]$ ; by [3], pp. 271-2, prop. 6,  $I$  is a non-zero ideal of  $F[x, y]$ . To prove  $I$  differential under  $D$ , let  $d$  be in  $D$ ,  $h$  be in  $I$ ,  $g$  be in  $O_x$ . Then  $h \cdot g$  is in  $F[x, y]$ ,  $d(h \cdot g)$  is in  $F[x, y]$ ,  $d(h)g + hd(g)$  is in  $F[x, y]$ . Now  $d(g)$  is in  $O_x$  by the lemma so  $hd(g)$  is in  $F[x, y]$  since  $h$  is in  $I$ . Then  $d(h)g$  is in  $F[x, y]$ ,  $I$  is differential under  $D$ . Then  $I = F[x, y]$  so  $1 \cdot O_x \subset F[x, y]$ ,  $O_x = F[x, y]$  as promised.

**REMARK.**  $D$  can always be taken to be finite since the derivations of  $F[x, y]$  into itself form a finite  $F[x, y]$ -module.

The converse of Theorem 1 is false, i.e., there are integrally closed finitely generated domains which are not differentially simple under any family of  $F$ -derivations. For example, let  $y^2 = x_1^2 + x_2^2$ . Then  $F[x, y] = O_x$  but is not differentially simple over  $F$ . In fact, the ideal  $(x_1, x_2, y)$  of  $F[x, y]$  is differential for any derivation, as is easy to see. But when  $n = 1$ , we do have the converse. (For background material, see [2];\*

pp. 83-88.)

**THEOREM 2.** *Let  $K$  be a function field in one variable over a field  $F$  of characteristic zero, and let  $x$  be an element of  $K$  transcendental over  $F$ . Let  $O_x$  denote the set of elements of  $K$  integral over  $x$ . Then  $O_x$  is differentiably simple with field of constants  $F$  under a family of two or fewer derivations.*

*Proof.* First we shall specify the derivations.  $O_x$  is a Dedekind ring, i.e., every ideal of  $O_x$  is invertible. Let  $K = F(x, y)$  with  $y$  integral over  $x$  and let  $f(x, y) = 0$  be the irreducible monic for  $y$ . Define  $d$  on  $K$  by

$$d(g(x, y)) = \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial g}{\partial y} \frac{\partial f}{\partial x} .$$

This is well-defined, and  $d$  sends  $O_x$  into itself by the lemma. Let  $J$  be the ideal of  $O_x$  generated by the values of  $d$  of integral elements.  $J$  is invertible, so there exist  $h_i(x, y)$  in  $K$ ,  $1 \leq i \leq q$ , such that  $h_i d$  sends  $O_x$  into itself and such that there exist  $u_i$  in  $O_x$ ,  $1 \leq i \leq q$ , with  $\sum_{i=1}^q h_i d(u_i) = 1$ . ( $q$  can be taken to be 2. For  $J$  is generated by  $f_x$  and  $f_y$ , since  $d(M(x, y)) = f_y M_x - f_x M_y$  for  $M$  in  $K$ .  $q$  can be taken to be 1 if and only if  $J$  is principal, which need not occur.) The family  $D$  is  $\{h_1 d, \dots, h_q d\}$ . To prove  $O_x$  differentiably simple under  $D$ , suppose the contrary. As in the preceding and following theorems,  $F$  may be assumed to be algebraically closed. If  $O_x$  has a non-zero differential ideal, it has a maximal differential ideal  $I$ , since  $O_x$  has a unit.  $O_x^2$  is not contained in  $I$ , so by Theorem 4 of [1],  $I$  is prime. But every prime ideal of  $O_x$  is maximal; in fact, if  $w$  belongs to  $O_x$ , there is a  $\lambda$  in  $F$  with  $w - \lambda$  in  $I$ . Since  $I$  is differential for  $D$ ,  $h_i d(w) - h_i d(\lambda)$  is in  $I$ ,  $1 \leq i \leq q$ ,  $h_i d(w)$  is in  $I$ ,  $1 \leq i \leq q$ . That is,  $h_i d(w)$  is in  $I$  for all  $w$  in  $O_x$ . Then  $\sum_{i=1}^q h_i d(u_i) = 1$ , 1 is in  $I$ ,  $I = O_x$ . This contradiction proves that  $O_x$  has no differential ideals. Its field of constants is  $F$ . For if  $u$  is in  $F(x, y)$  and  $d(u) = 0$  then  $(d/dx)(u) = 0$ , so that  $u$  belongs to  $F$ .

**THEOREM 3.** *Let  $K, F, x, O_x$  be as in the hypothesis of Theorem 2. Let  $R$  be an order of  $O_x$  and let  $y$  be an element of  $K$  integral over  $x$  with irreducible monic  $f$  such that  $K = F(x, y)$ . Then  $R = O_x$  if and only if  $y$  belongs to  $R$  and the ideal  $J$  in  $R$  generated by  $f_x$  and  $f_y$  is invertible.*

*Proof.* If  $R = O_x$ , then  $y$  belongs to  $R$  and every ideal in  $R$  is invertible. Conversely, suppose that  $y$  belongs to  $R$  and that  $J$ , the ideal generated in  $R$  by the values of  $d$ , is invertible. (Here  $d$  is the same derivation as in Theorem 2.) That is, assume that there exist  $h_i$

in  $K$ ,  $1 \leq i \leq q$ , with  $h_i d$  sending  $R$  into itself, and elements  $v_i$  in  $R$ ,  $1 \leq i \leq q$ , with  $1 = \sum_{i=1}^q h_i d(v_i)$ . We shall prove  $R$  differentiably simple under  $D = \{h_i d, \dots, h_q d\}$ . It is known that every prime ideal of  $R$  is maximal; it fact, if  $I$  is a prime ideal of  $R$ , and  $w$  is an element of  $R$ , there is a  $\lambda$  in  $F$  with  $w - \lambda$  in  $I$ . If  $R$  has a differential ideal, it has a maximal differential ideal, and one proceeds as in Theorem 2. So  $R$  is differentiably simple under  $D$ . By Theorem 1,  $R$  is integrally closed in  $K$ , i.e.,  $R = O_x$  as required.

As an illustration, let  $K = F(x, y)$  with  $f(x, y) = y^n - P(x) = 0$ ,  $n \geq 1$ ,  $P$  a polynomial in  $x$  with no repeated roots. Here  $R = F[x, y]$ . Let us examine the ideal in  $F[x, y]$  generated by  $f_x$  and  $f_y$ , i.e., by  $P'(x)$  and  $y^{n-1}$ . This ideal contains  $y^{n-1}y = y^n = P(x)$  and  $p'(x)$ .  $P(x)$  and  $P'(x)$  have no common factor, so there are polynomials  $Q(x)$  and  $S(x)$  with  $QP + SP' = 1$ . Then the ideal generated by  $f_x$  and  $f_y$  is  $F[x, y]$  and so is trivially invertible. We conclude  $F[x, y] = O_x$ .

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