

# Pacific Journal of Mathematics

## **A CLASS OF LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS**

MORTON LINCOLN SLATER AND HERBERT S. WILF

# A CLASS OF LINEAR DIFFERENTIAL- DIFFERENCE EQUATIONS

MORTON SLATER\* AND HERBERT S. WILF\*\*

**I. Introduction.** The purpose of this paper is to study the following integral equation:

$$(1) \quad \varphi(x) = \int_x^{x+1} K(y)\varphi(y)dy$$

or the differential-difference equation

$$(1') \quad \varphi'(x) = K(x+1)\varphi(x+1) - K(x)\varphi(x)$$

with the boundary condition

$$(2) \quad \lim_{x \rightarrow \infty} \varphi(x) = 1.$$

Equations of the type (1), (1') have been investigated in great generality by many authors. In particular, the interested reader is referred to Yates [6], and Cooke [2], for recent developments, and a bibliography of significant earlier work. The equations of the form (1) which we shall consider are related to the class of linear differential-difference equations with asymptotically constant coefficients, a class treated thoroughly by Wright [5], and Bellman [1].

The novelty of the results below arises from the boundary condition (2) which appears not to have been studied before, and which gives results of an essentially different character from those of the works cited above. The system (1), (2) is of interest in some problems connected with the theory of neutron slowing down (Placzek [3]).

A further departure from previous work is the fact that no use is made of complex variable methods or the asymptotic characteristic equation of the kernel  $K(y)$ .

Aside from some fairly obvious theorems concerning uniqueness, boundedness and positivity, our main results are the following:

- (a) necessary and sufficient conditions for the existence of a solution of (1), (2); this is achieved by constructing a minorant for the solution.
- (b) proof of the existence of  $\varphi(-\infty)$  under fairly general conditions.
- (c) an application of Fubini's theorem to exhibit a rather surpris-

---

Received February 4, 1960.

\* Nuclear Development Corporation of America, White Plains, N. Y.

\*\* The above research was done while the second author was employed by the Nuclear Development Corporation of America, White Plains, N. Y. At present his address is the University of Illinois, Urbana, Illinois.

ing relation between an integral of the solution over the real axis and its limits at  $\pm\infty$ . We assume

- H 1  $K(x)$  is measurable ,
- H 2  $0 < K(x) \leq 1$  , for almost all  $x$  ,
- H 3 For  $x \geq M$  ,  $K(x)$  increases ,
- H 4  $\lim_{x \rightarrow \infty} K(x) = 1$  ,

throughout the paper.

To summarize the results below, we shall give necessary and sufficient conditions for the existence (Theorem 4), uniqueness (Theorem 1), boundedness (Theorem 2), and positivity (Theorem 3) of the solution; a sufficient condition for its monotonicity (Theorem 5); a proof of the existence of  $\varphi(-\infty)$  (Theorem 6) and the evaluation of a definite integral involving the solution (Theorem 7).

By "solution" we shall always mean a function  $\varphi(x)$  satisfying both (1) and (2). All integrals are to be understood in the sense of Lebesgue.

## II. Existence and uniqueness of solutions.

**THEOREM 1.** *Under H 1 – H 4, the solution  $\varphi(x)$ , when it exists, is unique.*

*Proof.* If the theorem is false, there exists a function  $\psi(x)$  not identically zero which satisfies (1) and for which

$$\lim_{x \rightarrow \infty} \psi(x) = 0 .$$

Then by the continuity of  $\psi(x)$  there exist numbers  $\eta$  and  $x_0$  such that  $\eta > 0$ ,  $|\psi(x_0)| = \eta$  and for all  $x > x_0$ ,  $|\psi(x)| < \eta$ . But then

$$\eta = |\psi(x_0)| \leq \int_{x_0}^{x_0+1} |\psi(y)| dy < \eta$$

a contradiction, which completes the proof.

**THEOREM 2.** *With H1 – H4 we have, for any solution  $\varphi(x)$  of (1), (2),*

$$(3) \quad |\varphi(x)| \leq 1 \quad (-\infty < x < \infty) .$$

*Proof.* For if  $|\varphi(x)| > 1$  for some  $x$ , then by (2) and the continuity of  $|\varphi(x)|$  there is a  $C > 1$  and an  $x_0$  such that  $|\varphi(x_0)| = C$ , and for all  $x > x_0$ ,  $|\varphi(x)| < C$ . But then

$$|\varphi(x_0)| \leq \int_{x_0}^{x_0+1} |\varphi(y)| dy$$

implies  $C < C$ , which is a contradiction.

**THEOREM 3.** *Supposing H1 – H4, the solution  $\varphi(x)$  of (1) and (2), when it exists, is positive for all  $x$ , and is non-decreasing for  $x \geq M$ .*

*Proof.* We prove positivity first. If  $\varphi(x)$  is not  $> 0$  for all  $x$ , then by (2) and the continuity of  $\varphi(x)$  there is an  $x_0$  such that  $\varphi(x_0) = 0$  and for all  $x > x_0$ ,  $\varphi(x) > 0$ . Then

$$\varphi(x_0) = 0 = \int_{x_0}^{x_0+1} K(y)\varphi(y)dy ,$$

which is a contradiction by H2.

To prove the monotonicity part, we define

$$(4) \quad \psi_0(x) = 1 ,$$

and

$$(5) \quad \psi_{n+1}(x) = \int_x^{x+1} K(y)\psi_n(y)dy .$$

Since  $0 < K(y) \leq 1$ ,  $\psi_1(x) \leq \psi_0(x)$ , and since

$$(6) \quad \psi_n(x) - \psi_{n+1}(x) = \int_x^{x+1} K(y)[\psi_{n-1}(y) - \psi_n(y)]dy ,$$

we see by induction that  $\{\psi_n(x)\}$  is a decreasing sequence. But since  $\varphi(x) \leq 1 = \psi_0(x)$ , we see by a second induction that  $\psi_n(x) \geq \varphi(x)$  for all  $x$ . Hence the  $\psi_n(x)$  decrease to a limit function  $\psi(x)$  satisfying (1) by Lebesgue's dominated convergence theorem, and

$$\lim_{x \rightarrow \infty} \psi(x) = 1$$

since  $\varphi(x) \leq \psi(x) \leq 1$ . Now  $\psi_0(x)$  is non-decreasing for  $x \geq M$ , and thus so is  $\psi_1(x)$ , and again by induction,  $\psi_n(x)$  and hence  $\psi(x)$ . But by Theorem 1,  $\psi(x) = \varphi(x)$ , which proves the theorem.

**LEMMA 1.** *Under H1 – H4 and*

$$\text{H5: } 1 - K(x) \in \mathcal{L}(M, \infty)$$

*there is a function  $S(x)$  such that  $S(x) \geq 0$ ,  $S(x)$  is non-decreasing,  $\lim_{x \rightarrow \infty} S(x) = 1$ , and*

$$(7) \quad S(x) \leq \int_x^{x+1} K(y)S(y)dy \quad (-\infty < x < \infty) .$$

*Proof.* Define

$$(8) \quad S(x) = \begin{cases} 0 & x \leq M \\ C_n & M + \frac{n}{2} \leq x < M + \frac{n+1}{2} \end{cases} \quad (n = 0, 1, 2, \dots)$$

where the  $C_n$  are constants to be determined, and define

$$(9) \quad q_n = \int_{M+(n/2)}^{M+(n+1/2)} K(y)dy .$$

Now, requiring that  $S(x)$  satisfy (1) at the points  $M + (n/2)$  gives

$$C_n q_n + C_{n+1} q_{n+1} = C_n$$

that is

$$C_{n+1} = C_n \left[ \frac{1 - q_n}{q_{n+1}} \right],$$

and

$$(10) \quad C_{n+1} = \prod_{j=0}^n \left[ \frac{1 - q_j}{q_{j+1}} \right] C_0 .$$

But since

$$\frac{1 - q_j}{q_{j+1}} - 1 = \frac{1 - (q_j + q_{j+1})}{q_{j+1}} \geq 0$$

we see that the  $C_n$  form a non-decreasing sequence. Also

$$\frac{1 - q_j}{q_{j+1}} - 1 \leq \frac{1 - K\{M + (n/2)\}}{K(M)}$$

since  $K(y)$  increases. But then H5 implies that

$$\sum_{n=0}^{\infty} \{1 - K[M + (n/2)]\}$$

converges, and so the limit of the product in (10) exists. We can then choose  $C_0$  so that

$$\lim_{n \rightarrow \infty} C_n = 1 .$$

It remains to show that (7) is everywhere satisfied. If  $x_0 > M$  and  $x_0 \neq M + (n/2)$  for any  $n$ , let  $M + (n_0/2)$  be the largest of the  $M + (n/2)$  which is less than  $x_0$ . Then

$$\begin{aligned} & \int_{x_0}^{x_0+1} K(y)S(y)dy \\ &= \int_{M+(n_0/2)}^{M+(n_0+2/2)} K\left(y + x_0 - M - \frac{n_0}{2}\right)S\left(y + x_0 - M - \frac{n_0}{2}\right)dy \\ &\geq \int_{M+(n_0/2)}^{M+(n_0+2/2)} K(y)S(y)dy \\ &= C_{n_0} \\ &= S(x_0) , \end{aligned}$$

since  $K$  and  $S$  are positive and non-decreasing.

We can now prove

**THEOREM 4.** *Let H1 – H4 hold. Then, necessary and sufficient for the existence of a solution of (1), (2) is H5.*

*Proof.* Suppose  $\varphi(x)$  exists, then

$$\begin{aligned} \varphi(x) &= \int_x^{x+1} K(y)\varphi(y)dy \\ &= \int_x^{x+1} \varphi(y)dy - \int_x^{x+1} [1 - K(y)]\varphi(y)dy . \end{aligned}$$

Choose  $\varepsilon$  between 0 and 1 and  $x_0 > M$  such that  $\varphi(x) > 1 - \varepsilon$  for  $x \geq x_0$ . Then

$$(1 - \varepsilon) \int_{x_0}^{x_0+1} [1 - K(y)]dy \leq \varphi(x_0 + 1) - \varphi(x_0)$$

since  $\varphi(x)$  is non-decreasing (Theorem 3) for  $x \geq M$ . Replacing  $x_0$  by  $x_0 + 1$ , etc., and adding

$$\int_{x_0}^{\infty} [1 - K(y)]dy \leq 1 - \varphi(x_0) < \infty .$$

On the other hand, if H5 holds, consider again the  $\psi_n(x)$  of (4)–(5). Since  $\{\psi_n(x)\}$  is a decreasing sequence, and

$$\psi_{n+1}(x) - S(x) \geq \int_x^{x+1} K(y)[\psi_n(y) - S(y)]dy$$

we see that  $\psi_n(x) \geq S(x)$  for all  $n$  and  $x$ . Hence  $\psi_n(x)$  decreases to a limit  $\varphi(x)$ , satisfying (1), and since

$$1 \geq \varphi(x) \geq S(x)$$

we have (2) also.

**III. Monotonicity.** The solution  $\varphi(x)$  of (1), (2), when it exists, need not to be monotone on the whole real axis. In this section we will first illustrate the above statement, and then give sufficient conditions for the monotonicity of the solution. A lemma that will be of use in the illustration is

**LEMMA 2.** *Let  $K_a(x)$  and  $K_b(x)$  each satisfy H1–H5, and in addition suppose that for all  $x$*

$$K_a(x) \leq K_b(x) .$$

Then if  $\varphi_a(x)$ ,  $\varphi_b(x)$  are the corresponding solutions of (1), (2), we have

$$\varphi_a(x) \leq \varphi_b(x)$$

for all  $x$ .

*Proof.* First,

$$\begin{aligned} \varphi_a(x) &= \int_x^{x+1} K_a(y) \varphi_a(y) dy \\ &\leq \int_x^{x+1} K_b(y) \varphi_a(y) dy . \end{aligned}$$

Now let  $\varphi_{a,0}(x) = \varphi_a(x)$ , and define

$$\varphi_{a,n+1}(x) = \int_x^{x+1} K_b(y) \varphi_{a,n}(y) dy .$$

Then  $\{\varphi_{a,n}(x)\} \uparrow_n$  and is bounded above by 1. Hence the sequence converges to a solution of

$$\begin{cases} \varphi(x) = \int_x^{x+1} K_b(y) \varphi(y) dy \\ \lim_{x \rightarrow \infty} \varphi(x) = 1 . \end{cases}$$

The result then follows from Theorem 1 .

Now consider the family

$$K_a(x) = \frac{x^2 + a}{x^2 + 1} \quad (0 \leq a \leq 1) .$$

Clearly each  $K_a(x)$  satisfies H1-H5. Let  $\varphi_0(x)$  satisfy (1), (2) with  $K(x) = K_0(x)$ . Then

$$\varphi_0'(-1) = -K_0(-1)\varphi_0(-1) = -(1/2)\varphi_0(-1) < 0$$

by Theorem 3. Hence  $\varphi_0(x)$  is not monotone. In fact we can invoke Lemma 2 to show that there exists a number  $a^*\varepsilon(0,1)$  such that for  $a < a^*$   $\varphi_a(x)$  is not monotone. For if not, there exists a sequence  $\{a_n\} \downarrow 0$  such that  $\varphi_{a_n}(x)$  satisfies (1), (2) with  $K(x) = K_{a_n}(x)$  and  $\varphi_{a_n}(x)$  is monotone for each  $n$ . Since  $\{\varphi_{a_n}(x)\}$  decreases to a solution of (1), (2) with  $K(x) = K_0(x)$  (by Lemma 2 and Theorem 1) we must have  $\varphi_0(x)$  monotone which is a contradiction.

The following theorem, however, gives a sufficient condition for the monotonicity of  $\varphi(x)$ :

**THEOREM 5.** *With H1-H5, suppose that for almost all  $x$ ,*

$$(11) \quad K(x + 1) \geq K(x) \int_x^{x+1} K(y)dy .$$

Then  $\varphi(x)$  is non-decreasing on the real axis.

*Proof.* Let  $S_0(x)$  be the function  $S(x)$  of (8). Define

$$(12) \quad S_{n+1}(x) = \int_x^{x+1} K(y)S_n(y)dy \quad (n = 0, 1, \dots) .$$

Then, for all  $n$ ,

$$(13) \quad \begin{aligned} (a) \quad & 0 \leq S_n(x) \leq 1 \\ (b) \quad & \lim_{x \rightarrow \infty} S_n(x) = 1 \\ (c) \quad & S_n(x) \uparrow \varphi(x) . \end{aligned}$$

We show next that with (11), the subsequence  $\{S_{2n}(x)\}$  is a sequence of non-decreasing functions. Clearly  $S_0(x) \uparrow_x$  for all  $x$ . Now suppose that for all  $k \leq n$ ,  $S_{2k}(x) \uparrow_x$  for all  $x$ . Then

$$S'_{2n+2}(x) = K(x + 1)S_{2n+1}(x + 1) - K(x)S_{2n+1}(x)$$

a.e.

Now by (13)(c),

$$S_{2n+1}(x + 1) \geq S_{2n}(x + 1)$$

and since

$$S_{2n+1}(x) = \int_x^{x+1} K(y)S_{2n}(y)dy ,$$

it follows from the inductive hypothesis that

$$S_{2n+1}(x) \leq S_{2n}(x + 1) \int_x^{x+1} K(y)dy .$$

Hence

$$\begin{aligned} S'_{2n+2}(x) & \geq \left[ K(x + 1) - K(x) \int_x^{x+1} K(y)dy \right] S_{2n}(x + 1) \\ & \geq 0 \quad \text{a.e.} \end{aligned}$$

by (11), which proves the theorem, since  $S_{2n+2}(x)$  is absolutely continuous.

**IV. Behaviour for large negative values of  $x$ .** We wish now to explore the limiting behaviour of the solution  $\varphi(x)$  as  $x \rightarrow -\infty$ . We have seen that the solution will in general oscillate. We will establish below a sufficient condition for the existence of  $\varphi(-\infty)$ .



**THEOREM 6.** *Suppose  $\varphi(x)$  is a solution of (1), (2). Let  $K(x)$  satisfy H1-H4, and further suppose that*

$$(14) \quad \lim_{x \rightarrow -\infty} \int_x^{x+1} |K(t+1) - K(t)| dt = 0.$$

Then

$$(15) \quad \lim_{x \rightarrow -\infty} \varphi(x) \equiv \varphi(-\infty)$$

exists.

*Proof.* Let  $m$  (resp.  $M$ ) be the lim inf (resp. lim sup) of  $\varphi(x)$  as  $x \rightarrow -\infty$ , and write

$$k = \limsup_{x \rightarrow -\infty} \int_x^{x+1} |\varphi'(t)| dt.$$

Let  $\varepsilon > 0$  be given. Let  $-x_0 > 0$  be chosen so that  $\varphi(x_0) < m + \varepsilon$  and for  $x \leq x_0$ ,  $\int_x^{x+1} |\varphi'(t)| dt < k + \varepsilon$ . Let  $x_1$  be the first point to the left of  $x_0$  at which  $\varphi(x_1) = M - \varepsilon$ , so that  $\varphi(x) < M - \varepsilon$  on the interval  $x_1 < x \leq x_0$ . It follows that  $x_0 < x_1 + 1$  for otherwise a "proper" maximum for  $\varphi(x)$  on  $x_1 \leq x \leq x_1 + 1$  occurs at  $x_1$ , which is impossible. For the same reason there is a point  $x_2$  satisfying  $x_1 < x_0 < x_2 \leq x_1 + 1$  at which  $\varphi(x_2) = M - \varepsilon$ . Hence

$$\begin{aligned} k + \varepsilon &\geq \int_{x_1}^{x_1+1} |\varphi'(t)| dt \geq \int_{x_1}^{x_0} |\varphi'(t)| dt + \int_{x_0}^{x_2} |\varphi'(t)| dt \\ &\geq \left| \int_{x_1}^{x_0} \varphi'(t) dt \right| + \left| \int_{x_0}^{x_2} \varphi'(t) dt \right| \\ &= (M - m - \varepsilon) + (M - m - \varepsilon). \end{aligned}$$

Hence  $k \geq 2(M - m)$ .

However, since

$$\varphi'(x) = K(x+1)[\varphi(x+1) - \varphi(x)] + \varphi(x)[K(x+1) - K(x)],$$

we find, using (14)  $k \leq M - m$ . Thus  $M = m$ , which proves the theorem, and incidently,  $k = 0$ .

**REMARK.**  $\int_x^{x+1} |K(t+1) - K(t)| dt \leq \int_x^{x+2} |1 - K(t)| dt$ ; thus in the above theorem, (14) may be replaced by  $1 - K(x) \in \mathcal{L}(-\infty, \infty)$ , and the conclusion is still valid.

We are now able to prove the following integral relationship.

**THEOREM 7.** *Suppose  $\varphi(x)$  is a solution of (1), (2). Let  $K(x)$  satisfy H1-H4, and suppose further*

$$(16) \quad 1 - K(x) \in \mathcal{L}(-\infty, \infty).$$

*Then*

$$(17) \quad \int_{-\infty}^{\infty} [1 - K(y)]\varphi(y)dy = \frac{1 - \varphi(-\infty)}{2}.$$

*Proof.* Put

$$F(x) = \int_0^1 \varphi(x - y)ydy.$$

Then

$$\begin{aligned} F'(x) &= \int_0^1 \varphi'(x - y)ydy = -\varphi(x - 1) + \int_0^1 \varphi(x - y)dy \\ &= \int_0^1 \varphi(x - y)[1 - K(x - y)]dy. \end{aligned}$$

Since  $\varphi(x)$  is bounded and  $1 - K(x) \in \mathcal{L}(-\infty, \infty)$ , it follows from Fubini's theorem (see reference 4, p. 87) that  $F'(x) \in \mathcal{L}(-\infty, \infty)$ , and

$$F(\infty) - F(-\infty) = \int_{-\infty}^{\infty} [1 - K(t)]\varphi(t)dt.$$

But since  $\varphi(x)$  satisfies (2),  $F(\infty) = (1/2)$ , and by the remark following Theorem 6,  $F(-\infty) = (1/2)\varphi(-\infty)$ . This completes the proof.

## REFERENCES

1. R. Bellman, *On the existence and boundedness of solutions of non-linear differential-difference equations*, Ann. Math. **50** (1949), 347-355.
2. K. L. Cooke, *The asymptotic behavior of the solutions of linear and nonlinear differential-difference equations*, Trans. Amer. Math. Soc., **75** (1953), 80-105.
3. G. Plackzek, *On the theory of the slowing down of neutrons in heavy substances*, Physical Review, **69** (1946), 423-438.
4. S. Saks, *Theory of the Integral*, 1937.
5. E. M. Wright, *The linear difference-differential equation with asymptotically constant coefficients*, Amer. J. Math. **70** (1948), 221-238.
6. B. Yates, *The linear difference-differential equation with linear coefficients*, Trans. Amer. Math. Soc. **80** (1955), 281-298.



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

DAVID GILBARG  
Stanford University  
Stanford, California

F. H. BROWNELL  
University of Washington  
Seattle 5, Washington

A. L. WHITEMAN  
University of Southern California  
Los Angeles 7, California

L. J. PAIGE  
University of California  
Los Angeles 24, California

## ASSOCIATE EDITORS

E. F. BECKENBACH  
T. M. CHERRY  
D. DERRY

E. HEWITT  
A. HORN  
L. NACHBIN

M. OHTSUKA  
H. L. ROYDEN  
M. M. SCHIFFER

E. SPANIER  
E. G. STRAUS  
F. WOLF

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE COLLEGE  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE COLLEGE  
UNIVERSITY OF WASHINGTON  
\* \* \*

AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
HUGHES AIRCRAFT COMPANY  
SPACE TECHNOLOGY LABORATORIES  
NAVAL ORDNANCE TEST STATION

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

M. Altman, <i>An optimum cubically convergent iterative method of inverting a linear bounded operator in Hilbert space</i> .....	1107
Nesmith Cornett Ankeny, <i>Criterion for <math>r</math>th power residuacity</i> .....	1115
Julius Rubin Blum and David Lee Hanson, <i>On invariant probability measures I</i> .....	1125
Frank Featherstone Bonsall, <i>Positive operators compact in an auxiliary topology</i> .....	1131
Billy Joe Boyer, <i>Summability of derived conjugate series</i> .....	1139
Delmar L. Boyer, <i>A note on a problem of Fuchs</i> .....	1147
Hans-Joachim Bremermann, <i>The envelopes of holomorphy of tube domains in infinite dimensional Banach spaces</i> .....	1149
Andrew Michael Bruckner, <i>Minimal superadditive extensions of superadditive functions</i> .....	1155
Billy Finney Bryant, <i>On expansive homeomorphisms</i> .....	1163
Jean W. Butler, <i>On complete and independent sets of operations in finite algebras</i> .....	1169
Lucien Le Cam, <i>An approximation theorem for the Poisson binomial distribution</i> .....	1181
Paul Civin, <i>Involutions on locally compact rings</i> .....	1199
Earl A. Coddington, <i>Normal extensions of formally normal operators</i> .....	1203
Jacob Feldman, <i>Some classes of equivalent Gaussian processes on an interval</i> .....	1211
Shaul Foguel, <i>Weak and strong convergence for Markov processes</i> .....	1221
Martin Fox, <i>Some zero sum two-person games with moves in the unit interval</i> .....	1235
Robert Pertsch Gilbert, <i>Singularities of three-dimensional harmonic functions</i> .....	1243
Branko Grünbaum, <i>Partitions of mass-distributions and of convex bodies by hyperplanes</i> .....	1257
Sidney Morris Harmon, <i>Regular covering surfaces of Riemann surfaces</i> .....	1263
Edwin Hewitt and Herbert S. Zuckerman, <i>The multiplicative semigroup of integers modulo <math>m</math></i> .....	1291
Paul Daniel Hill, <i>Relation of a direct limit group to associated vector groups</i> .....	1309
Calvin Virgil Holmes, <i>Commutator groups of monomial groups</i> .....	1313
James Fredrik Jakobsen and W. R. Utz, <i>The non-existence of expansive homeomorphisms on a closed 2-cell</i> .....	1319
John William Jewett, <i>Multiplication on classes of pseudo-analytic functions</i> .....	1323
Helmut Klingen, <i>Analytic automorphisms of bounded symmetric complex domains</i> .....	1327
Robert Jacob Koch, <i>Ordered semigroups in partially ordered semigroups</i> .....	1333
Marvin David Marcus and N. A. Khan, <i>On a commutator result of Taussky and Zassenhaus</i> .....	1337
John Glen Marica and Steve Jerome Bryant, <i>Unary algebras</i> .....	1347
Edward Peter Merkes and W. T. Scott, <i>On univalence of a continued fraction</i> .....	1361
Shu-Teh Chen Moy, <i>Asymptotic properties of derivatives of stationary measures</i> .....	1371
John William Neuberger, <i>Concerning boundary value problems</i> .....	1385
Edward C. Posner, <i>Integral closure of differential rings</i> .....	1393
Marian Reichaw-Reichbach, <i>Some theorems on mappings onto</i> .....	1397
Marvin Rosenblum and Harold Widom, <i>Two extremal problems</i> .....	1409
Morton Lincoln Slater and Herbert S. Wilf, <i>A class of linear differential-difference equations</i> .....	1419
Charles Robson Storey, Jr., <i>The structure of threads</i> .....	1429
J. François Treves, <i>An estimate for differential polynomials in <math>\partial/\partial z_1, \dots, \partial/\partial z_n</math></i> .....	1447
J. D. Weston, <i>On the representation of operators by convolutions integrals</i> .....	1453
James Victor Whittaker, <i>Normal subgroups of some homeomorphism groups</i> .....	1469