

Pacific Journal of Mathematics

GENERALIZED TWISTED FIELDS

A. A. ALBERT

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1. Introduction. Consider a finite field \mathfrak{K} . If V is any automorphism of \mathfrak{K} we define \mathfrak{K}_V to be the *fixed field* of \mathfrak{K} under V . Let S and T be any automorphism of \mathfrak{K} and define F to be the fixed field

$$(1) \quad \mathfrak{F} = \mathfrak{F}_q = (\mathfrak{K}_S)_T = (\mathfrak{K}_T)_S,$$

under both S and T . Then \mathfrak{F} is the field of $q = p^a$ elements, where p is the characteristic of \mathfrak{K} , and \mathfrak{K} is a field of degree n over \mathfrak{F} . We shall assume that

$$(2) \quad n > 2, \quad q > 2.$$

Then the period of a primitive element of \mathfrak{K} is $q^n - 1$ and there always exist elements c in \mathfrak{K} such that $c \neq k^{q-1}$ for any element k of \mathfrak{K} . Indeed we could always select c to be a primitive element of \mathfrak{K} .

Define a product (x, y) on the additive abelian group \mathfrak{K} , in terms of the product xy of the field \mathfrak{K} , by

$$(3) \quad (x, y) = xA_y = yB_x = xy - c(xT)(yS),$$

for c in \mathfrak{K} . Then

$$(4) \quad A_y = R_y - TR_{c(yS)}, \quad B_x = R_x - SR_{c(xT)},$$

where the transformation $R_y = R[y]$ is defined for all y in \mathfrak{K} by the product $xy = xR_y$ of \mathfrak{K} . Then the condition that $(x, y) \neq 0$ for all $xy \neq 0$ is equivalent to the property that

$$(5) \quad c \neq \frac{x}{xT} \frac{y}{yS},$$

for any nonzero x and y of \mathfrak{K} . But the definition of a generating automorphism U of \mathfrak{K} over \mathfrak{F} by $xU = x^q$ implies that

$$(6) \quad S = U^\beta, \quad T = U^\gamma.$$

We shall assume that $S \neq I$, $T \neq I$, so that

$$(7) \quad 0 < \beta < n, \quad 0 < \gamma < n.$$

Then $xy[(xS)(yT)]^{-1} = z^{q-1}$, where

$$(8) \quad 1 - q^\beta = (q - 1)^\delta, \quad 1 - q^\gamma = (q - 1)^\epsilon, \quad z = x^\delta y^\epsilon.$$

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Thus the condition that $c \neq k^{q-1}$ is sufficient to insure the property that $(x, y) \neq 0$ whenever $xy \neq 0$.

For every c satisfying (5) we can define a division ring $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, T, c)$, with unity quantity $f = e - c$, where e is the unity quantity of \mathfrak{R} . It is the same additive group as K and we define the product $x \cdot y$ of D by

$$(9) \quad xA_c \cdot yB_c = (x, y) .$$

These rings may be seen to generalize the twisted fields defined in an earlier paper.¹

We shall show that \mathfrak{D} is isomorphic to \mathfrak{R} if and only if $S = T$. Indeed we shall derive the following result.

THEOREM 1. *Let $S \neq I$, $T \neq I$, $S \neq T$. Then the right nucleus of $\mathfrak{D}(\mathfrak{R}, S, T, c)$ is $f\mathfrak{R}_s$ and the left nucleus of $\mathfrak{D}(\mathfrak{R}, S, T, c)$ is $f\mathfrak{R}_t$. If \mathfrak{S} is the set of all elements g of \mathfrak{R} such that $gS = gT$ then $gA_c = gB_c$ and $\mathfrak{L}A_c = \mathfrak{L}B_c$ is the middle nucleus of \mathfrak{D} .*

The result above implies that $f\mathfrak{S}$ is the center of $\mathfrak{D}(\mathfrak{R}, S, T, c)$. Since it is known² that isotopic rings have isomorphic right (left and middle) nuclei, our results imply that the (generalized) twisted fields $\mathfrak{D}(\mathfrak{R}, S, T, c)$ are new whenever the group generated by either S or T is not the group generated by S and T . In this case our new twisted fields define new finite non-Desarguesian projective planes.³

2. The fundamental equation. Consider the equation

$$(9) \quad A_x A_c^{-1} A_y = A_z ,$$

for x, y and z in \mathfrak{R} . Assume that the degree of \mathfrak{R} over \mathfrak{R}_T is m , where we shall now assume that

$$(10) \quad m > 2 .$$

¹ For earlier definitions of twisted fields see the case $c = -1$ in *On nonassociative division algebras*, Trans. Amer. Math. Soc. **72** (1952), 296-309 and the general case in *Finite noncommutative division algebras*, Proc. Amer. Math. Soc. **9** (1958), 928-932. In those papers we defined a product $[x, y] = x(yT) - cy(xT)$ so that $(x, y) = [x, yT^{-1}] = xy - c(yS)(xT)$ is the product (3) with $S = T^{-1}$.

² This result was originally given for loops by R. H. Bruck. It is easy to show that, if \mathfrak{D} and \mathfrak{D}_0 are isotopic rings with isotopy defined by the relation $QR_{xP} = R_x^{(c)}QR_z$, then the mapping $x \rightarrow (zx)P^{-1}$ induces an isomorphism of the right nucleus \mathfrak{D} onto that of \mathfrak{D}_0 , and the mapping $x \rightarrow (xz)P^{-1}$ induces an isomorphism of the middle nucleus of \mathfrak{D} onto that of \mathfrak{D}_0 .

³ Two finite projective planes $\mathfrak{M}(\mathfrak{D})$ and $\mathfrak{M}(\mathfrak{D}_0)$ coordinatized by division rings \mathfrak{D} and \mathfrak{D}_0 respectively are known to be isomorphic if and only if \mathfrak{D} and \mathfrak{D}_0 are isotopic. See the author's *Finite division algebras and finite planes*, Proceedings of Symposia in Applied Mathematics; vol. 10, pp. 53-70.

Then the *norm* in \mathfrak{K} over \mathfrak{K}_T of any element k of \mathfrak{K} is

$$(11) \quad \nu(k) = k(kT) \cdots (kT^{m-1}),$$

and $\nu(k)$ is in \mathfrak{K}_T , that is,

$$(12) \quad \nu(k) = [\nu(k)]T$$

for every k of \mathfrak{K} . Thus

$$(13) \quad I - (TR_c)^m = I - R_{\nu(e)} = R_d,$$

where

$$(14) \quad d = e - \nu(c) = dT.$$

Now

$$(15) \quad A_e = I - TR_c, \quad B_e = I - SR_c,$$

and we obtain

$$(16) \quad A_e[I + TR_c + (TR_c)^2 + \cdots + (TR_c)^{m-1}] = R_d,$$

so that

$$(17) \quad I + TR_c + (TR_c)^2 + \cdots + (TR_c)^{m-1} = A_e^{-1}R_d.$$

Our definition (4) implies that

$$(18) \quad R_a A_y = A_y R_a, \quad R_b B_x = B_x R_b$$

for every x and y of K , providing that

$$(19) \quad a = aT, \quad b = bS.$$

In particular, $R_a A_y = A_y R_a$, and so (9) is equivalent to

$$(20) \quad A_x[I + (TR_c) + (TR_c)^2 + \cdots + (TR_c)^{m-1}]A_y = A_z R_d.$$

It is well known that distinct automorphisms of any field \mathfrak{K} are linearly independent in the field of right multiplications of \mathfrak{K} . Thus we can equate the coefficients of the distinct powers of T in the equation (20). The right member of (20) is $R_{za} - TR_{ca(zS)}$ and so does not contain the term in T^{m-1} when $m > 2$. It follows that

$$(21) \quad R_x[(TR_c)^{m-1}R_y - (TR_c)^{m-2}(TR_c)R_{yS}] \\ - TR_{c(xS)}[(TR_c)^{m-2}R_y - (TR_c)^{m-3}(TR_c)R_{yS}] = 0.$$

This equation is equivalent to

$$(22) \quad xT^{m-1}(y - yS) = xST^{m-2}(y - yS),$$

and so to the relation

$$(23) \quad [(x - xST^{-1})T^{m-1}](y - yS) = 0 .$$

By symmetry we have the following result.

LEMMA 1. *Let T have period $m > 2$. Then the equation $A_x A_e^{-1} A_y = A_z$ holds for some x, y, z in \mathfrak{R} only if $y = yS$ or $x = xST^{-1}$. If S has period $m_0 > 2$ the equation $B_y B_e^{-1} B_x = B_z$ holds for some x, y, z in \mathfrak{R} only if $x = xT$ or $y = yST^{-1}$.*

3. The nuclei. The ring $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, T, c)$ has its product defined by

$$(24) \quad x \cdot y = xR_y^{(c)} = yL_x^{(c)},$$

where

$$(25) \quad R_{yB_e}^{(c)} = A_e^{-1} A_y, \quad L_{xA_e}^{(c)} = B_e^{-1} B_x .$$

When $S = T$ our formula (3) becomes $(x, y) = xy - c[(xy)S] = xy(I - SR_c)$. But then the ring \mathfrak{D}_0 , defined by the product (x, y) , is isotopic to the field \mathfrak{R} . Since $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, S, c)$ is isotopic to \mathfrak{D}_0 it is isotopic to \mathfrak{R} , and it is well known that \mathfrak{D} is then also isomorphic to \mathfrak{R} . Assume henceforth that

$$(26) \quad S \neq T .$$

The right nucleus of \mathfrak{D} is the set \mathfrak{N}_ρ of all elements z_ρ in \mathfrak{R} such that

$$(27) \quad (x \cdot y) \cdot z_\rho = x \cdot (y \cdot z_\rho) ,$$

for every x and y of \mathfrak{R} . Suppose that $b = bS$ so that

$$(28) \quad A_b = R_b - TR_{c(bS)} = (I - TR_c)R_b, \quad A_e^{-1} A_b = R_b .$$

By (18) we know that $R_b B_x = B_x R_b$, and so $R_b (B_e^{-1} B_x) = (B_e^{-1} B_x) R_b$ for every x of \mathfrak{R} . By (25) this implies that the transformation

$$(29) \quad R_b = A_e^{-1} A_b = R_{bB_e}^{(c)}$$

commutes with every $L_x^{(c)}$. However, (27) is equivalent to

$$(30) \quad L_x^{(c)} R_{z_\rho}^{(c)} = R_{z_\rho}^{(c)} L_x^{(c)} .$$

Thus $bB_e = b(I - SR_c) = b(e - c) = bf$ is in \mathfrak{N}_ρ . We have proved that the right nucleus of $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, T, c)$ contains the field $f\mathfrak{R}_s$, a subring of \mathfrak{D} isomorphic to \mathfrak{R}_s .

The left nucleus \mathfrak{N}_λ of \mathfrak{D} consists of all z_λ such that

$$(31) \quad (z_\lambda \cdot y) \cdot x = z_\lambda \cdot (y \cdot x)$$

for all x and y of \mathfrak{R} . This equation is equivalent to

$$(32) \quad L_{z_\lambda}^{(c)} R_x^{(c)} = R_x^{(c)} L_{z_\lambda}^{(c)}$$

for every x of \mathfrak{R} . If $a = aT$ then $B_a = (I - SR_c)R_a$, $B^{-1}B_a = R_a = L_{aA_e}^{(c)}$ commutes with every A_y and every $R_x^{(c)}$, and we see that the left nucleus of $\mathfrak{D}(\mathfrak{R}, S, T, c)$ contains the field $f\mathfrak{R}_T$ isomorphic to \mathfrak{R}_T .

The *middle nucleus* of $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, T, c)$ is the set \mathfrak{N}_μ of all z_μ of \mathfrak{R} such that

$$(33) \quad (x \cdot z_\mu) \cdot y = x \cdot (z_\mu \cdot y)$$

for every x and y of \mathfrak{R} . This equation is equivalent to

$$(34) \quad R_z^{(c)} R_y^{(c)} = R_{z \cdot y}^{(c)},$$

where $z = z_\mu$. However, we can observe that the assumption that

$$(35) \quad R_z^{(c)} R_y^{(c)} = R_v^{(c)},$$

for some v in \mathfrak{R} , implies that $(f \cdot z) \cdot y = f \cdot v = v = z \cdot y$, Hence (34) holds for every y in \mathfrak{R} if and only if

$$(36) \quad A_g A_e^{-1} A_y = A_v,$$

for every y of \mathfrak{R} , where v is in \mathfrak{R} and

$$(37) \quad gB_e = z = z_\mu.$$

If $gS = gT$ then $A_g = R_g - TR_{c(gS)} = R_g - TR_{c(gT)} = R_g - R_g TR_c = R_g A_e$. Then (36) becomes

$$(38) \quad R_g A_y = R_g (R_g - TR_{c(gS)}) = R_{gy} - TR_{c(ySgT)} = A_{gy}.$$

Hence $gB_e = g(I - SR_c) = g - (gS)c = g - (gT)c = gA_e$, and \mathfrak{N}_μ contains the field of all elements gB_e for $gS = gT$.

We are now able to derive the converse of these results. We first observe that (27) is equivalent to

$$(39) \quad R_y^{(c)} R_z^{(c)} = R_{y \cdot z}^{(c)},$$

for every y of \mathfrak{R} , where $z = z_\rho$. This equation is equivalent to

$$(40) \quad A_y A_e^{-1} A_u = A_v,$$

where $z = uB_e$. If the period of T is $m > 2$ we use Lemma 1 to see that, if we take $y \neq yST^{-1}$, then $u = uS$, $z = uB_e = fu$. The stated choice of y is always possible since we assuming that $S \neq T$ and so some element of \mathfrak{R} is not left fixed by ST^{-1} . Thus $\mathfrak{N} = f\mathfrak{R}_S$. Similarly, if the period of S is not two then $\mathfrak{N}_\lambda = f\mathfrak{R}_T$. Assume that one of S and T has period two.

The automorphisms S and T cannot both have period two. For the group G of automorphisms of \mathfrak{R} is a cyclic group and has a unique subgroup \mathfrak{S} of order two. This group contains I and only one other automorphism. If S and T both had period two we would have $S = T$ and so $m = n = 2$, contrary to hypothesis. Thus we may assume that one of S and T has period two. There is clearly no loss of generality if we assume that T has period two, so that the period of S is at least three. By the argument already given we have $\mathfrak{R}_\lambda = f\mathfrak{R}_T$. We are then led to study (40) as holding for all elements y of \mathfrak{R} , where $z_p = uB_e$. Now

$$(41) \quad A_e = I - TR_c, \quad A_e(I + TR_c) = R_a, \quad d = e - c(cT) = dT.$$

But then (40) becomes

$$(42) \quad [R_y - TR_{c(yS)}](I + TR_c)[R_u - TR_{c(uS)}] = R_{vd} - TR_{cd(vS)}.$$

This yields the equations

$$(43) \quad y[u - c(cT)(uS)] - (yST)[c(cT)](u - uS) = vd,$$

$$(44) \quad yT(u - uS) - yS[u - (uS)c(cT)] = -d(vS).$$

Hence

$$\begin{aligned} d(yS)[uS - (cS)(cST)(uS^2)] - yS^2T(cS)(cST)(uS - uS^2)d &= vS(dS)d \\ &= (dS)yS[u - (uS)c(cT)] - yT(u - uS)(dS). \end{aligned}$$

Since this holds for all y we have the transformation equation

$$(45) \quad SR[d(uS) - d(cS)(cST)uS^2] - S^2TR[d(cS)(cST)(uS - uS^2)] \\ = SR[dSu - (dS)(uS)c(cT)] - TR[(u - uS)dS].$$

Since $S^2 \neq I$ and $T \neq S$, S^2T we know that the coefficient of S^2T is zero. Thus $(u - uS)dS = 0$ and $u = uS$ as desired. This shows that $\mathfrak{R}_p = f\mathfrak{R}_S$.

The *middle nucleus* condition (36) implies that $gS = gT$ if T does not have period two. When T does have period two but S does not have period two the analogous property

$$(46) \quad L_{x,z}^{(c)} = L_z^{(c)}L_x^{(c)}$$

is equivalent to

$$(47) \quad B_\sigma B_e^{-1} B_x = B_v,$$

and we see again that $gS = gT$. This completes our proof of the theorem stated in the introduction.

4. Commutativity. It is known⁴ that $\mathfrak{D} = (\mathfrak{R}, S, S^{-1}, c)$ is commutative if and only if $c = -1$. There remains the case where

$$(48) \quad S \neq I, T \neq I, ST \neq I, S \neq T.$$

Any $\mathfrak{D}(\mathfrak{R}, S, T, c)$ is commutative if and only if $R_x^{(c)} = L_x^{(c)}$ for every x of \mathfrak{R} . Assume first that $\mathfrak{R}_s \neq \mathfrak{R}_t$. There is clearly no loss of generality if we assume that there is an element b in \mathfrak{R}_s and not in \mathfrak{R}_t , since the roles of S and T can be interchanged when $\mathfrak{D}(\mathfrak{R}, S, T, c)$ is commutative. Thus we have $b = bS \neq bT$. By (28) we know that $A_b = A_e R_b$ and so we have $R_{bf}^{(c)} = R_b$. Then $L_{bf}^{(c)} = B_e^{-1} B_y = R_b$, where $y = (bf)A_e^{-1}$. It follows that

$$(49) \quad B_y = R_y - SR_{c(yT)} = B_e R_b = (I - SR_c)R_b.$$

Then $R_y = R_b$, $y = b$, $c(yT) = c(bT) = cb$, and $b = bT$ contrary to hypothesis.

We have shown that if $\mathfrak{D}(\mathfrak{R}, S, T, c)$ is commutative the automorphisms S and T have the same fixed fields, that is, $b = bS$ if and only if $b = bT$, b is in \mathfrak{F} . Thus S and T both generate the cyclic automorphism group \mathfrak{G} of order n of \mathfrak{R} over \mathfrak{F} , and S is a power of T . Since $T^{-1} = T^{n-1} \neq S$ there exists an integer r such that

$$(50) \quad 0 < r < n - 1, S = T^r.$$

We now use the fact that $R_x^{(c)} = L_x^{(c)}$ for every x of K to see that $A_e^{-1} A_x = B_e^{-1} B_y$ for every x of \mathfrak{R} , where $y = xB_e A_e^{-1}$. Also $(TR_c)^n = (SR_c)^n = R_{\nu(c)}$, and our condition becomes

$$(51) \quad [I + TR_c + (TR_c)^2 + \cdots + (TR_c)^{n-1}][R_x - TR_{c(xS)}] \\ = [I + SR_c + (SR_c)^2 + \cdots + (SR_c)^{n-1}][R_y - SR_{c(yT)}],$$

where we have used the fact that $d = e - \nu(c) = dT = dS$. Compute the constant term to obtain the equation

$$(52) \quad R_x - (TR_c)^n R_{xS} = R_y - (SR_c)_u R_{yT}.$$

This is equivalent to the relation $x - [\nu(c)](xS) = y - [\nu(c)]yT$ for every x of K , where $y = xB_e A_e^{-1}$. Thus (52) is equivalent to

$$(53) \quad I - SR_{\nu(c)} = B_e A_e^{-1} [I - TR_{\nu(c)}].$$

We also compute the term in T^r in (51). Since $r < n - 1$ the left member of this term is $(TR_c)^r R_x - (TR_c)^r R_{xS}$, which is equal to $R^r R_{gc}(R_x - R_{xS})$, where $g = (cT)(cT)^2 \cdots (cT)^{r-1}$. The right member is the term in S , and this is $SR_c(R_y - R_{yT})$. Hence $(x - xS)g = y - yT$, a result equivalent to

⁴ See footnote 1.

$$(54) \quad (I - S)R_g = B_e A_e^{-1}(I - T).$$

Since the transformations $I - T$ and $I - TR_{\nu(c)}$ commute we may use (53) to obtain

$$(55) \quad (I - S)R_g[I - TR_{\nu(c)}] = [I - SR_{\nu(c)}](I - T).$$

By (48) we may equate coefficients of I, S, T and ST , respectively. The constant term yields $g = e$. The term in S then yields $\nu(c) = e$ which is impossible when S and T generate the same group and $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, T, c)$ is a division algebra.

We have proved the following result.

THEOREM 2. *Let $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, T, c)$ be a division algebra defined for $S \neq I, T \neq I, S \neq T$. Then \mathfrak{D} is commutative if and only if $ST = I$ and $c = -1$.*

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