

# Pacific Journal of Mathematics

**GENERALIZED TWISTED FIELDS**

A. A. ALBERT

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**1. Introduction.** Consider a finite field  $\mathfrak{K}$ . If  $V$  is any automorphism of  $\mathfrak{K}$  we define  $\mathfrak{K}_V$  to be the *fixed field* of  $\mathfrak{K}$  under  $V$ . Let  $S$  and  $T$  be any automorphism of  $\mathfrak{K}$  and define  $F$  to be the fixed field

$$(1) \quad \mathfrak{F} = \mathfrak{F}_q = (\mathfrak{K}_S)_T = (\mathfrak{K}_T)_S,$$

under both  $S$  and  $T$ . Then  $\mathfrak{F}$  is the field of  $q = p^a$  elements, where  $p$  is the characteristic of  $\mathfrak{K}$ , and  $\mathfrak{K}$  is a field of degree  $n$  over  $\mathfrak{F}$ . We shall assume that

$$(2) \quad n > 2, \quad q > 2.$$

Then the period of a primitive element of  $\mathfrak{K}$  is  $q^n - 1$  and there always exist elements  $c$  in  $\mathfrak{K}$  such that  $c \neq k^{q-1}$  for any element  $k$  of  $\mathfrak{K}$ . Indeed we could always select  $c$  to be a primitive element of  $\mathfrak{K}$ .

Define a product  $(x, y)$  on the additive abelian group  $\mathfrak{K}$ , in terms of the product  $xy$  of the field  $\mathfrak{K}$ , by

$$(3) \quad (x, y) = xA_y = yB_x = xy - c(xT)(yS),$$

for  $c$  in  $\mathfrak{K}$ . Then

$$(4) \quad A_y = R_y - TR_{c(yS)}, \quad B_x = R_x - SR_{c(xT)},$$

where the transformation  $R_y = R[y]$  is defined for all  $y$  in  $\mathfrak{K}$  by the product  $xy = xR_y$  of  $\mathfrak{K}$ . Then the condition that  $(x, y) \neq 0$  for all  $xy \neq 0$  is equivalent to the property that

$$(5) \quad c \neq \frac{x}{xT} \frac{y}{yS},$$

for any nonzero  $x$  and  $y$  of  $\mathfrak{K}$ . But the definition of a generating automorphism  $U$  of  $\mathfrak{K}$  over  $\mathfrak{F}$  by  $xU = x^q$  implies that

$$(6) \quad S = U^\beta, \quad T = U^\gamma.$$

We shall assume that  $S \neq I$ ,  $T \neq I$ , so that

$$(7) \quad 0 < \beta < n, \quad 0 < \gamma < n.$$

Then  $xy[(xS)(yT)]^{-1} = z^{q-1}$ , where

$$(8) \quad 1 - q^\beta = (q - 1)^\delta, \quad 1 - q^\gamma = (q - 1)^\epsilon, \quad z = x^\delta y^\epsilon.$$

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Thus the condition that  $c \neq k^{a-1}$  is sufficient to insure the property that  $(x, y) \neq 0$  whenever  $xy \neq 0$ .

For every  $c$  satisfying (5) we can define a division ring  $\mathfrak{D} = \mathfrak{D}(\mathfrak{K}, S, T, c)$ , with unity quantity  $f = e - c$ , where  $e$  is the unity quantity of  $\mathfrak{K}$ . It is the same additive group as  $K$  and we define the product  $x \cdot y$  of  $D$  by

$$(9) \quad xA_e \cdot yB_e = (x, y).$$

These rings may be seen to generalize the twisted fields defined in an earlier paper.<sup>1</sup>

We shall show that  $\mathfrak{D}$  is isomorphic to  $\mathfrak{K}$  if and only if  $S = T$ . Indeed we shall derive the following result.

**THEOREM 1.** *Let  $S \neq I$ ,  $T \neq I$ ,  $S \neq T$ . Then the right nucleus of  $\mathfrak{D}(\mathfrak{K}, S, T, c)$  is  $f\mathfrak{R}_S$  and the left nucleus of  $\mathfrak{D}(\mathfrak{K}, S, T, c)$  is  $f\mathfrak{R}_T$ . If  $\mathfrak{L}$  is the set of all elements  $g$  of  $\mathfrak{K}$  such that  $gS = gT$  then  $gA_e = gB_e$  and  $\mathfrak{L}A_e = \mathfrak{L}B_e$  is the middle nucleus of  $\mathfrak{D}$ .*

The result above implies that  $f\mathfrak{Z}$  is the center of  $\mathfrak{D}(\mathfrak{K}, S, T, c)$ . Since it is known<sup>2</sup> that isotopic rings have isomorphic right (left and middle) nuclei, our results imply that the (generalized) twisted fields  $\mathfrak{D}(\mathfrak{K}, S, T, c)$  are new whenever the group generated by either  $S$  or  $T$  is not the group generated by  $S$  and  $T$ . In this case our new twisted fields define new finite non-Desarguesian projective planes.<sup>3</sup>

## 2. The fundamental equation. Consider the equation

$$(9) \quad A_x A_e^{-1} A_y = A_z,$$

for  $x, y$  and  $z$  in  $\mathfrak{K}$ . Assume that the degree of  $\mathfrak{K}$  over  $\mathfrak{K}_T$  is  $m$ , where we shall now assume that

$$(10) \quad m > 2.$$

<sup>1</sup> For earlier definitions of twisted fields see the case  $c = -1$  in *On nonassociative division algebras*, Trans. Amer. Math. Soc. **72** (1952), 296-309 and the general case in *Finite noncommutative division algebras*, Proc. Amer. Math. Soc. **9** (1958), 928-932. In those papers we defined a product  $[x, y] = x(yT) - cy(xT)$  so that  $(x, y) = [x, yT^{-1}] = xy - c(yS)(xT)$  is the product (3) with  $S = T^{-1}$ .

<sup>2</sup> This result was originally given for loops by R. H. Bruck. It is easy to show that, if  $\mathfrak{D}$  and  $\mathfrak{D}_0$  are isotopic rings with isotopy defined by the relation  $QR_xP = R_x^{(c)}QR_z$ , then the mapping  $x \rightarrow (zx)P^{-1}$  induces an isomorphism of the right nucleus  $\mathfrak{D}$  onto that of  $\mathfrak{D}_0$ , and the mapping  $x \rightarrow (xz)P^{-1}$  induces an isomorphism of the middle nucleus of  $\mathfrak{D}$  onto that of  $\mathfrak{D}_0$ .

<sup>3</sup> Two finite projective planes  $\mathfrak{M}(\mathfrak{D})$  and  $\mathfrak{M}(\mathfrak{D}_0)$  coordinatized by division rings  $\mathfrak{D}$  and  $\mathfrak{D}_0$  respectively are known to be isomorphic if and only if  $\mathfrak{D}$  and  $\mathfrak{D}_0$  are isotopic. See the author's *Finite division algebras and finite planes*, Proceedings of Symposia in Applied Mathematics; vol. 10, pp. 53-70.

Then the *norm* in  $\mathfrak{K}$  over  $\mathfrak{K}_T$  of any element  $k$  of  $\mathfrak{K}$  is

$$(11) \quad \nu(k) = k(kT) \cdots (kT^{m-1}),$$

and  $\nu(k)$  is in  $\mathfrak{K}_T$ , that is,

$$(12) \quad \nu(k) = [\nu(k)]T$$

for every  $k$  of  $\mathfrak{K}$ . Thus

$$(13) \quad I - (TR_c)^m = I - R_{\nu(e)} = R_d,$$

where

$$(14) \quad d = e - \nu(e) = dT.$$

Now

$$(15) \quad A_e = I - TR_c, \quad B_e = I - SR_c,$$

and we obtain

$$(16) \quad A_e[I + TR_c + (TR_c)^2 + \cdots + (TR_c)^{m-1}] = R_d,$$

so that

$$(17) \quad I + TR_c + (TR_c)^2 + \cdots + (TR_c)^{m-1} = A_e^{-1}R_d.$$

Our definition (4) implies that

$$(18) \quad R_a A_y = A_y R_a, \quad R_b B_x = B_x R_b$$

for every  $x$  and  $y$  of  $K$ , providing that

$$(19) \quad a = aT, \quad b = bS.$$

In particular,  $R_a A_y = A_y R_a$ , and so (9) is equivalent to

$$(20) \quad A_x[I + (TR_c) + (TR_c)^2 + \cdots + (TR_c)^{m-1}]A_y = A_z R_d.$$

It is well known that distinct automorphisms of any field  $\mathfrak{K}$  are linearly independent in the field of right multiplications of  $\mathfrak{K}$ . Thus we can equate the coefficients of the distinct powers of  $T$  in the equation (20). The right member of (20) is  $R_{za} - TR_{ca(zS)}$  and so does not contain the term in  $T^{m-1}$  when  $m > 2$ . It follows that

$$(21) \quad R_x[(TR_c)^{m-1}R_y - (TR_c)^{m-2}(TR_c)R_{yS}] \\ - TR_{c(xS)}[(TR_c)^{m-2}R_y - (TR_c)^{m-3}(TR_c)R_{yS}] = 0.$$

This equation is equivalent to

$$(22) \quad xT^{m-1}(y - yS) = xST^{m-2}(y - yS),$$

and so to the relation

$$(23) \quad [(x - xST^{-1})T^{m-1}](y - yS) = 0 .$$

By symmetry we have the following result.

**LEMMA 1.** *Let  $T$  have period  $m > 2$ . Then the equation  $A_x A_e^{-1} A_y = A_x$  holds for some  $x, y, z$  in  $\mathfrak{R}$  only if  $y = yS$  or  $x = xST^{-1}$ . If  $S$  has period  $m_0 > 2$  the equation  $B_y B_e^{-1} B_x = B_x$  holds for some  $x, y, z$  in  $\mathfrak{R}$  only if  $x = xT$  or  $y = yST^{-1}$ .*

**3. The nuclei.** The ring  $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, T, c)$  has its product defined by

$$(24) \quad x \cdot y = xR_y^{(c)} = yL_y^{(c)},$$

where

$$(25) \quad R_{yB_e}^{(c)} = A_e^{-1} A_y, \quad L_{xA_e}^{(c)} = B_e^{-1} B_x .$$

When  $S = T$  our formula (3) becomes  $(x, y) = xy - c[(xy)S] = xy(I - SR_c)$ . But then the ring  $\mathfrak{D}_0$ , defined by the product  $(x, y)$ , is isotopic to the field  $\mathfrak{R}$ . Since  $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, S, c)$  is isotopic to  $\mathfrak{D}_0$  it is isotopic to  $\mathfrak{R}$ , and it is well known that  $\mathfrak{D}$  is then also isomorphic to  $\mathfrak{R}$ . Assume henceforth that

$$(26) \quad S \neq T .$$

The right nucleus of  $\mathfrak{D}$  is the set  $\mathfrak{N}_\rho$  of all elements  $z_\rho$  in  $\mathfrak{R}$  such that

$$(27) \quad (x \cdot y) \cdot z_\rho = x \cdot (y \cdot z_\rho) ,$$

for every  $x$  and  $y$  of  $\mathfrak{R}$ . Suppose that  $b = bS$  so that

$$(28) \quad A_b = R_b - TR_{c(bS)} = (I - TR_c)R_b, \quad A_e^{-1} A_b = R_b .$$

By (18) we know that  $R_b B_x = B_x R_b$ , and so  $R_b (B_e^{-1} B_x) = (B_e^{-1} B_x) R_b$  for every  $x$  of  $\mathfrak{R}$ . By (25) this implies that the transformation

$$(29) \quad R_b = A_e^{-1} A_b = R_{bB_e}^{(c)}$$

commutes with every  $L_x^{(c)}$ . However, (27) is equivalent to

$$(30) \quad L_x^{(c)} R_{z_\rho}^{(c)} = R_{z_\rho}^{(c)} L_x^{(c)} .$$

Thus  $bB_e = b(I - SR_c) = b(e - c) = bf$  is in  $\mathfrak{N}_\rho$ . We have proved that the right nucleus of  $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, T, c)$  contains the field  $f\mathfrak{R}_S$ , a subring of  $\mathfrak{D}$  isomorphic to  $\mathfrak{R}_S$ .

The left nucleus  $\mathfrak{N}_\lambda$  of  $\mathfrak{D}$  consists of all  $z_\lambda$  such that

$$(31) \quad (z_\lambda \cdot y) \cdot x = z_\lambda \cdot (y \cdot x)$$

for all  $x$  and  $y$  of  $\mathfrak{R}$ . This equation is equivalent to

$$(32) \quad L_{z_\lambda}^{(c)} R_x^{(c)} = R_x^{(c)} L_{z_\lambda}^{(c)}$$

for every  $x$  of  $\mathfrak{R}$ . If  $a = aT$  then  $B_a = (I - SR_c)R_a$ ,  $B^{-1}B_a = R_a = L_{\alpha A_e}^{(c)}$  commutes with every  $A_y$  and every  $R_x^{(c)}$ , and we see that the left nucleus of  $\mathfrak{D}(\mathfrak{R}, S, T, c)$  contains the field  $f\mathfrak{R}_T$  isomorphic to  $\mathfrak{R}_T$ .

The middle nucleus of  $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, T, c)$  is the set  $\mathfrak{N}_\mu$  of all  $z_\mu$  of  $\mathfrak{R}$  such that

$$(33) \quad (x \cdot z_\mu) \cdot y = x \cdot (z_\mu \cdot y)$$

for every  $x$  and  $y$  of  $\mathfrak{R}$ . This equation is equivalent to

$$(34) \quad R_z^{(c)} R_y^{(c)} = R_{z \cdot y}^{(c)},$$

where  $z = z_\mu$ . However, we can observe that the assumption that

$$(35) \quad R_z^{(c)} R_y^{(c)} = R_v^{(c)},$$

for some  $v$  in  $\mathfrak{R}$ , implies that  $(f \cdot z) \cdot y = f \cdot v = v = z \cdot y$ , Hence (34) holds for every  $y$  in  $\mathfrak{R}$  if and only if

$$(36) \quad A_y A_e^{-1} A_y = A_v,$$

for every  $y$  of  $\mathfrak{R}$ , where  $v$  is in  $\mathfrak{R}$  and

$$(37) \quad gB_e = z = z_\mu.$$

If  $gS = gT$  then  $A_g = R_g - TR_{c(gS)} = R_g - TR_{c(gT)} = R_g - R_g TR_c = R_g A_e$ . Then (36) becomes

$$(38) \quad R_g A_y = R_g (R_g - TR_{c(yS)}) = R_{gy} - TR_{c(ySgT)} = A_{gy}.$$

Hence  $gB_e = g(I - SR_c) = g - (gS)c = g - (gT)c = gA_e$ , and  $\mathfrak{N}_\mu$  contains the field of all elements  $gB_e$  for  $gS = gT$ .

We are now able to derive the converse of these results. We first observe that (27) is equivalent to

$$(39) \quad R_y^{(c)} R_z^{(c)} = R_{y \cdot z}^{(c)},$$

for every  $y$  of  $\mathfrak{R}$ , where  $z = z_\rho$ . This equation is equivalent to

$$(40) \quad A_y A_e^{-1} A_u = A_v,$$

where  $z = uB_e$ . If the period of  $T$  is  $m > 2$  we use Lemma 1 to see that, if we take  $y \neq yST^{-1}$ , then  $u = uS$ ,  $z = uB_e = fu$ . The stated choice of  $y$  is always possible since we assuming that  $S \neq T$  and so some element of  $\mathfrak{R}$  is not left fixed by  $ST^{-1}$ . Thus  $\mathfrak{N} = f\mathfrak{R}_S$ . Similarly, if the period of  $S$  is not two then  $\mathfrak{N}_\lambda = f\mathfrak{R}_T$ . Assume that one of  $S$  and  $T$  has period two.

The automorphisms  $S$  and  $T$  cannot both have period two. For the group  $G$  of automorphisms of  $\mathfrak{R}$  is a cyclic group and has a unique subgroup  $\mathfrak{S}$  of order two. This group contains  $I$  and only one other automorphism. If  $S$  and  $T$  both had period two we would have  $S = T$  and so  $m = n = 2$ , contrary to hypothesis. Thus we may assume that one of  $S$  and  $T$  has period two. There is clearly no loss of generality if we assume that  $T$  has period two, so that the period of  $S$  is at least three. By the argument already given we have  $\mathfrak{R}_\lambda = f\mathfrak{R}_T$ . We are then led to study (40) as holding for all elements  $y$  of  $\mathfrak{R}$ , where  $z_p = uB_e$ . Now

$$(41) \quad A_e = I - TR_c, \quad A_e(I + TR_c) = R_a, \quad d = e - c(cT) = dT.$$

But then (40) becomes

$$(42) \quad [R_y - TR_{c(yS)}](I + TR_c)[R_u - TR_{c(uS)}] = R_{vd} - TR_{cd(vS)}.$$

This yields the equations

$$(43) \quad y[u - c(cT)(uS)] - (yST)[c(cT)](u - uS) = vd,$$

$$(44) \quad yT(u - uS) - yS[u - (uS)c(cT)] = -d(vS).$$

Hence

$$\begin{aligned} d(yS)[uS - (cS)(cST)(uS^2)] - yS^2T(cS)(cST)(uS - uS^2)d &= vS(dS)d \\ &= (dS)yS[u - (uS)c(cT)] - yT(u - uS)(dS). \end{aligned}$$

Since this holds for all  $y$  we have the transformation equation

$$(45) \quad SR[d(uS) - d(cS)(cST)uS^2] - S^2TR[d(cS)(cST)(uS - uS^2)] \\ = SR[dSu - (dS)(uS)c(cT)] - TR[(u - uS)dS].$$

Since  $S^2 \neq I$  and  $T \neq S$ ,  $S^2T$  we know that the coefficient of  $S^2T$  is zero. Thus  $(u - uS)dS = 0$  and  $u = uS$  as desired. This shows that  $\mathfrak{R}_p = f\mathfrak{R}_S$ .

The *middle nucleus* condition (36) implies that  $gS = gT$  if  $T$  does not have period two. When  $T$  does have period two but  $S$  does not have period two the analogous property

$$(46) \quad L_{x_z}^{(c)} = L_z^{(c)}L_x^{(c)}$$

is equivalent to

$$(47) \quad B_\theta B_e^{-1} B_x = B_\theta,$$

and we see again that  $gS = gT$ . This completes our proof of the theorem stated in the introduction.

**4. Commutativity.** It is known<sup>4</sup> that  $\mathfrak{D} = (\mathfrak{R}, S, S^{-1}, c)$  is commutative if and only if  $c = -1$ . There remains the case where

$$(48) \quad S \neq I, T \neq I, ST \neq I, S \neq T.$$

Any  $\mathfrak{D}(\mathfrak{R}, S, T, c)$  is commutative if and only if  $R_x^{(e)} = L_x^{(c)}$  for every  $x$  of  $\mathfrak{R}$ . Assume first that  $\mathfrak{R}_S \neq \mathfrak{R}_T$ . There is clearly no loss of generality if we assume that there is an element  $b$  in  $\mathfrak{R}_S$  and not in  $\mathfrak{R}_T$ , since the roles of  $S$  and  $T$  can be interchanged when  $\mathfrak{D}(\mathfrak{R}, S, T, c)$  is commutative. Thus we have  $b = bS \neq bT$ . By (28) we know that  $A_b = A_e R_b$  and so we have  $R_{bf}^{(c)} = R_b$ . Then  $L_{bf}^{(c)} = B_e^{-1} B_y = R_b$ , where  $y = (bf)A_e^{-1}$ . It follows that

$$(49) \quad B_y = R_y - SR_{c(yT)} = B_e R_b = (I - SR_c)R_b.$$

Then  $R_y = R_b$ ,  $y = b$ ,  $c(yT) = c(bT) = cb$ , and  $b = bT$  contrary to hypothesis.

We have shown that if  $\mathfrak{D}(\mathfrak{R}, S, T, c)$  is commutative the automorphisms  $S$  and  $T$  have the same fixed fields, that is,  $b = bS$  if and only if  $b = bT$ ,  $b$  is in  $\mathfrak{F}$ . Thus  $S$  and  $T$  both generate the cyclic automorphism group  $\mathfrak{G}$  of order  $n$  of  $\mathfrak{R}$  over  $\mathfrak{F}$ , and  $S$  is a power of  $T$ . Since  $T^{-1} = T^{n-1} \neq S$  there exists an integer  $r$  such that

$$(50) \quad 0 < r < n - 1, S = T^r.$$

We now use the fact that  $R_x^{(c)} = L_x^{(c)}$  for every  $x$  of  $K$  to see that  $A_e^{-1} A_x = B_e^{-1} B_y$  for every  $x$  of  $\mathfrak{R}$ , where  $y = xB_e A_e^{-1}$ . Also  $(TR_c)^n = (SR_c)^n = R_{v(c)}$ , and our condition becomes

$$(51) \quad [I + TR_c + (TR_c)^2 + \cdots + (TR_c)^{n-1}][R_x - TR_{c(xS)}] \\ = [I + SR_c + (SR_c)^2 + \cdots + (SR_c)^{n-1}][R_y - SR_{c(yT)}],$$

where we have used the fact that  $d = e - \nu(c) = dT = dS$ . Compute the constant term to obtain the equation

$$(52) \quad R_x - (TR_c)^n R_{xS} = R_y - (SR_c)_u R_{yT}.$$

This is equivalent to the relation  $x - [\nu(c)](xS) = y - [\nu(c)]yT$  for every  $x$  of  $K$ , where  $y = xB_e A_e^{-1}$ . Thus (52) is equivalent to

$$(53) \quad I - SR_{v(c)} = B_e A_e^{-1} [I - TR_{v(c)}].$$

We also compute the term in  $T^r$  in (51). Since  $r < n - 1$  the left member of this term is  $(TR_c)^r R_x - (TR_c)^r R_{xS}$ , which is equal to  $R^r R_{gc}(R_x - R_{xS})$ , where  $g = (cT)(cT)^2 \cdots (cT)^{r-1}$ . The right member is the term in  $S$ , and this is  $SR_c(R_y - R_{yT})$ . Hence  $(x - xS)g = y - yT$ , a result equivalent to

<sup>4</sup> See footnote 1.



$$(54) \quad (I - S)R_g = B_e A_e^{-1}(I - T) .$$

Since the transformations  $I - T$  and  $I - TR_{\nu(c)}$  commute we may use (53) to obtain

$$(55) \quad (I - S)R_g[I - TR_{\nu(c)}] = [I - SR_{\nu(c)}](I - T) .$$

By (48) we may equate coefficients of  $I, S, T$  and  $ST$ , respectively. The constant term yields  $g = e$ . The term in  $S$  then yields  $\nu(c) = e$  which is impossible when  $S$  and  $T$  generate the same group and  $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, T, c)$  is a division algebra.

We have proved the following result.

**THEOREM 2.** *Let  $\mathfrak{D} = \mathfrak{D}(\mathfrak{R}, S, T, c)$  be a division algebra defined for  $S \neq I, T \neq I, S \neq T$ . Then  $\mathfrak{D}$  is commutative if and only if  $ST = I$  and  $c = -1$ .*

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