

# Pacific Journal of Mathematics

## **ASYMPTOTICS. II. LAPLACE'S METHOD FOR MULTIPLE INTEGRALS**

WATSON BRYAN FULKS AND J. O. SATHER

## ASYMPTOTICS II: LAPLACE'S METHOD FOR MULTIPLE INTEGRALS

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Laplace's method is a well known and important tool for studying the rate of growth of an integral of the form

$$I(h) = \int_a^b e^{-hf} g dx$$

as  $h \rightarrow \infty$ , where  $f$  has a single minimum in  $[a, b]$ . Its extension to multiple integrals has been studied by L. C. Hsu in a series of papers starting in 1948, and by P. G. Rooney (see bibliography). These authors establish what amount to a first term of an asymptotic expansion. All but one (see [7]) of these results are under fairly heavy smoothness conditions.

In this paper we examine multiple integrals of the form

$$I(h) = \int_R e^{-hf} g dx$$

where  $f$  and  $g$  are measurable functions defined on a set  $R$  in  $E_p$ . Without making any smoothness assumptions on  $f$  and  $g$ , and using only the existence of  $I(h)$  and, of course, asymptotic expansions of  $f$  and  $g$  near the minimum point of  $f$  we obtain an asymptotic expansion of  $I$ . The special features of our procedure are the lack of smoothness assumptions and the fact that we get a complete expansion.

Without loss of generality we may assume that the essential infimum of  $f$  occurs at the origin, and that this minimal value is zero. We introduce polar coordinates:  $x = (\rho, \Omega)$  where

$$\rho = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_p^2},$$

and where  $\Omega = x/|x|$  is a point on the surface,  $S_{p-1}$ , of the unit sphere.

Our hypothesis are the following:

- (1) The origin is an interior point of  $R$ .
- (2) For each  $\rho_0 > 0$  there is an  $A > 0$  such that  $f(\rho, \Omega) \geq A$  if  $\rho \geq \rho_0$ . (This says that  $f$  can be close to zero only at the origin.)
- (3) There is an  $n \geq 0$  and  $n + 1$  continuous functions  $f_k(\Omega)$ ,  $k = 0, 1, 2, \cdots, n$ , defined on  $S_{p-1}$  with  $f_0 > 0$  for which

$$f(\rho, \Omega) = \rho^\nu \sum_{k=0}^n f_k(\Omega) \rho^k + o(\rho^{n+\nu}) \text{ as } \rho \rightarrow 0$$

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where  $\nu > 0$ . (This is meant in the following sense: for each  $\varepsilon > 0$  there is a  $\rho_0 > 0$  for which

$$|f(\rho, \Omega) - \rho^\nu \sum_{k=0}^n f_k(\Omega)\rho^k| < \varepsilon\rho^{n+\nu}$$

whenever  $\rho \leq \rho_0$ . Besides giving the asymptotic behavior of  $f$  near the origin (3) implies that the infimum of  $f$  in  $R$  is indeed zero.)

(4) There are  $n + 1$  functions  $g_k(\Omega), k = 0, 1, \dots, n$ , for which

$$g = \rho^{\lambda-p} \sum_{k=0}^n g_k(\Omega)\rho^k + o(\rho^{n+\lambda-k}) \text{ as } \rho \rightarrow 0$$

where  $\lambda > 0$ . (Thus  $g$  is permitted a mild singularity at the origin. The expansion is meant in the same sense as the one in (3).)

Under these conditions we will prove that if there is a  $h_0$  for which  $I(h)$  exists then it exists for all  $h \geq h_0$  and

$$I(h) = \sum_{k=0}^n c_k h^{-(k+\lambda)/\nu} + o(h^{-(n+\lambda)/\nu})$$

where the  $c_k$ 's are constants depending only on the  $f_j$ 's and  $g_j$ 's for  $j \leq k$ . Their evaluation will be described in the proof of this result. In particular

$$C_0 = \frac{\Gamma((\lambda + 1)/\nu)}{\lambda} \int_{S_{p-1}} g_0(\Omega)/[f_0(\Omega)]^{\lambda/\nu} d\Omega$$

where  $d\Omega$  is the element of  $(p - 1)$ -dimensional measure on  $S_{p-1}$ .

In the course of the proof we will use the following lemmas, which are given now so as to not interrupt the main thread of the argument.

**LEMMA 1.** *Let  $f$  be a measurable function on a set  $R$  in  $E_p$ , and let  $g \in L_1(R)$ . Then the function  $G(z)$  defined by*

$$G(z) = \int_{\{f \leq z\}} g dx$$

*has bounded variation on  $\{-\infty < z < \infty\}$ .*

*Proof.* Let  $g = g_1 - g_2$ , where

$$g_1(x) = \begin{cases} g(x), & g(x) \geq 0 \\ 0, & g(x) < 0 \end{cases}; \quad g_2(x) = \begin{cases} 0, & g(x) \geq 0 \\ -g(x), & g(x) < 0, \end{cases}$$

and define  $G_1$  and  $G_2$  by

$$G_1(z) = \int_{\{f \leq z\}} g_1 dx, \quad G_2(z) = \int_{\{f \leq z\}} g_2 dx.$$

Clearly  $G_1$  and  $G_2$  are increasing and bounded on  $\{-\infty < z < \infty\}$ , and  $G = G_1 - G_2$ .

**LEMMA 2.** *Let  $F(t)$  be a continuous function defined on a possibly infinite interval  $\{a < t < b\}$ , and let  $f$  be a measurable function on a set  $R$  in  $E_p$  taking values in the interval  $\{a < t < b\}$ . If  $g \in L_1(R)$ , and  $F(f)g \in L_1(R)$  and  $G$  is defined as in Lemma 1, then*

$$\int_R F(f)gdx = \int_a^b F(t)dG(t).$$

*Proof.* Suppose first that  $a$  and  $b$  are finite, and that  $g \geq 0$ . Form a partition:  $a = t_0 < t_1 < \dots < t_n = b$ , and set

$$E_j = \{x \mid t_{j-1} < f \leq t_j\},$$

and let  $M_j = \sup_{\{t_{j-1} \leq t \leq t_j\}} F(t)$  and  $m_j = \inf_{\{t_{j-1} \leq t \leq t_j\}} F(t)$ .

Then

$$\begin{aligned} \int_R F(f)gdx &= \sum_{j=1}^n \int_{E_j} F(f)gdx \leq \sum_{j=1}^n M_j \int_{E_j} gdx \\ &= \sum_{j=1}^n M_j [G(t_j) - G(t_{j-1})]. \end{aligned}$$

Similarly

$$\int_R F(f)gdx \geq \sum_{j=1}^n m_j [G(t_j) - G(t_{j-1})].$$

If we let  $n \rightarrow \infty$  so that  $\max_{1 \leq j \leq n} (t_j - t_{j-1}) \rightarrow 0$  then both

$$\sum_{j=1}^n M_j [G(t_j) - G(t_{j-1})] \text{ and } \sum_{j=1}^n m_j [G(t_j) - G(t_{j-1})]$$

converge to  $\int_a^b F(t)dG(t)$ , since  $F$  is continuous and  $G$  monotone.

If  $g$  is not positive we can write  $g = g_1 - g_2$  as in Lemma 1, apply the proof just completed to each of  $g_1$  and  $g_2$ , and combine the results to complete the proof for the case where  $a$  and  $b$  are finite.

Suppose for example  $b$  is infinite. Then for any finite  $b'$ ,

$$\begin{aligned} \int_R F(f)gdx &= \lim_{b' \rightarrow \infty} \int_{\{f \leq b'\}} F(f)gdx = \lim_{b' \rightarrow \infty} \int_a^{b'} F(t)dG(t) \\ &= \int_a^\infty F(t)dG(t). \end{aligned}$$

A similar argument applies if  $a = -\infty$ .

We now return to the proof of the main theorem. First we note that if  $h \geq h_0$  then  $e^{-h_0 f} g$  forms a dominating function for  $e^{-h f} g$ , so that

$I(h)$  exists.

For each  $\varepsilon > 0$  we define the two functions  $f_+(\rho, \Omega)$  and  $f_-(\rho, \Omega)$  by

$$f_{\pm}(\rho, \Omega) = \rho^\nu \sum_{k=0}^n f_k(\Omega) \rho^k \pm \varepsilon \rho^{n+\nu} .$$

These functions are defined in all of  $E_\rho$ . Now given an  $\varepsilon > 0$  there is a  $\rho_0$  so that

- (i)  $|f(\rho, \Omega) - \rho^\nu \sum_{k=0}^n f_k(\Omega) \rho^k| < \varepsilon \rho^{n+\nu}$
- (ii)  $|g(\rho, \Omega) - \rho^{\lambda-p} \sum_{k=0}^n g_k(\Omega) \rho^k| < \varepsilon \rho^{n+\lambda-p}$  for  $\rho < \rho_0$ ,

and so that

(iii) both the functions  $f_{\pm}(\rho, \Omega)$  are increasing in  $\rho$  for  $\{0 \leq \rho \leq \rho_0\}$  for each  $\Omega \in S_{p-1}$ . This can easily be achieved since  $f_0$  is positive (and therefore bounded away from zero) and the other  $f_k$ 's are bounded.

(iv) the sphere  $\{\rho \leq \rho_0\}$  is in  $R$ .

We denote  $\{\rho \leq \rho_0\}$  by  $R_0$  and write  $I(h)$  in the form

$$I(h) = \int_{R_0} e^{-hf} g dx + \int_{R-R_0} e^{-hf} g dx \equiv I_1(h) + I_2(h)$$

respectively. We proceed to estimate  $I_2$ : by hypothesis (2) there is an  $A > 0$  so that  $f \geq A$  if  $\rho \geq \rho_0$ . Thus

$$\begin{aligned} |I_2(h)| &\leq \int_{R-R_0} e^{-hf} |g| dx \leq e^{-(h-h_0)A} \int_{R-R_0} e^{-h_0f} |g| dx \\ &= C e^{-hA} \text{ where } C \text{ is a constant.} \end{aligned}$$

That is,

$$I_2(h) = O(e^{-hA}) \text{ as } h \rightarrow \infty ,$$

so it is clear that the dominant part of  $I(h)$  must arise from  $I_1(h)$ . The remainder of the proof is largely concerned with estimating  $I_1$ .

In  $R_0$  we define  $r(\rho, \Omega)$  by

$$g(\rho, \Omega) = \rho^{\lambda-p} \sum_0^n g_k(\Omega) \rho^k + r(\rho, \Omega) \rho^{n+\lambda-p} .$$

Let

$$g_k^+(\Omega) = \begin{cases} g_k(\Omega), & g_k(\Omega) \geq 0 \\ 0, & g_k(\Omega) < 0 \end{cases}, \quad g_k^-(\Omega) = \begin{cases} 0, & g_k(\Omega) \geq 0 \\ -g_k(\Omega), & g_k(\Omega) > 0 \end{cases}$$

and

$$r^+(\rho, \Omega) = \begin{cases} r(\rho, \Omega), & r(\rho, \Omega) \geq 0 \\ 0, & r(\rho, \Omega) < 0 \end{cases}; \quad r^-(\rho, \Omega) = \begin{cases} 0, & r(\rho, \Omega) \geq 0 \\ -r(\rho, \Omega), & r(\rho, \Omega) < 0 \end{cases} .$$

In  $R_0$  we now define  $g^+(\rho, \Omega)$  and  $g^-(\rho, \Omega)$  by

$$g^+(\rho, \Omega) = \rho^{\lambda-p} \sum_{k=0}^n g_k^+(\Omega) \rho^k + r^+(\rho, \Omega) \rho^{n+\lambda-p}$$

and

$$g^-(\rho, \Omega) = \rho^{\lambda-p} \sum_{k=0}^n g_k^-(\Omega) \rho^k + r^-(\rho, \Omega) \rho^{n+\lambda-p}.$$

Then  $g = g^+ - g^-$  and

$$I_1 = \int_{R_0} e^{-hf} g^+ dx - \int_{R_0} e^{-hf} g^- dx.$$

Thus we may assume that  $g \geq 0$  in  $R_0$ .

We recall the definition of  $f_+$  and  $f_-$  and define  $I_+(h)$  and  $I_-(h)$  by

$$I_+(h) = \int_{R_0} e^{-hf_+} g dx, \quad I_-(h) = \int_{R_0} e^{-hf_-} g dx.$$

Since  $g \geq 0$  we conclude

$$I_+(h) \leq I_1(h) \leq I_-(h).$$

Next we turn our attention to  $I_+$ : Let  $R_t = \{x | f_+ \leq t\}$  and choose  $a$  so small that  $R_a \subset R_0$ . Then

$$I_+(h) = \int_{R_a} e^{-hf_+} g dx + \int_{R_0 - R_a} e^{-hf_+} g dx = I'_+ + I''_+,$$

respectively. Now  $f_+$  is bounded away from zero in  $R_0$  outside any neighborhood of the origin. Thus by the same argument used on  $I_2$  we get

$$I''_+ = O(e^{-ha}).$$

Furthermore  $e^{-hf_+}$  is bounded away from zero in  $R_a$ , since  $f_+$  is bounded there. Thus  $e^{-hf_+} g \in L_1(R_a)$  and by Lemma 2,

$$I'_+ = \int_0^a e^{-ht} dG(t),$$

where  $G(t) = \int_{R_t} g dx$ . Integrating by parts we get

$$\begin{aligned} I'_+ &= e^{-ha} G(a) + h \int_0^a e^{-ht} G(t) dt \\ &= h \int_0^a e^{-ht} G(t) dt + O(e^{-ha}). \end{aligned}$$

We next do some preliminary calculations, preparatory to estimating  $G(t)$ . For each  $t$ ,  $0 \leq t \leq a$ , the equation  $t = f_+(\rho, \Omega)$  has a unique solution for  $\rho$  which is continuous in  $\Omega$ , since  $f_+$  is increasing in  $\rho$ .

Thus the solution defines a star-shaped curve (or surface) given by  $\rho = \rho(t, \Omega)$ . We proceed to estimate  $\rho(t, \Omega)$ . Set  $t = U^\nu$  then  $t = f_+(\rho, \Omega)$  can be written in the form

$$U^\nu = \rho^\nu \left[ \sum_0^n f_k(\Omega) \rho^k + \varepsilon \rho^n \right]$$

or

$$U = \rho [f_0(\Omega) + f_1(\Omega)\rho + \cdots (f_n(\Omega) + \varepsilon)\rho^n]^{1/\nu}.$$

From here on we assume  $n > 0$ , for if  $n = 0$ , we can solve directly for  $\rho$  and the estimates are considerably simpler than those which follow.

Now the right hand side of the last equation is a monotone function of  $\rho$ ,  $0 \leq \rho \leq a$ , hence an inverse function exists. Since, for each fixed  $\Omega$ ,  $U$  is an  $(n+2)$ -times differentiable (it's even analytic!) function of  $\rho$ ,  $0 \leq \rho \leq a$ , then  $\rho$  is an  $(n+2)$ -times differentiable function of  $U$ , and it can therefore be expanded in a Taylor series with remainder. Thus since  $f_0(\Omega) > 0$  we get

$$\rho = \psi_1(\Omega)U + \psi_2(\Omega)U^2 + \cdots + \psi_{n+1}(\Omega, \varepsilon)U^{n+1} + \psi_{n+2}(\Omega, \varepsilon, U)U^{n+2}$$

where  $\psi_1(\Omega) = 1/[f_0(\Omega)]^{1/\nu}$ . Since the  $\psi_k$ 's are expressible in terms of the  $f_k$ 's it is easy to check that  $\psi_k$  depends only on  $f_j$ 's for  $j \leq k$ , that  $\psi_k$  is independent of  $\varepsilon$  for  $k \leq n$ , that  $\psi_{n+1}$  depends only linearly on  $\varepsilon$  and finally that  $\psi_{n+2}$  is uniformly bounded for  $\Omega \in S_{p-1}$ ,  $0 \leq \varepsilon \leq 1$ , and  $0 \leq U \leq a^{1/\nu}$ .

Since  $U = t^{1/\nu}$  we express  $\rho$  in terms of  $t$ ,  $\Omega$ , and  $\varepsilon$  by

$$\begin{aligned} \rho(t, \Omega) = & \psi_1(\Omega)t^{1/\nu} + \psi_2(\Omega)t^{2/\nu} + \cdots + \psi_{n+1}(\Omega, \varepsilon)t^{(n+1)/\nu} \\ & + \psi_{n+2}(\Omega, \varepsilon, U)t^{(n+2)/\nu} \end{aligned}$$

By definition  $G(t) = \int_{R_t} g dx$ , which we can write as

$$G(t) = \int_{S_{p-1}} \int_0^{\rho(t, \Omega)} g(\rho, \Omega) \rho^{p-1} d\rho d\Omega,$$

where  $d\Omega$  represents the element of measure on the sphere  $S_{p-1} : \{\rho = 1\}$ . We proceed to compute:

$$\begin{aligned} G(t) &= \int_{S_{p-1}} \int_0^{\rho(t, \Omega)} \left( \sum_0^n g_k(\Omega) \rho^{k+\lambda-1} + o(\rho^{n+\lambda-1}) \right) d\rho d\Omega \\ &= \int_{S_{p-1}} \left[ \rho^\lambda(t, \Omega) \left( \sum_0^n \frac{g_k(\Omega)}{k+\lambda} \rho^k(t, \Omega) \right) + o(\rho^{n+\lambda}(t, \Omega)) \right] d\Omega. \end{aligned}$$

If we substitute for  $\rho(t, \Omega)$  the expression previously computed for it, the preceding integral can be written in the form

$$G(t) = \int_{S_{p-1}} \left[ t^{\lambda/\nu} \sum_0^{n-1} \gamma_k(\Omega) t^{k/\nu} + \gamma_n(\Omega, \varepsilon) t^{(n+\lambda)/\nu} + o(t^{(n+\lambda)/\nu}) \right] d\Omega$$

where  $\gamma_k$  is independent of  $\varepsilon$  for  $k = 0, 1, 2, \dots, n-1$ , and  $\gamma_n$  is linear in  $\varepsilon$ . We may also note that each of the  $g_j$ 's enter the  $\gamma_k$ 's linearly. In particular

$$\gamma_0 = g_0(\Omega)/[f_0(\Omega)]^{\lambda/\nu}.$$

Now if we write  $\gamma_n(\Omega, \varepsilon) = \gamma_n(\Omega) - \varepsilon\gamma'_n(\Omega)$  we have

$$\begin{aligned} G(t) &= \int_{S_{p-1}} \left( \sum_0^n \gamma_k(\Omega) t^{(k+\lambda)/\nu} - \varepsilon\gamma'_n(\Omega) t^{(n+\lambda)/\nu} \right) d\Omega + o(t^{(n+\lambda)/\nu}), \\ &= \sum_0^n \eta_k t^{(k+\lambda)/\nu} - \varepsilon\eta'_n t^{(n+\lambda)/\nu} + o(t^{(n+\lambda)/\nu}) \end{aligned}$$

where  $\eta_k = \int_{S_{p-1}} \gamma_k(\Omega) d\Omega$ . In particular  $\eta_0 = (1/\lambda) \int_{S_{p-1}} [g_0(\Omega)/[f_0(\Omega)]^{\lambda/\nu}] d\Omega$ .

Now by Watson's lemma we can multiply this asymptotic formula for  $G$  by  $e^{-ht}$  and integrate termwise to get

$$I_+ = \sum_0^n c_k h^{-(k+\lambda)/\nu} - \varepsilon c'_n h^{-(n+\lambda)/\nu} + o(h^{-(n+\lambda)/\nu})$$

where  $c_k = \eta_k \Gamma((k+\lambda+1)/\nu)$ . In particular  $c_0 = \eta_0 \Gamma((\lambda+1)/\nu)$ . Since  $I_+ = I_+ + I_+'' = I_+ + o(e^{-hA'})$ , we have also

$$I_+ = \sum_0^n c_k h^{-(k+\lambda)/\nu} - \varepsilon c'_n h^{-(n+\lambda)/\nu} + o(h^{-(n+\lambda)/\nu}).$$

By the same argument, since  $I_-$  differs from  $I_+$  only in the sign of  $\varepsilon$ , we get

$$I_- = \sum_0^n c_k h^{-(k+\lambda)/\nu} + \varepsilon c'_n h^{-(n+\lambda)/\nu} + o(h^{-(n+\lambda)/\nu}).$$

Now as we have shown before

$$I_+(h) \leq I_1(h) \leq I_-(h).$$

Thus

$$I_+ - \sum_0^n c_k h^{-(k+\lambda)/\nu} \leq I_1(h) - \sum_0^n c_k h^{-(k+\lambda)/\nu} \leq I_- - \sum_0^n c_k h^{-(k+\lambda)/\nu}.$$

If we multiply through by  $h^{(n+\lambda)/\nu}$  and let  $h \rightarrow \infty$  we get

$$-\varepsilon c'_n \leq \underline{\lim} \left[ (I_1(h) - \sum_0^n c_k h^{-(k+\lambda)/\nu}) h^{(n+\lambda)/\nu} \right] \leq \varepsilon c'_n.$$

But  $I(h) = I_1(h) + o(e^{-hA})$  so that we have also

$$-\varepsilon c'_n \leq \underline{\lim} \left[ (I(h) - \sum_0^n c_k h^{-(k+\lambda)/\nu}) h^{(n+\lambda)/\nu} \right] \leq \varepsilon c'_n,$$



for every  $\varepsilon > 0$ . Let  $\varepsilon \rightarrow 0$  to complete the proof for  $g \geq 0$ .

If  $g$  may change sign near the origin we can decompose  $g$  with  $g^+$  and  $g^-$  as described earlier. The proof just completed applies to each of these. We can then subtract the results to obtain the result for  $g$ . Also since  $g_j$ 's enter into the  $c_k$ 's linearly, the same formula for the  $c$ 's applies whether  $g$  is one signed or has a variable sign near the origin.

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