ON THE ACTION OF A LOCALLY COMPACT GROUP ON $E_n$

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It is known [2, p. 208] that if a locally compact group acts effectively and differentiably on $E_n$ then it is a Lie group. The object of this note is to show that if the differentiability requirements are replaced by some weaker restrictions, given later on, the theorem is still true. Let $G$ be a locally compact group acting on $E_n$ and let the coordinate functions of the action be given by $f_i(g, x_i, \cdots, x_n), \, 1 \leq i \leq n$. For economy we introduce the following notation

$$Q_{ij}(g, t, x) = \frac{f_i(g, x_1, \cdots, x_j + t, \cdots, x_n) - f_i(g, x_1, \cdots, x_j, \cdots, x_n)}{t}.$$ 

We denote by $\sigma(Q_{ij}(e, 0, x))$ the oscillation of $Q_{ij}(g, t, x)$ at the point $(e, 0, x)$.

Before proceeding there is one simple remark to be made on matrices. If $A = (a_{ij})$ is an $n \times n$ matrix such that $|a_{ij} - \delta_{ij}| < (1/n)$ then $A$ is non-singular. If $A$ were singular there would be a vector $x$ such that $\sum_i x_i^2 = 1$ and $Ax = 0$. From the Schwarz inequality it follows that $x_i^2 = (\sum_i (a_{ij} - \delta_{ij})x_j)^2 < (1/n)$ and consequently $1 = \sum x_i^2 < 1$ which is impossible. If $|a_{ij} - \delta_{ij}| \leq (\alpha/n)$, where $0 < \alpha < 1$, then the determinant of $A$ is bounded away from zero since the determinant is a continuous function and the set $\{a_{ij}: |a_{ij} - \delta_{ij}| \leq (\alpha/n)\}$ is compact in $E_{n^2}$.

**Theorem 1.** If $T$ is a pointwise periodic homeomorphism of $E_n$ then $T$ is periodic.

*Proof.* [2, p. 224.]

**Theorem 2.** If $G$ is a compact, zero dimensional, monothetic group acting effectively on $E_n$ and satisfying

$$(*) \quad \sigma(Q_{ij}(e, 0, x)) < \frac{\varepsilon}{n}, \quad 0 < \varepsilon < 1, \quad \text{for each } x \in E_n;$$

then $G$ is a finite cyclic group.

*Proof.* Since $G$ is monothetic, let $a$ be an element whose powers are dense in $G$. It is enough to show that there is a power of $a$ which leaves $E_n$ pointwise fixed since the action of $G$ is effective.

Received April 12, 1960. The author is a National Science Foundation Fellow.
If $q$ is a positive integer we let
\[ T_i^q(g, x) = x_i + f_i(g, x) + \cdots + f_i(g^{q-1}, x). \]
If $y = (y_i)$ and $x = (x_i)$ let
\[ T_i^q(g, x, y) = \frac{T_i^q(g, x_1, \cdots, x_{j-1}, y_j, \cdots, y_n) - T_i^q(g, x_1, \cdots, x_j, y_{j+1}, \cdots, y_n)}{y_j - x_j} \]
for $y_j \neq x_j$ and zero otherwise. If we let $y = f(g, x)$ then we obtain
\[
\begin{align*}
    f_i(g^q, x) - x_i &= T_i^q(g, y) - T_i^q(g, x) \\
    &= \sum_{j=1}^n T_i^q(g, x, y)(y_j - x_j) \\
    &= q \cdot \sum_{i=1}^n \frac{1}{q} T_i^q(g, x, y)(y_j - x_j).
\end{align*}
\]
Because of the fact that $f_i(e, x) = x_i$ and because of (*) it follows that there is a compact neighborhood $U(x)$ of the identity of $G$ such that if $g, \cdots, g^q \in U(x)$ then $| (1/q) T_i^q(g, x, y) - \delta_{i,j} | \leq (\alpha/n)$, $0 < \epsilon < \alpha < 1$. It follows that if $T$ is the matrix with entries $(1/q) T_i^q(g, x, y)$ then $T$ is non-singular and its determinant is bounded away from zero uniformly in $q$, so the determinant of the inverse is bounded uniformly in $q$; thus
\[
(f(g, x) - x) = (y - x) = \left(\delta_{i,j} \frac{1}{q}\right) \cdot T^{-1} \cdot (f(g^q, x) - x).
\]

Since $G$ is monothetic and zero dimensional there is a power of $a$ such that if $g = a^p$ then all the powers of $g$ lie in $U(x)$. Since $U(x)$ is compact it follows that the vectors $f(g^q, x) - x$ are bounded uniformly in $q$ and thus $f(g, x) - x = f(a^p, x) - x = 0$. Hence $a$ is pointwise periodic on $E_n$ and it follows from Theorem 1 that it is periodic and consequently has a power leaving $E_n$ pointwise fixed.

From this it follows quickly that if $G$ is a locally compact group acting effectively on $E_n$ and satisfying (*) then it is a Lie group. This follows from the fact that since $G$ is effective it must be finite dimensional [1] and then if $G$ is not a Lie group it must contain a compact, non-finite zero dimensional subgroup $H$ [2, p. 237] which acts effectively. $H$ has small subgroups which act effectively and it follows from Newman’s theorem [3, 4] that $H$ cannot have arbitrarily small elements of finite order. Thus $H$ has an element $a$ of infinite order such that the compact subgroup generated by $a$ acts effectively on $E_n$ and satisfies (*) but by Theorem 2 this is impossible.

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