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ON SIMILARITY INVARIANTS OF CERTAIN OPERATORS IN
 L_p

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The purpose of this paper is to extend the result of Corollary, Theorem 2 of the author's paper on Volterra operators (Annals of Math., 66, 1957, pp. 481-494 quoted as A ; we shall use the definitions and notations of that paper) to the most general situation applicable: We are dealing with operators T_F where $F(x, y) = (y - x)^{m-1} aG(x, y)$ is a function defined on the triangle $0 \leq x \leq y \leq 1$, where m is a positive integer, a a complex number of absolute value 1, G is a complex valued function which is continuously differentiable and $G(x, x)$ is positive real. We recall that if $f \in L_p [0, 1]$, then $(T_F)(f)(x) = \int_x^1 F(x, y)f(y)dy$ is again in $L_p [0, 1]$. The only difference from A is the presence of the constant a which affects none of results except Theorem 2 and its Corollary. Theorems 1 and 2 of the present paper fill the gap. Theorem 3 shows that differentiability conditions imposed on F cannot be abandoned entirely—and also that the integral equation (1) of A cannot be solved unless K (which corresponds to our F) has at least first derivatives near $y = x$.

If c is constant and E is the function identically equal to 1, we define T_E^c as T_H which $H(x, y) = (y - x)^{c-1}/\Gamma(c)$ (fractional integration of order c).

THEOREM 1. *Let c_1 and c_2 be complex numbers and let r_1 and r_2 be real numbers such that $r_i \geq 1$, then $c_1 T_E^{r_1}$ is similar to $c_2 T_E^{r_2}$ if and only if $c_1 = c_2$ and $r_1 = r_2$.*

Proof. The first part of the Proof of Theorem 2 of A applies and implies that $r_1 = r_2$ ($= r$) and $|c_1| = |c_2|$. Thus suppose that $c_1 T_E^r$ is similar to $c_2 T_E^r$ or that $c T_E^r$ is similar to

$$(1) \quad T_E^r = P c T_E^r P^{-1} \text{ for } |c| = 1$$

where P is a bounded linear transformation of $L_p [0, 1]$ onto itself with the bounded linear inverse P^{-1} . If T is similar to $S = P T P^{-1}$, then $f(T)$ is similar to

$$(2) \quad f(S) = P f(T) P^{-1}$$

for polynomials and even analytic functions f . Let

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$$f(z) = \sum_{i=0}^{\infty} a_i z^{i+1}$$

Then

$$f(cT_E^r) = \sum_{i=0}^{\infty} a_i c^{i+1} T_E^{r(i+1)} = T_{g_1(y-x)}$$

where $g_1(t) = ct^{r-1}g(ct^r)$ where we have written t for $y - x$ and where

$$g(z) = \sum_{i=0}^{\infty} b_i z^i$$

with $b_i = a_i / \Gamma(r(i + 1))$. Equations (1) and (2) imply that $\|f(T_E^r)\| \leq \|P\| \|P^{-1}\| \|f(cT_E^r)\|$. The definition of the norm of a linear transformation in a Banach space implies the following inequality:

$$\|f(T_E^r)\| = \|T_{t^{r-1}g(ct^r)}\| \geq \left\| \int_x^1 (y-x)^{r-1} g((y-x)^r) k(y) dy \right\|_p$$

for all $k \in L_p [0, 1]$ such that $\|k\|_p = 1$. On the other hand, Lemma 2 of *A* implies that

$$\|T_{t^{r-1}g(ct^r)}\| \leq \|ct^{r-1}g(ct^r)\|_1 = \|t^{r-1}g(ct^r)\|_1.$$

Thus if $k(y) = 1$, we obtain

$$\begin{aligned} L &= \left\| \int_x^1 (y-x)^{r-1} g((y-x)^r) dy \right\|_p \leq \|f(T_E^r)\| \\ (3) \quad &\leq \|P\| \|P^{-1}\| \|f(cT_E^r)\| \\ &\leq \|P\| \|P^{-1}\| \|t^{r-1}g(ct^r)\|_1 = R. \end{aligned}$$

We shall find a family of functions g_v (and correspondingly f_v) depending on a positive parameter v such that if we use the notations L_v and R_v for the corresponding left and right hand sides of (3), $L_v \rightarrow \infty$ and $R_v \rightarrow 0$ as $v \rightarrow \infty$ contradicting the inequality (3): this contradiction then proves our theorem.

Let us first consider the case where the real part of c , $Re(c)$, is less than 0. Let $g_v(t) = \exp(vt)$. Since T_E^r is generalized nilpotent for $r \geq 1$, the corresponding function $f_v(T_E^r)$ exists and (1) indeed implies (2) for $S = T_E^r$ and $T = cT_E^r$. Then

$$R_v = \|t^{r-1}g_v(ct^r)\|_1 = \int_0^1 |t^{r-1} \exp(vct^r)| dt$$

and $R_v \rightarrow 0$ as $v \rightarrow \infty$. On the other hand

$$L_v = (1/r^p) \int_0^1 (\exp(v(1-x)) - 1/v)^p dx \rightarrow \infty$$

as $v \rightarrow \infty$. If finally $Re(c) \geq 0$ and $c \neq 1$, then there exist a positive

integer n such that $Re(c^n) < 0$. But then (1) implies that $c^n T_E^{nr}$ is similar to $T_E^{nr} = P c^n T_E^{nr} P^{-1}$ which contradicts the preceding result and the proof of the theorem is complete.

THEOREM 2. *Let $F(x, y) = (y - x)^{m-1} a G(x, y)$ satisfy, in addition to the general hypotheses stated above, one of the following:*

- (1) G is analytic in a suitable region and m is arbitrary;
- (2) $G(x, y) = G(y - x)$, $G(0) \neq 0$, $G \in C^2$ and m is arbitrary;
- (3) $G \in C^2$ and $m = 1$. Let A be a complex number. Then $AI + T_F$ and $AI + T_F^*$ are similar to the unique operator $AI + caT_E^m$ and $AI + c\bar{a}T_E^m$ respectively where $c = \left(\int_0^1 (G(u, u)^{1/m} du \right)^m$.

Here I is the identity operator and T_K^* , the adjoint of T_K , is defined by

$$(T_K^*)(f)(x) = \int_0^x \overline{K(y, x)} f(y) dy .$$

Proof. Note first that A implies that $AI + T_F$ is similar to $AI + caT_E^m$ and that $AI + T_F^*$ is similar to $AI + c\bar{a}T_E^{*m}$ (see Cor. Theorem 2 of A). Observe next that $T_E^* f(x) = \int_0^x f(y) dy$ and

$$T_E^{*m} f(x) = (1/\Gamma(m)) \int_0^x (x - y)^{m-1} f(y) dy$$

and that if $(S_{1-x} f)(x) = f(1 - x)$ then S_{1-x} is an isometry of $L_p [0, 1]$ onto itself and $S_{1-x} T_E^m S_{1-x}^{-1} = T_E^{*m}$. It remains to show uniqueness. Suppose that $A_1 I + c_1 a_1 T_E^{m_1}$ is similar to $A_2 I + c_2 a_2 T_E^{m_2}$. Then $A_1 = A_2$ (because of the complete continuity of T_E) and $c_1 a_1 T_E^{m_1}$ is similar to $c_2 a_2 T_E^{m_2}$ which by Theorem 1 implies that $c_1 = c_2$, $a_1 = a_2$, $m_1 = m_2$.

THEOREM 3. *The linear transformation $T_E + T_E^{1+a}$ where $0 < a < 1$ of $L_p [0, 1]$ into itself is not similar to any linear transformation cT_E^r for complex c and real $r \geq 1$.*

Proof. Preliminaries. 1. If two linear transformations S and T are similar, i.e., if there exists P such that $S = PTP^{-1}$, then there exists a constant K such that

$$(4) \quad 1/K \leq \| T^n \| / \| S^n \| \leq K ,$$

for all positive integers n . It suffices to take $K = \| P \| \| P^{-1} \|$.

2. The following inequality is a consequence of the fact that if $0 \leq F_1(x, y) \leq F_2(x, y)$ then $\| T_{F_1} \| \leq \| T_{F_2} \|$:

$$(5) \quad \|(T_E + T_E^{1+a})^n\| \geq n \|T_E^{n+a}\|$$

for all positive integers n .

3. Our next task is to find estimates for $\|T_E^n\|$. An estimate from above is the following:

$$(6) \quad \|T_E^n\| \leq 1/(n\Gamma(n)p^{1/p})$$

for all positive integers n . An estimate from below is furnished by the following Proposition:

Given the real positive number e there exists a positive number $K = K(e)$ and a positive integer $N = N(e)$ such that for all integers $n \geq N$,

$$(7) \quad \|T_E^n\| \geq K/(n^{1+e}\Gamma(n)).$$

Proof of (6). If $f \in L_p[0, 1]$,

$$T_E^n f(x) = \int_x^1 [(y-x)^{n-1}/\Gamma(n)] f(y) dy.$$

If $(1/p) + (1/q) = 1$, Hölder's inequality yields

$$\begin{aligned} \int_x^1 (y-x)^{n-1} f(y) dy &\leq \left(\int_x^1 (y-x)^{(n-1)q} dy \right)^{1/q} \|f\|_p \\ &= (1-x)^{((n-1)q+1)/q} \|f\|_p / ((n-1)q+1)^{1/q} \end{aligned}$$

so that

$$\begin{aligned} \|T_E^n f\|_p^p &= \int_0^1 |(T_E^n f)(x)|^p dx \\ &= (1/\Gamma(n))^p \int_0^1 \left| \int_x^1 (y-x)^{n-1} f(y) dy \right|^p dx \\ &\leq (1/\Gamma(n))^p (1/((n-1)q+1)^{p/q}) \int_0^1 (1-x)^{((n-1)p+(p/q))} dx \|f\|_p^p \\ &= (1/\Gamma(n))^p (1/((n-1)q+1)^{p/q}) (1/((n-1)p+(p/q)+1)) \|f\|_p^p \end{aligned}$$

which implies that

$$\|T_E^n\| \leq (1/\Gamma(n))(1/((n-1)q+1)^{1/q})(1/((n-1)p+(p/q)+1)^{1/p})$$

which in turn implies (6).

Proof of (7). We first observe that elementary considerations concerning the gamma function imply that given c such that $0 < c < 1$ and given a positive real number d there exists an integer N depending on c and d such that for all integers $n \geq N$

$$(8) \quad \Gamma(n + c) < (n + c)^{c+a}\Gamma(n).$$

Consider next the function $f(x) = r(1 - x)^{-s} \in L_p [0, 1]$ such that $\|f\|_p = 1$, i.e., $r^p = 1 - sp$ and $0 < s < 1/p$. Then

$$T_E^n f(x) = r\Gamma(1 - s)(1 - x)^{n-s}/\Gamma(n + 1 - s)$$

and

$$\|T_E^n\| \geq r\Gamma(1 - s)/\Gamma(n + 1 - s)(p(n - s) + 1)^{1/p}.$$

We now choose s (and hence r) such that for the positive real number e of (7), $0 < (1/p) - s < e$ and then we choose d such that $0 < d < e + s - (1/p)$ and finally by virtue of (8) we obtain N as a function of e such that for all integers $n \geq N$, $\Gamma(n + 1 - s) < (n + 1 - s)^{1-s+a}\Gamma(n)$ whence

$$\|T_E^n\| \geq r\Gamma(1 - s)/(n + 1 - s)^{1-s+a}\Gamma(n)(p(n - s) + 1)^{1/p}$$

which upon choosing $K = K(e)$ properly implies (7).

After these preliminaries, we turn to the proof of the theorem. We distinguish several cases. Let $T = T_E + T_E^{1+a}$.

Case 1. $|c| \leq 1$. Consider

$$h_n = \|(cT_E^n)\|/\|T^n\| \leq \|T_E^n\|/(n\|T_E^{n+a}\|)$$

where we have used (5) and the fact that $r \geq 1$. Take now positive real numbers e and d such that $a + e + d < 1$. Then there exists by (7) a positive constant K and an integer N such that for all integers $n \geq N$

$$(9) \quad \begin{aligned} h_n &\leq (n + a)^{1+e}\Gamma(n + a)/(n^2\Gamma(n)p^{1/p}K) \\ &\leq (n + a)^{1+e+a+d}\Gamma(n)/(n^2\Gamma(n)p^{1/p}K) \end{aligned}$$

where we have made use of (8) and (6). The last inequality implies that $h_n \rightarrow 0$ which in conjunction with (4) implies the truth of our theorem in the case under consideration.

Case 2. $r < 1$. Using the notations and making similar choices as under Case 1, (9) becomes

$$h_n \leq |c|^n(n + a)^{1+e+a+d}\Gamma(n)/(n^2r\Gamma(rn)p^{1/p}K)$$

which, since $|c|^n\Gamma(n)/\Gamma(rn)$ is bounded (in fact converges to 0) for $r > 1$ as $n \rightarrow \infty$, again proves the truth of the theorem in the present case.

Case 3. $r = 1, |c| > 1$. This time we consider the quotient

$$\begin{aligned}
 k_n &= \| T^n \| \| (cT_E)^n \| \\
 (10) \quad &\leq \sum_{i=0}^n \binom{n}{i} \| T_E^{n+a(n-i)} \| \| (c|)^n \| T_E^n \| \\
 &\leq ((n^{1+e}\Gamma(n))/(|c|^n K p^{1/p})) \sum_{i=0}^n \binom{n}{i} / (\Gamma(n + a(n - i) + 1)) ,
 \end{aligned}$$

which is valid for sufficiently large n ; again we used (6) and (7).

In order to complete the proof of our theorem, we need the following fact:

Given any positive real number e and given the positive real number $a < 1$, there exists an integer $N = N(e; a)$ such that for all integers i and n such that $0 \leq i \leq n \leq N$

$$(11) \quad \Gamma(n)/\Gamma(n + a(n - i) + 1) \leq 2e^{n-i} .$$

Proof. The case $i = 0$ results from elementary considerations about the gamma function. If $i = 1$, we find N_1 so that (11) is valid for $i = 0$ and $n \geq N_1$. We then find N_2 so that (8) is true for some arbitrary but fixed d , for $c = a$ and for $n \geq N_2$. Then $\Gamma(n)/\Gamma(n + (n - 1)a + 1) \leq (\Gamma(n)/\Gamma(n + na + 1))/(n + na + 1)^{a+a}$ which for $n \geq \max(N_1, N_2, e^{-1/a}) = N_3$ implies (11) for $i = 2$ and $n \geq N_3$. The remaining cases are settled by induction (except $i = n$ which is obvious); note that we never have to go above N_3 at any point. This completes the proof of (11).

The proof is now completed by substituting (11) into (10):

$$k_n \leq 2n^{1+e}(1 + e_1)^n / |c|^n K p^{1/p}$$

where e_1 is the constant e of (11). Thus $k_n \rightarrow 0$ upon proper choice of e_1 and our theorem is again true in view of (4). This completes the proof of Theorem 3.

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A. A. Albert, <i>Generalized twisted fields</i>	1
Richard Arens, <i>Operational calculus of linear relations</i>	9
John Herbert Barrett, <i>Disconjugacy of a self-adjoint differential equation of the fourth order</i>	25
Paul Richard Beesack, <i>Hardy's inequality and its extensions</i>	39
Julius Rubin Blum and David Lee Hanson, <i>On invariant probability measures. II</i>	63
Robert Allen Bonic, <i>Symmetry in group algebras of discrete groups</i>	73
R. Creighton Buck, <i>Multiplication operators</i>	95
Jack Gary Ceder, <i>Some generalizations of metric spaces</i>	105
Meyer Dwass, <i>Random crossings of cumulative distribution functions</i>	127
Albert Edrei, Wolfgang H. J. Fuchs and Simon Hellerstein, <i>Radial distribution and deficiencies of the values of a meromorphic function</i>	135
William Cassidy Fox, <i>Harmonic functions with arbitrary local singularities</i>	153
Theodore Thomas Frankel, <i>Manifolds with positive curvature</i>	165
Avner Friedman, <i>A strong maximum principle for weakly subparabolic functions</i>	175
Watson Bryan Fulks and J. O. Sather, <i>Asymptotics. II. Laplace's method for multiple integrals</i>	185
Adriano Mario Garsia and Eugene Richard Rodemich, <i>An embedding of Riemann surfaces of genus one</i>	193
Irving Leonard Glicksberg, <i>Weak compactness and separate continuity</i>	205
Branko Grünbaum, <i>On a conjecture of H. Hadwiger</i>	215
Frank J. Hahn, <i>On the action of a locally compact group on E_n</i>	221
Magnus R. Hestenes, <i>Relative hermitian matrices</i>	225
G. K. Kalisch, <i>On similarity invariants of certain operators in L_p</i>	247
Yitzhak Katznelson and Walter Rudin, <i>The Stone-Weierstrass property in Banach algebras</i>	253
Samir A. Khabbaz, <i>The subgroups of a divisible group G which can be represented as intersections of divisible subgroups of G</i>	267
Marvin Isadore Knopp, <i>Construction of a class of modular functions and forms</i>	275
Charles Alan McCarthy, <i>Commuting Boolean algebras of projections</i>	295
T. M. MacRobert, <i>Transformations of series of E-functions</i>	309
Heinz Renggli, <i>An inequality for logarithmic capacities</i>	313
M. S. Robertson, <i>Applications of the subordination principle to univalent functions</i>	315
David Sachs, <i>Partition and modulated lattices</i>	325
Frank S. Scalora, <i>Abstract martingale convergence theorems</i>	347
Elbert A. Walker, <i>Torsion endomorphic images of mixed Abelian groups</i>	375
Morgan Ward, <i>The prime divisors of Fibonacci numbers</i>	379
Charles R. B. Wright, <i>On the nilpotency class of a group of exponent four</i>	387