ON SIMILARITY INVARIANTS OF CERTAIN OPERATORS IN $L_p$

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The purpose of this paper is to extend the result of Corollary, Theorem 2 of the author's paper on Volterra operators (Annals of Math., 66, 1957, pp. 481-494 quoted as $A$; we shall use the definitions and notations of that paper) to the most general situation applicable: We are dealing with operators $T_F$ where $F(x, y) = (y - x)^{m-1} aG(x, y)$ is a function defined on the triangle $0 \leq x \leq y \leq 1$, where $m$ is a positive integer, $a$ a complex number of absolute value 1, $G$ is a complex valued function which is continuously differentiable and $G(x, x)$ is positive real. We recall that if $f \in L_p[0,1]$, then $(T_F(f))(x) = \int_x^1 F(x, y)f(y)dy$ is again in $L_p[0,1]$. The only difference from $A$ is the presence of the constant $a$ which affects none of results except Theorem 2 and its Corollary. Theorems 1 and 2 of the present paper fill the gap. Theorem 3 shows that differentiability conditions imposed on $F$ cannot be abandoned entirely—and also that the integral equation (1) of $A$ cannot be solved unless $K$ (which corresponds to our $F$) has at least first derivatives near $y = x$.

If $c$ is constant and $E$ is the function identically equal to 1, we define $T_E^r$ as $T_H$ which $H(x, y) = (y - x)^{-1/\Gamma(c)}$ (fractional integration of order $c$).

**Theorem 1.** Let $c_1$ and $c_2$ be complex numbers and let $r_1$ and $r_2$ be real numbers such that $r_1 \geq 1$, then $c_1 T_E^{r_1}$ is similar to $c_2 T_E^{r_2}$ if and only if $c_1 = c_2$ and $r_1 = r_2$.

**Proof.** The first part of the Proof of Theorem 2 of $A$ applies and implies that $r_1 = r_2 (= r)$ and $|c_1| = |c_2|$. Thus suppose that $c_1 T_E^{r_1}$ is similar to $c_2 T_E^{r_2}$ or that $c T_E^r$ is similar to

$$T_E^r = PcT_E^rP^{-1} \text{ for } |c| = 1$$

where $P$ is a bounded linear transformation of $L_p[0,1]$ onto itself with the bounded linear inverse $P^{-1}$. If $T$ is similar to $S = PTP^{-1}$, then $f(T)$ is similar to

$$f(S) = Pf(T)P^{-1}$$

for polynomials and even analytic functions $f$. Let

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Then

\[ f(z) = \sum_{i=0}^{\infty} a_i z^{i+1} \]

Then

\[ f(cT^r_E) = \sum_{i=0}^{\infty} a_i c^{i+1} T^r_C^{(i+1)} = T^r_{c(y-x)} \]

where \( g_i(t) = c^{t-i}g(ct) \) where we have written \( t \) for \( y - x \) and where

\[ g(x) = \sum_{i=0}^{\infty} b_i x^i \]

with \( b_i = a_i / r^i(i+1) \). Equations (1) and (2) imply that \( \| f(T^r_E) \| \leq \| P \| \| P^{-1} \| \| f(cT^r_C) \| \). The definition of the norm of a linear transformation in a Banach space implies the following inequality:

\[ \left\| T^{r-1} \right\| \| k \| \| g(\xi) \| \| T^r_C \| \leq \left\| \int_0^1 (y - x)^{r-1} g((y - x)^r) k(y) dy \right\|_p \]

for all \( k \in L_p [0,1] \) such that \( \| k \|_p = 1 \). On the other hand, Lemma 2 of \( A \) implies that

\[ \| T^{r-1} \| \| g(\xi) \| \| T^r_C \| \leq \left\| \int_0^1 (y - x)^{r-1} g((y - x)^r) k(y) dy \right\|_p \]

Thus if \( k(y) = 1 \), we obtain

\[ L = \left\| \int_0^1 (y - x)^{r-1} g((y - x)^r) dy \right\| \leq \left\| f(T^r_C) \right\| \]

\[ \leq \| P \| \| P^{-1} \| \| f(cT^r_C) \| \]

\[ \leq \| P \| \| P^{-1} \| \| t^{r-1}g(ct^r) \|_1 = R \].

We shall find a family of functions \( g_v \) (and correspondingly \( f_v \)) depending on a positive parameter \( v \) such that if we use the notations \( L_v \) and \( R_v \) for the corresponding left and right hand sides of (3), \( L_v \to \infty \) and \( R_v \to 0 \) as \( v \to \infty \) contradicting the inequality (3): this contradiction then proves our theorem.

Let us first consider the case where the real part of \( c \), \( Re(c) \), is less than 0. Let \( g_v(t) = \exp(vt) \). Since \( T^r_C \) is generalized nilpotent for \( r \geq 1 \), the corresponding function \( f_v(T^r_C) \) exists and (1) indeed implies (2) for \( S = T^r_C \) and \( T = cT^r_C \). Then

\[ R_v = \left\| t^{r-1} g_v(ct^r) \right\|_1 = \int_0^1 | t^{r-1} \exp(vct^r) | dt \]

and \( R_v \to 0 \) as \( v \to \infty \). On the other hand

\[ L_v = (1/r^p) \int_0^1 (\exp(v(1-x)) - 1/v)x \to \infty \]

as \( v \to \infty \). If finally \( Re(c) \geq 0 \) and \( c \neq 1 \), then there exist a positive
integer $n$ such that $Re(e^n) < 0$. But then (1) implies that $c^n T^m_{E}$ is similar to $T^m_{E} = P e^n T^m_{E} P^{-1}$ which contradicts the preceding result and the proof of the theorem is complete.

**Theorem 2.** Let $F(x, y) = (y - x)^{m-1} G(x, y)$ satisfy, in addition to the general hypotheses stated above, one of the following:

1. $G$ is analytic in a suitable region and $m$ is arbitrary;
2. $G(x, y) = G(y - x)$, $G(0) \neq 0$, $G \in C^2$ and $m$ is arbitrary;
3. $G \in C^2$ and $m = 1$. Let $A$ be a complex number. Then $AI + T_{E}$ and $AI + T_{E}^*$ are similar to the unique operator $AI + caT_{E}^m$ and $AI + caT_{E}^m$ respectively where $c = \left( \int_0^1 (G(u, u)^{1/m} du)^m \right)$.

Here $I$ is the identity operator and $T_{E}^*$, the adjoint of $T_{E}$, is defined by

$$(T_{E}^*)(f)(x) = \int_0^x K(y, x)f(y)dy.$$ 

**Proof.** Note first that $A$ implies that $AI + T_{E}$ is similar to $AI + caT_{E}^m$ and that $AI + T_{E}^*$ is similar to $AI + caT_{E}^m$ (see Cor. Theorem 2 of $A$). Observe next that $T_{E}^{*m}f(x) = \int_0^x f(y)dy$ and

$$T_{E}^{*m}f(x) = (1/\Gamma(m)) \int_0^x (x - y)^{m-1} f(y)dy$$

and that if $(S_{1-x}f)(x) = f(1 - x)$ then $S_{1-x}$ is an isometry of $L_p[0, 1]$ onto itself and $S_{1-x} T_{E}^{*m} S_{1-x} = T_{E}^{*m}$. It remains to show uniqueness. Suppose that $A_1 I + ca_1 T_{E}^{m_1}$ is similar to $A_2 I + ca_2 T_{E}^{m_2}$. Then $A_1 = A_2$ (because of the complete continuity of $T_{E}$) and $ca_1 T_{E}^{m_1}$ is similar to $ca_2 T_{E}^{m_2}$ which by Theorem 1 implies that $c_1 = c_2$, $a_1 = a_2$, $m_1 = m_2$.

**Theorem 3.** The linear transformation $T_{E} + T_{E}^{*+a}$ where $0 < a < 1$ of $L_p[0, 1]$ into itself is not similar to any linear transformation $c T_{E}$ for complex $c$ and real $r \geq 1$.

**Proof.** Preliminaries. 1. If two linear transformations $S$ and $T$ are similar, i.e., if there exists $P$ such that $S = PTP^{-1}$, then there exists a constant $K$ such that

$$(4) \quad 1/K \leq || T^n ||/|| S^n || \leq K,$$

for all positive integers $n$. It suffices to take $K = || P ||/|| P^{-1} ||$.

2. The following inequality is a consequence of the fact that if $0 \leq F_1(x, y) \leq F_2(x, y)$ then $|| T_{E_1} || \leq || T_{E_2} ||$: 

...
for all positive integers $n$.

3. Our next task is to find estimates for $\| T^n \|$. An estimate from above is the following:

\[
\| T^n \| \leq 1/(n\Gamma(n)p^{1/p})
\]

for all positive integers $n$. An estimate from below is furnished by the following Proposition:

Given the real positive number $e$ there exists a positive number $K = K(e)$ and a positive integer $N = N(e)$ such that for all integers $n \geq N$,

\[
\| T^n \| \geq K/(n^{1+e}\Gamma(n))
\]

Proof of (6). If $f \in L_p[0,1]$,

\[
T^n f(x) = \int_0^1 [(y - x)^{n-1}/\Gamma(n)]f(y)dy
\]

If $(1/p) + (1/q) = 1$, Hölder's inequality yields

\[
\int_0^1 (y - x)^{n-1}f(y)dy \leq \left( \int_0^1 (y - x)^{(n-1)q}dy \right)^{1/q} \| f \|_p
\]

so that

\[
\| T^n f \|_p \leq (1 - x)^{(n-1)q+1/q} \| f \|_p/((n - 1)q + 1)^{1/q}
\]

which implies that

\[
\| T^n \| \leq (1/\Gamma(n))(1/((n - 1)q + 1)^{1/q})(1/(n - 1)p + (p/q + 1)) \| f \|_p
\]

which in turn implies (6).

Proof of (7). We first observe that elementary considerations concerning the gamma function imply that given $c$ such that $0 < c < 1$ and given a positive real number $d$ there exists an integer $N$ depending on $c$ and $d$ such that for all integers $n \geq N$
Consider next the function \( f(x) = r(1 - x)^{-s} \in L_p [0, 1] \) such that \( \|f\|_p = 1 \), i.e., \( r^p = 1 - sp \) and \( 0 < s < 1/p \). Then
\[
T^n f(x) = r \Gamma(1 - s)(1 - x)^{n-1}/\Gamma(n + 1 - s)
\]
and
\[
\|T^n\| \geq r \Gamma(1 - s)/(n + 1 - s)(p(n - s) + 1)^{1/p}.
\]
We now choose \( s \) (and hence \( r \)) such that for the positive real number \( e \) of (7), \( 0 < (1/p) - s < e \) and then we choose \( d \) such that \( 0 < d < e + s - (1/p) \) and finally by virtue of (8) we obtain \( N \) as a function of \( e \) such that for all integers \( n \geq N \), \( \Gamma(n + 1 - s) < (n + 1 - s)^{1-s+a} \Gamma(n) \) whence
\[
\|T^n\| \geq r \Gamma(1 - s)/(n + 1 - s)^{1-s+a} \Gamma(n)(p(n - s) + 1)^{1/p}
\]
which upon choosing \( K = K(e) \) properly implies (7).

After these preliminaries, we turn to the proof of the theorem. We distinguish several cases. Let \( T = T^E + T^{1+a} \).

**Case 1.** \( |c| \leq 1 \). Consider
\[
h_n = \|(cT^n)^n\|/\|T^n\| \leq \|T^n\|/(n \|T^{n+a}\|)
\]
where we have used (5) and the fact that \( r \geq 1 \). Take now positive real numbers \( e \) and \( d \) such that \( a + e + d < 1 \). Then there exists by (7) a positive constant \( K \) and an integer \( N \) such that for all integers \( n \geq N \)
\[
h_n \leq (n + a)^{1-e+a} \Gamma(n + a)/(n^2 \Gamma(n) p^{1/p} K)
\]
\[
\leq (n + a)^{1-e+a} \Gamma(n)/(n^2 \Gamma(n) p^{1/p} K)
\]
where we have made use of (8) and (6). The last inequality implies that \( h_n \rightarrow 0 \) which in conjunction with (4) implies the truth of our theorem in the case under consideration.

**Case 2.** \( r < 1 \). Using the notations and making similar choices as under Case 1, (9) becomes
\[
h_n \leq |c|^n(n + a)^{1-e+a} \Gamma(n)/(n^2 r \Gamma(rn) p^{1/p} K)
\]
which, since \( |c|^n \Gamma(n)/\Gamma(rn) \) is bounded (in fact converges to 0) for \( r > 1 \) as \( n \rightarrow \infty \), again proves the truth of the theorem in the present case.

**Case 3.** \( r = 1, \ |c| > 1 \). This time we consider the quotient
which is valid for sufficiently large \( n \); again we used (6) and (7).

In order to complete the proof of our theorem, we need the following fact:

Given any positive real number \( e \) and given the positive real number \( a < 1 \), there exists an integer \( N = N(e; a) \) such that for all integers \( i \) and \( n \) such that \( 0 \leq i \leq n \leq N \) \( i \)

(11) \[ \Gamma(n)/\Gamma(n + a(n - i) + 1) \leq 2e^{n-i}. \]

Proof. The case \( i = 0 \) results from elementary considerations about the gamma function. If \( i = 1 \), we find \( N_1 \) so that (11) is valid for \( i = 0 \) and \( n \geq N_1 \). We then find \( N_2 \) so that (8) is true for some arbitrary but fixed \( \epsilon \), for \( c = a \) and for \( n \geq N_2 \). Then \( \Gamma(n)/\Gamma(n + (n-1)a + 1) \leq \frac{\Gamma(n)/\Gamma(n + na + 1)\Gamma(n + (n-1)a + 1)}{(n + na + 1)^{a + \epsilon}} \) which for \( n \geq \max (N_1, N_2, e^{-1/\epsilon}) = N_3 \) implies (11) for \( i = 2 \) and \( n \geq N_3 \). The remaining cases are settled by induction (except \( i = n \) which is obvious); note that we never have to go above \( N_3 \) at any point. This completes the proof of (11).

The proof is now completed by substituting (11) into (10):

\[
k_n \leq 2n^{1+\epsilon}(1 + e_1)^n || c \n K^{-1/\epsilon} \]

where \( e_1 \) is the constant \( e \) of (11). Thus \( k_n \to 0 \) upon proper choice of \( e_1 \) and our theorem is again true in view of (4). This completes the proof of Theorem 3.

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