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**THE SUBGROUPS OF A DIVISIBLE GROUP G WHICH CAN BE
REPRESENTED AS INTERSECTIONS OF DIVISIBLE
SUBGROUPS OF G**

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Introduction. In [1], page 70, L. Fuchs asks the following question: Which are those subgroups of a divisible group G that can be represented as intersections of divisible subgroups of G ?

The main purpose of this paper is to give an answer to this question.

NOTATION.

- N1: If H is a primary p -group, let $S(H)$ denote the subgroup of elements of H whose orders are 1 or p .
- N2: If G is Abelian, let $T(G)$ be the torsion subgroup of G ; let G_p denote the primary p -component of $T(G)$; and, in case G is divisible, let $F(G)$ denote a maximal torsion free subgroup of G .
- N3: Let the symbol \oplus denote a direct sum. Let the symbol $<$ denote "properly contained in." Let \subset denote "contained in." Let $N \setminus M$ denote "the set of elements in N and not in M ." Let \cong denote "is isomorphic to." Let \exists denote "there exists (exist)." Let \ni denote "such that." Let $(N_a)_{a \in A}$ denote a family of sets N_a indexed by members of the set A . Finally if Q is a subset of a group, let $\langle Q \rangle$ denote the subgroup of that group generated by the elements of Q .
- N4: Let R denote the additive group of rationals. Let P denote the set of primes. Let the group $C(p^\infty)$ be the indecomposable divisible primary p -group.
- N5: Let $C = C(2^\infty) \oplus C(3^\infty) \oplus C(5^\infty) \oplus \dots$; and if $S \subset P$, let $C_S = \bigoplus_{p \in S} C(p^\infty)$.
- N6: If G is a group, let $P(G)$ be the set of $p \in P$, such that $\exists x \in G$ with order $x = p$.
- N7: Finally, we recall the following convenient and succinct classification of the subgroups of R [see Kurosh I, page 208]. Let p_1, p_2, p_3, \dots be the sequence of primes in natural order. A characteristic is a sequence $a = (a_1, a_2, a_3, \dots)$, where $a_i = a$ non-negative integer or ∞ . A type is a class of equivalent characteristics, two characteristic $a = (a_1, a_2, a_3, \dots)$ and $b = (b_1, b_2, b_3, \dots)$ being equivalent if and only if $\sum_{i=1}^{\infty} |a_i - b_i| < \infty$, where $\infty - \infty = 0$.

$A \subset R$ has type a if and only if it is isomorphic to the subgroup

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of R consisting of those rationals whose denominators in the reduced form are divisible by no higher power of the prime p_i than the a_i th if $a_i < \infty$, and by every power of p_i if $a_i = \infty$.

Define $a \geq b$ if and only if $a_i \geq b_i$, for $i = 1, 2, 3, \dots$.

N8: Let $S \subset P$. We shall say that A above has type T_s is and only if $a_i = 0$ for $p_i \in S$ and $a_i = \infty$ otherwise. Then it is well known that $R/B \subset C_s$ if and only if B contains a subgroup A of type T_s , and that the intersection of two subgroups of R containing subgroups of type T_s again contains a subgroup of type T_s .

N9: Let the symbol \bigcap_{σ} stand for the phrase ‘‘an intersection of divisible subgroups of G .’’

LEMMA 1. (Kulikov):

- a. *A divisible group M is a minimal divisible group containing the subgroup L if and only if $H \subset M$ and $H \cap L = 0$ imply $H = 0$, H being a subgroup.*
- b. *If M is a minimal divisible group containing L , then M/L is torsion and divisible.*

LEMMA 2. *Let G be divisible and L a subgroup of G . Let M be a minimal divisible subgroup of G containing L . Thus, using the notational in N2, we may write, $G = M \oplus E = M \oplus T(E) \oplus F(E)$. We have:*

- a. *If M is minimal divisible containing L , then $S(M_p) = S(L_p)$ for each $p \in P$, and $T(M)$ is minimal divisible containing $T(L)$.*
- b. *Kulikov: If L is torsion, then M is minimal divisible containing L if and only if $S(L_p) = S(M_p)$ for all $p \in P$.*
- c. *If L is \bigcap_{σ} then $T(L)$ is $\bigcap_{T(G)}$ and hence \bigcap_{σ} .*

Proof.

- a. Let $x \in S(M_p) \setminus S(L_p)$. Then $\{x\} \cap S(L_p) = 0$, and therefore $\{x\} \cap L = 0$. By Lemma 1a, $x = 0$. Next, $T(M)$ is divisible, and contains $T(L)$. If $N \subset T(M)$ is divisible and contains $T(L)$, $T(M)$ can be written as $T(M) = N \oplus K$, where $K \cap T(L) = 0$; and hence $K \cap L = 0$, so that by Lemma 1a, $K = 0$; and hence, $N = T(M)$.
- b. The ‘‘only if’’ part is contained in part a. For the ‘‘if’’ part, assume $N \subset M$ is divisible and contains L . Then we may write $M = N \oplus K$. Then, by hypothesis, K cannot have elements of prime order, and must therefore be 0.
- c. Assume $L = \bigcap_{a \in A} M_a$, where each M_a is divisible and contains L . Then each $T(M_a)$ is divisible and contains $T(L)$. Moreover, we have $\bigcap_{a \in A} T(M_a) = T(\bigcap_{a \in A} M_a) = T(L)$. Hence $T(L)$ is $\bigcap_{T(G)}$ and hence \bigcap_{σ} .

LEMMA 3. *Let G be a minimal divisible group containing the*

subgroup L , and having a representation of the form $G = \bigoplus_{a \in A} G_a$. Then G/L (which by Lemma 1b is divisible and torsion) contains a subgroup isomorphic to $C(p^\infty)$ if and only if for some $a \in A$, $G_a/G_a \cap L$ contains a subgroup of the same kind. In other words: $P(G/L) = \bigcup_{a \in A} P(G_a/G_a \cap L) = \bigcup_{a \in A} P(G_a + L/L)$.

Proof. Because of the divisibility of all the groups concerned, it suffices to check the existence of elements of order p . Suppose $x \in G_a/G_a \cap L$ has order p . Then $G/L \supset G_a + L/L \cong G_a/(G_a \cap L)$. Hence, G/L has an element of order p . Conversely, suppose that for no $a \in A$ does $G_a/(G_a \cap L)$ contain an element of order p . Then no $G_a + L/L$ contains an element of order p . Hence, the subgroup of G/L generated by the $(G_a + L)/L$ contains no elements of order p . But since the G_a 's generate G , the $G_a + L/L$'s generate all of G/L .

THEOREM 1. Let G be a divisible group; let L be a subgroup of G ; let M be a minimal divisible subgroup of G containing L . Suppose G has a representation of the form $G = M \oplus E$, then $L = M \cap (\bigcap_{\omega \in \Omega} M_\omega)$, where M_ω is a divisible subgroup of G containing L for each $\omega \in \Omega$, if and only if there exists homomorphisms $\underline{h}_\omega: M \rightarrow E$ for each $\omega \in \Omega$ such that $\bigcap_{\omega \in \Omega} \ker h_\omega = L$.

Proof. To prove the "if" part, let I be the identity map of M ; and for each $\omega \in \Omega$ let $g_\omega: M \rightarrow G$ be defined by $g_\omega = I + h_\omega$. Let $M_\omega = g_\omega(M)$. Then $L \subset M_\omega$ since $h_\omega(L) = 0$, and therefore M_ω is divisible since it is a homomorphic image of M . Finally, $x \in M_\omega \cap M$ implies $x \in \ker h_\omega$ (since $x = y + h_\omega(y)$ implies $h_\omega(y) = x - y \in M \cap E = 0$); and hence, $L \subset \bigcap_{\omega \in \Omega} M_\omega \cap M = \bigcap_{\omega \in \Omega} \ker h_\omega = L$.

To prove the "only if" part, suppose $L = M \cap (\bigcap_{\omega} M_\omega)$. It can be assumed that each M_ω is minimal divisible containing L and, therefore, also minimal divisible containing $M \cap M_\omega$. Also, M is minimal divisible containing $M \cap M_\omega$. Also, M is minimal divisible containing $M \cap M_\omega$; so there is an isomorphism $i_\omega: M \rightarrow M_\omega$ which is the identity on $M \cap M_\omega$. Note that $i_\omega(x) \in M \Rightarrow i_\omega(i_\omega(x)) = i_\omega(x) \Rightarrow i_\omega(x) = x \Rightarrow x \in M \cap M_\omega$. Let $p: G \rightarrow E$ be the projection determined by the decomposition $G = M \oplus E$. Let h_ω be defined by $h_\omega = pi_\omega$. Then $h_\omega(x) = 0$, $i_\omega(x) \in M$, and $x \in M \cap M_\omega$ are equivalent. Thus $\bigcap_{\omega \in \Omega} \ker h_\omega = \bigcap_{\omega \in \Omega} M \cap M_\omega = L$.

REMARK. The underlined portion of Theorem 1 may be replaced by $\underline{h}_\omega: M \rightarrow T(E)$.

COROLLARY 1. Let G be a divisible group; let L be a subgroup of G ; and let M be a subgroup of G which is minimal with respect to being divisible and containing L . Thus, we may write $G = M \oplus E =$

$M \oplus T(E) \oplus F(E)$, and $P(E) = P(T(E))$. Then L is \bigcap_{σ} if and only if $P(M/L) \subset P(E)$.

Proof. The condition $P(M/L) \subset P(E)$ is easily seen to be equivalent to the existence of the family $\{h_{\omega}\}_{\omega \in \sigma}$ of homomorphisms in Theorem 1.

REMARK. Let G be divisible and torsion free, then $L \subset G$ is \bigcap_{σ} if and only if L is divisible or, equivalently, is a direct summand of G .

COROLLARY 2. Let G be divisible, and let L be a torsion subgroup of G . Then L is \bigcap_{σ} if and only if for each $p \in P$, $S(L_p) < S(G_p)$ whenever $S(L_p) \neq 0$, and L_p is not divisible.

Proof. If L is \bigcap_{σ} then by Lemma 2c for each $p \in P$ obviously L_p is \bigcap_{σ_p} ; and, hence, to prove that our condition is necessary, we may assume that G is primary, and L is not divisible, in which case the necessity becomes obvious in view of the fact that otherwise G would be the only minimal divisible subgroup of itself containing L ; and, consequently, $L = G$, since L is \bigcap_{σ} , contrary to L being not divisible.

To Prove the "only if" part, note that $p \in P(M/L)$ implies $M_p/L_p = (M/L)_p \neq 0$, since L is torsion. Thus, by hypothesis, $p \in P(M/L) \Rightarrow S(L_p) < S(G_p) \Rightarrow S(M_p) < S(G_p) = S(M_p) \oplus S(E_p) \Rightarrow S(E_p) \neq 0 \Rightarrow p \in P(E)$.

COROLLARY 3. Let G be divisible and $L \subset G$ be torsion, reduced and \bigcap_{σ} . Then every subgroup of L is \bigcap_{σ} .

COROLLARY 4. Let G be divisible and L be \bigcap_{σ} . Let M be a minimal divisible subgroup of G containing L . Let K be a subgroup of G such that $L \subset K \subset M$. Then K is \bigcap_{σ} .

Proof. If $G = M \oplus E$, then $P(M/K) \subset P(M/L) \subset P(E)$.

COROLLARY 5. Let K be any Abelian group of arbitrary cardinal number A . Then K can be embedded in a divisible group G of power $A \aleph_0$ in such a way that any subgroup of K can be represented as an intersection of two divisible subgroups of G .

Proof. Let M be a minimal divisible group containing K and let E be a group isomorphic to M/K . Let G be the direct sum of M and E . The cardinality of G is clearly $A \aleph_0$ and the isomorphism of M/K into E induces a homomorphism $h: M \rightarrow E$ with $\ker h = K$. Thus, Theorem 1 gives the required conclusion.

REMARK. Let $L \subset G$, and let $L = L_1 \oplus L_2$, where L_2 is divisible and

L_1 is reduced. Then, also, $G = L_2 \oplus K$, where K may be chosen to contain L_1 . It is easy to see that L is \bigcap_{σ} if and only if L_1 is \bigcap_K . Thus, in order to avoid excessive wording, we may in the following theorems assume without loss of generality that L is reduced.

THEOREM 2. *Assume $L \subset G$ is reduced, then L is \bigcap_{σ} if and only if $T(L)$ is \bigcap_{σ} and $P(G/L) \subset P(G)$, equality holding if L is \bigcap_{σ} .*

Proof. Let $G = M \oplus E$, where M and E are as in Theorem 1. Then $P(G) = P(T(M)) \cup P(E)$, and $P(G/L) = P(M/L) \cup P(E)$, because $G/L \cong (M/L) \oplus E$. Note that if $T(L)$ is \bigcap_{σ} , then $T(L)$ is $\bigcap_{T(G)}$ and hence $P(T(M)/T(L)) \subset P(T(E)) = P(E)$, by Corollary 1. But since $T(L)$ is reduced, $P(T(M)/T(L)) = P(T(M))$. Thus the assumption that $T(L)$ is \bigcap_{σ} implies $P(G) = P(E)$ and, therefore, that the conditions $P(G/L) = P(G)$, $P(G/L) \subset P(G)$, and $P(M/L) \subset P(E)$ are equivalent. This observation, together with Lemma 2c and Corollary 1, proves Theorem 2.

COROLLARY 6. *Let G be any divisible group. Let C be as defined in N5, and let $\bar{G} = C \oplus G$. Then any subgroup $K \subset \bar{G}$ is $\bigcap_{\bar{\sigma}}$.*

REMARK. In Corollary 6, C may be replaced by any Abelian group containing it.

COROLLARY 7. *Any torsion free subgroup T of \bar{G} above is $\bigcap_{\bar{\sigma}}$.*

Proof. T is contained in a direct summand of \bar{G} whose complementary direct summand contains a subgroup isomorphic to C .

REMARK. The following example shows that if $L \subset G$ is \bigcap_{σ} and if $\bar{L} \subset G$ is isomorphic to L , \bar{L} need not be \bigcap_{σ} . Let $G = \bigoplus_{i=1}^{\infty} C_i$ where $C_i \cong C(p^{\infty})$ and where p is fixed. Then,

$$S\left(\bigoplus_{i=1}^{\infty} C_i\right) \cong S\left(\bigoplus_{i=2}^{\infty} C_i\right);$$

however, $S\left(\bigoplus_{i=1}^{\infty} C_i\right)$ is not \bigcap_{σ} , while $S\left(\bigoplus_{i=2}^{\infty} C_i\right)$ is \bigcap_{σ} .

In this connection we have:

COROLLARY 8. *Let $L \subset G$ be \bigcap_{σ} , and let $\bar{L} \subset G$ be isomorphic to L . Then if $T(\bar{L})$ is \bigcap_{σ} —this is in particular the case if $\bar{L} \subset L - \bar{L}$ is also \bigcap_{σ} . Thus, \bar{L} is \bigcap_{σ} if and only if $T(\bar{L})$ is \bigcap_{σ} .*

Proof. For the proof we may assume L is reduced. By Theorem 2, it suffices to show that $P(G/\bar{L}) \subset P(G)$. Let M and \bar{M} be minimal divisible subgroup of G containing L and \bar{L} , respectively, so that $G =$

$M \oplus N = \bar{M} \oplus \bar{N}$. Then we have $M/L \cong \bar{M}/\bar{L}$ [see Kurosh I, page 168]. Thus, $P(G/\bar{L}) = P(\bar{M}/\bar{L}) \cup P(\bar{N})$
 $= P(M/L) \cup P(\bar{N})$
 $\subset P(G/L) \cup P(G)$
 $\subset P(G)$, since L is \bigcap_{σ} .

THEOREM 3. *Let G be a divisible group and let $L \subset G$ be reduced. Then the following statements are equivalent:*

- (a) L is \bigcap_{σ} .
- (b) $T(L)$ is \bigcap_{σ} and for any subgroup \bar{R} of G isomorphic to R , either $L \cap \bar{R}$ is zero or of type $\geq T_{P(G)}$.
- (c) $S(L_p) < S(G_p)$ if $p \in P(G)$ and for any subgroup \bar{R} of G isomorphic to R , either $L \cap \bar{R}$ is zero or of type $\geq T_{P(G)}$.
- (d) $T(L)$ is \bigcap_{σ} and $L \subset (\bigoplus_{a \in A} R_a) \oplus T(G) \subset G$, where $R_a \cong R$ and each $L \cap R_a$ is either zero or of $\geq T_{P(G)}$.

Proof. By Theorem 2, (a) is equivalent to the conditions $T(L)$ is \bigcap_{σ} and $P(G/L) \subset P(G)$. These conditions imply (b) since G/L contains a subgroup isomorphic to $\bar{R}/\bar{R} \cap L$, so that $P(\bar{R}/\bar{R} \cap L) \subset P(G/L) \subset P(G)$ and therefore $\bar{R} \cap L$ is zero or of type $\geq T_{P(G)}$ by N8. Properties (b) and (c) are equivalent by Lemma 2c and Corollary 2. Also (b) implies (d). Finally, suppose (d) holds. Then, $P(T(G)/(T(G) \cap L)) = P(T(G)/T(L)) \subset P(T(G)) = P(G)$ by Theorem 2. Let $G = (\bigoplus_{b \in B} R_b) \oplus (\bigoplus_{a \in A} R_a) \oplus T(G)$. Then $R_b \cap L$ is 0 for all $b \in B$, and for each $a \in A$, $R_a \cap L$ is 0 or has type $\geq T_{P(G)}$ by hypothesis. Thus, by Lemma 3, $P(G/L) = \bigcap_{b \in B} P(R_b/(R_b \cap L)) \cup \bigcup_{a \in A} P(R_a/(R_a \cap L)) \cup P(T(G)/(T(G) \cap L)) \subset P(G)$. By Theorem 2, this implies (a).

DEFINITION. Define a subset $(x_a)_{a \in A}$ of elements of an Abelian group H to be independent if and only if

$$n_1 x_{a_1} + n_2 x_{a_2} + \dots + n_m x_{a_m} = 0$$

implies $n_1 = n_2 = \dots = n_m = 0$ where each $a_i \in A$ and the n_i 's are integers.

COROLLARY 10. Assume $L \subset G$ is reduced, then L is \bigcap_{σ} if and only if $T(L)$ is \bigcap_{σ} and L contains a subgroup H which contains a maximal independent subset of L and which has the form $\bigoplus_{a \in A} S_a$, where each S_a is isomorphic to a subgroup of R having type $T_{P(G)}$.

Proof. Assume L is \bigcap_{σ} , then by Lemma 2c $T(L)$ is \bigcap_{σ} . Let $M \subset G$ be minimal divisible containing L , and assume that $M = T(M) \oplus F(M) = T(M) \oplus (\bigoplus_{a \in A} R_a)$, where $R_a \cong R$ for all $a \in A$. Then, by Theorem 3c and Lemma 1a, each $L \cap R_a$ contains a subgroup S_a of

type $T_{P(G)}$. Then it is easy to see that $\bigoplus_{a \in A} S_a$ exists; and that, if $x_a \in S_a$, then $(x_a)_{a \in A}$ is a maximal independent subset of M and therefore, of L .

Next assume the condition holds and let M be as usual. For each S_a , let $R_a \supset S_a$ be a subgroup of M of type R . Since any two non-zero subgroups of R have a non-zero intersection, and since $\bigoplus_{a \in A} S_a$ exists, also $\bigoplus_{a \in A} R_a$ exists. Let $x_a \in S_a$; then $(x_a)_{a \in A}$ is a maximal independent subset of H ; and since H contains a maximal independent subset of L , $(x_a)_{a \in A}$ is also a maximal independent subset of L . Thus, we must have $M = T(M) \oplus (\bigoplus_{a \in A} R_a)$. Theorem 3d and the fact that $T(L)$ is \bigcap_G imply that L is \bigcap_G .

REMARK. Concerning the last definition given above, it is well known that H contains a maximal independent subset and that if $(x_a)_{a \in A}$ is independent, then $H/\{(x_a)_{a \in A}\}$ is torsion if and only if $(x_a)_{a \in A}$ is also maximal independent. Thus, Corollary 10 may be worded as follows: Assume L is reduced; then L is \bigcap_G if and only if $T(L)$ is \bigcap_G and L contains a subgroup H which has the form $\bigoplus_{a \in A} S_a$ where S_a is isomorphic to a subgroup of R of type $T_{P(G)}$ and such that L/H is torsion.

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