THE SUBGROUPS OF A DIVISIBLE GROUP $G$ WHICH CAN BE REPRESENTED AS INTERSECTIONS OF DIVISIBLE SUBGROUPS OF $G$

SAMIR A. KHABBAZ
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Introduction. In [1], page 70, L. Fuchs asks the following question: Which are those subgroups of a divisible group $G$ that can be represented as intersections of divisible subgroups of $G$?

The main purpose of this paper is to give an answer to this question.

NOTATION.

N1: If $H$ is a primary $p$-group, let $S(H)$ denote the subgroup of elements of $H$ whose orders are 1 or $p$.

N2: If $G$ is Abelian, let $T(G)$ be the torsion subgroup of $G$; let $G_p$ denote the primary $p$-component of $T(G)$; and, in case $G$ is divisible, let $F(G)$ denote a maximal torsion free subgroup of $G$.

N3: Let the symbol $\oplus$ denote a direct sum. Let the symbol $<$ denote "properly contained in." Let $\subset$ denote "contained in." Let $N \setminus M$ denote "the set of elements in $N$ and not in $M$." Let $\cong$ denote "is isomorphic to." Let $\exists$ denote "there exists (exist)." Let $\exists^*$ denote "such that." Let $(N_a)_{a \in A}$ denote a family of sets $N_a$ indexed by members of the set $A$. Finally if $Q$ is a subset of a group, let $\{Q\}$ denote the subgroup of that group generated by the elements of $Q$.

N4: Let $R$ denote the additive group of rationals. Let $P$ denote the set of primes. Let the group $C(p^\infty)$ be the indecomposable divisible primary $p$-group.

N5: Let $C = C(2^\infty) \oplus C(3^\infty) \oplus C(5^\infty) \oplus \cdots$; and if $S \subset P$, let $C_S = \bigoplus_{p \in S} C(p^\infty)$.

N6: If $G$ is a group, let $P(G)$ be the set of $p \in P$, such that $\exists x \in G$ with order $x = p$.

N7: Finally, we recall the following convenient and succinct classification of the subgroups of $R$ [see Kurosh I, page 208]. Let $p_1, p_2, p_3, \cdots$ be the sequence of primes in natural order. A characteristic is a sequence $\alpha = (a_1, a_2, a_3, \cdots)$, where $a_i = a$ non-negative integer or $\infty$. A type is a class of equivalent characteristics, two characteristic $\alpha = (a_1, a_2, a_3, \cdots)$ and $\beta = (b_1, b_2, b_3, \cdots)$ being equivalent if and only if $\sum_{i=1}^{\infty} |a_i - b_i| < \infty$, where $\infty - \infty = 0$.

$A \subset R$ has type $\alpha$ if and only if it is isomorphic to the subgroup

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of $R$ consisting of those rationals whose denominators in the reduced form are divisible by no higher power of the prime $p_i$ than the $a_i$th if $a_i < \infty$, and by every power of $p_i$ if $a_i = \infty$.

Define $a \geq b$ if and only if $a_i \geq b_i$, for $i = 1, 2, 3, \cdots$.

**N8:** Let $S \subseteq P$. We shall say that $A$ above has type $T_s$ if and only if $\alpha_i = 0$ for $p_i \in S$ and $\alpha_i = \infty$ otherwise. Then it is well known that $R/B \subset C_s$ if and only if $B$ contains a subgroup $A$ of type $T_s$, and that the intersection of two subgroups of $R$ containing subgroups of type $T_s$ again contains a subgroup of type $T_s$.

**N9:** Let the symbol $\prod_\sigma$ stand for the phrase “an intersection of divisible subgroups of $G$.”

**Lemma 1.** (Kulikov):

a. A divisible group $M$ is a minimal divisible group containing the subgroup $L$ if and only if $H \subseteq M$ and $H \cap L = 0$ imply $H = 0$, $H$ being a subgroup.

b. If $M$ is a minimal divisible group containing $L$, then $M/L$ is torsion and divisible.

**Lemma 2.** Let $G$ be divisible and $L$ a subgroup of $G$. Let $M$ be a minimal divisible subgroup of $G$ containing $L$. Thus, using the notational in N2, we may write, $G = M \oplus E = M \oplus T(E) \oplus F(E)$. We have:

a. If $M$ is minimal divisible containing $L$, then $S(M_p) = S(L_p)$ for each $p \in P$, and $T(M)$ is minimal divisible containing $T(L)$.

b. Kulikov: If $L$ is torsion, then $M$ is minimal divisible containing $L$ if and only if $S(L_p) = S(M_p)$ for all $p \in P$.

c. If $L = \prod_\alpha$ then $T(L)$ is $\prod_{T(\alpha)}$ and hence $\prod_\sigma$.

**Proof.**

a. Let $x \in S(M_p) \setminus S(L_p)$. Then $\{x\} \cap S(L_p) = 0$, and therefore $\{x\} \cap L = 0$. By Lemma 1a, $x = 0$. Next, $T(M)$ is divisible, and contains $T(L)$. If $N \subset T(M)$ is divisible and contains $T(L)$, $T(M)$ can be written as $T(M) = N \oplus K$, where $K \cap T(L) = 0$; and hence $K \cap L = 0$, so that by Lemma 1a, $K = 0$; and hence, $N = T(M)$.

b. The “only if” part is contained in part a. For the “if” part, assume $N \subset M$ is divisible and contains $L$. Then we may write $M = N \oplus K$. Then, by hypothesis, $K$ cannot have elements of prime order, and must therefore be 0.

c. Assume $L = \bigcap_{a \in A} M_a$, where each $M_a$ is divisible and contains $L$. Then each $T(M_a)$ is divisible and contains $T(L)$. Moreover, we have $\bigcap_{a \in A} T(M_a) = T(\bigcap_{a \in A} M_a) = T(L)$. Hence $T(L)$ is $\bigcap_{T(\alpha)}$ and hence $\bigcap_\sigma$.

**Lemma 3.** Let $G$ be a minimal divisible group containing the
subgroup $L$, and having a representation of the form $G = \bigoplus_{a \in A} G_a$. Then $G/L$ (which by Lemma 1b is divisible and torsion) contains a subgroup isomorphic to $C(p^\infty)$ if and only if for some $a \in A$, $G_a/G_a \cap L$ contains a subgroup of the same kind. In other words: $P(G/L) = \bigcup_{a \in A} P(G_a/G_a \cap L) = \bigcup_{a \in A} P(G_a + L/L)$.

**Proof.** Because of the divisibility of all the groups concerned, it suffices to check the existence of elements of order $p$. Suppose $x \in G_a/G_a \cap L$ has order $p$. Then $G/L \supset G_a + L/L \cong G_a/(G_a \cap L)$. Hence, $G/L$ has an element of order $p$. Conversely, suppose that for no $a \in A$ does $G/(G_a \cap L)$ contain an element of order $p$. Then no $G_a + L/L$ contains an element of order $p$. Hence, the subgroup of $G/L$ generated by the $(G_a + L)/L$ contains no elements of order $p$. But since the $G_a$'s generate $G$, the $G_a + L/L$'s generate all of $G/L$.

**Theorem 1.** Let $G$ be a divisible group; let $L$ be a subgroup of $G$; and let $M$ be a minimal divisible subgroup of $G$ containing $L$. Thus, we may write $G = M \oplus E$, then $L = M \cap (\bigcap_{\omega \in \Omega} M_\omega)$, where $M_\omega$ is a divisible subgroup of $G$ containing $L$ for each $\omega \in \Omega$, if and only if there exists homomorphisms $h_\omega : M \to E$ for each $\omega \in \Omega$ such that $\bigcap_{\omega \in \Omega} \ker h_\omega = L$.

**Proof.** To prove the "if" part, let $I$ be the identity map of $M$; and for each $\omega \in \Omega$ let $g_\omega : M \to G$ be defined by $g_\omega = I + h_\omega$. Let $M_\omega = g_\omega(M)$. Then $L \subset M_\omega$ since $h_\omega(L) = 0$, and therefore $M_\omega$ is divisible since it is a homomorphic image of $M$. Finally, $x \in M_\omega \cap M$ implies $x \in \ker h_\omega \quad (\text{since } x = y + h_\omega(y) \text{ implies } h_\omega(y) = x - y \in M \cap E = 0)$; and hence, $L \subset \bigcap_{\omega \in \Omega} M_\omega \cap M = \bigcap_{\omega \in \Omega} \ker h_\omega = L$.

To prove the "only if" part, suppose $L = M \cap (\bigcap_{\omega \in \Omega} M_\omega)$. It can be assumed that each $M_\omega$ is minimal divisible containing $L$ and, therefore, also minimal divisible containing $M \cap M_\omega$. Also, $M$ is minimal divisible containing $M \cap M_\omega$. Also, $M$ is minimal divisible containing $M \cap M_\omega$; so there is an isomorphism $i_\omega : M \to M_\omega$ which is the identity on $M \cap M_\omega$. Note that $i_\omega(x) \in M \Rightarrow i_\omega(i_\omega(x)) = i_\omega(x) \Rightarrow i_\omega(x) = x \Rightarrow x \in M \cap M_\omega$. Let $p : G \to E$ be the projection determined by the decomposition $G = M \oplus E$. Let $h_\omega$ be defined by $h_\omega = pi_\omega$. Then $h_\omega(x) = 0$, $i_\omega(x) \in M$, and $x \in M \cap M_\omega$ are equivalent. Thus $\bigcap_{\omega \in \Omega} \ker h_\omega = \bigcap_{\omega \in \Omega} M \cap M_\omega = L$.

**Remark.** The underlined portion of Theorem 1 may be replaced by $h_\omega : M \to T(E)$.

**Corollary 1.** Let $G$ be a divisible group; let $L$ be a subgroup of $G$; and let $M$ be a subgroup of $G$ which is minimal with respect to being divisible and containing $L$. Thus, we may write $G = M \oplus E = \bigoplus_{a \in A} G_a$. Then $GIL$ (which by Lemma 1b is divisible and torsion) contains a subgroup isomorphic to $C(p^\infty)$ if and only if for some $a \in A$, $G_a/G_a \cap L$ contains a subgroup of the same kind. In other words: $P(G/L) = \bigcup_{a \in A} P(G_a/G_a \cap L) = \bigcup_{a \in A} P(G_a + L/L)$.
Then $L$ is $\bigcap_\theta$ if and only if $P(M/L) \subseteq P(E)$.

**Proof.** The condition $P(M/L) \subseteq P(E)$ is easily seen to be equivalent to the existence of the family $\{h_\omega\}_{\omega \in \Omega}$ of homomorphisms in Theorem 1.

**Remark.** Let $G$ be divisible and torsion free, then $L \subseteq G$ is $\bigcap_\theta$ if and only if $L$ is divisible or, equivalently, is a direct summand of $G$.

**Corollary 2.** Let $G$ be divisible, and let $L$ be a torsion subgroup of $G$. Then $L$ is $\bigcap_\theta$ if and only if for each $p \in P$, $S(L_p) < S(G_p)$ whenever $S(L_p) \neq 0$, and $L_p$ is not divisible.

**Proof.** If $L$ is $\bigcap_\theta$ then by Lemma 2c for each $p \in P$ obviously $L_p$ is $\bigcap_{\theta_p}$; and, hence, to prove that our condition is necessary, we may assume that $G$ is primary, and $L$ is not divisible, in which case the necessity becomes obvious in view of the fact that otherwise $G$ would be the only minimal divisible subgroup of itself containing $L$; and, consequently, $L = G$, since $L$ is $\bigcap_\theta$, contrary to $L$ being not divisible.

To prove the "only if" part, note that $p \in P(M/L)$ implies $M_p/L_p = (M/L)_p \neq 0$, since $L$ is torsion. Thus, by hypothesis, $p \in P(M/L) \Rightarrow S(L_p) < S(G_p) \Rightarrow S(M_p) = S(M_p) \oplus S(E_p) = S(E_p) \neq 0 \Rightarrow p \in P(E)$.

**Corollary 3.** Let $G$ be divisible and $L \subseteq G$ be torsion, reduced and $\bigcap_\theta$. Then every subgroup of $L$ is $\bigcap_\theta$.

**Corollary 4.** Let $G$ be divisible and $L$ be $\bigcap_\theta$. Let $M$ be a minimal divisible subgroup of $G$ containing $L$. Let $K$ be a subgroup of $G$ such that $L \subseteq K \subseteq M$. Then $K$ is $\bigcap_\theta$.

**Proof.** If $G = M \oplus E$, then $P(M/K) \subseteq P(M/L) \subseteq P(E)$.

**Corollary 5.** Let $K$ be any Abelian group of arbitrary cardinal number $A$. Then $K$ can be embedded in a divisible group $G$ of power $A \mathcal{N}_0$ in such a way that any subgroup of $K$ can be represented as an intersection of two divisible subgroups of $G$.

**Proof.** Let $M$ be a minimal divisible group containing $K$ and let $E$ be a group isomorphic to $M/K$. Let $G$ be the direct sum of $M$ and $E$. The cardinality of $G$ is clearly $A \mathcal{N}_0$ and the isomorphism of $M/K$ into $E$ induces a homomorphism $h: M \rightarrow E$ with $\ker h = K$. Thus, Theorem 1 gives the required conclusion.

**Remark.** Let $L \subseteq G$, and let $L = L_1 \oplus L_2$, where $L_2$ is divisible and
$L_1$ is reduced. Then, also, $G = L_2 \oplus K$, where $K$ may be chosen to contain $L_1$. It is easy to see that $L$ is $\bigcap_\beta$ if and only if $L_1$ is $\bigcap_\kappa$. Thus, in order to avoid excessive wording, we may in the following theorems assume without loss of generality that $L$ is reduced.

**Theorem 2.** Assume $L \subset G$ is reduced, then $L$ is $\bigcap_\beta$ if and only if $T(L)$ is $\bigcap_\beta$ and $P(G/L) \subset P(G)$, equality holding if $L$ is $\bigcap_\beta$.

*Proof.* Let $G = M \oplus E$, where $M$ and $E$ are as in Theorem 1. Then $P(G) = P(T(M)) \cup P(E)$, and $P(G/L) = P(M/L) \cup P(E)$, because $G/L \cong (M/L) \oplus E$. Note that if $T(L)$ is $\bigcap_\beta$, then $T(L)$ is $\bigcap_{\tau_2}$ and hence $P(T(M)/T(L)) \subset P(T(E)) = P(E)$, by Corollary 1. But since $T(L)$ is reduced, $P(T(M)/T(L)) = P(T(M))$. Thus the assumption that $T(L)$ is $\bigcap_\beta$ implies $P(G) = P(E)$ and, therefore, that the conditions $P(G/L) = P(G)$, $P(G/L) \subset P(G)$, and $P(M/L) \subset P(E)$ are equivalent. This observation, together with Lemma 2c and Corollary 1, proves Theorem 2.

**Corollary 6.** Let $G$ be any divisible group. Let $C$ be as defined in N5, and let $\bar{G} = C \oplus G$. Then any subgroup $K \subset G$ is $\bigcap_{\bar{\beta}}$.

*Remark.* In Corollary 6, $C$ may be replaced by any Abelian group containing it.

**Corollary 7.** Any torsion free subgroup $T$ of $\bar{G}$ above is $\bigcap_{\bar{\beta}}$.

*Proof.* $T$ is contained in a direct summand of $\bar{G}$ whose complementary direct summand contains a subgroup isomorphic to $C$.

*Remark.* The following example shows that if $L \subset G$ is $\bigcap_\beta$ and if $\bar{L} \subset G$ is isomorphic to $L$, $\bar{L}$ need not be $\bigcap_\beta$. Let $G = \bigoplus_{i=1}^{\infty} C_i$ where $C_i \cong C(p^\infty)$ and where $p$ is fixed. Then,

$$S(\bigoplus_{i=1}^{\infty} C_i) \cong S(\bigoplus_{i=2}^{\infty} C_i);$$

however, $S(\bigoplus_{i=1}^{\infty} C_i)$ is not $\bigcap_\beta$, while $S(\bigoplus_{i=2}^{\infty} C_i)$ is $\bigcap_\beta$.

In this connection we have:

**Corollary 8.** Let $L \subset G$ be $\bigcap_\beta$, and let $\bar{L} \subset G$ be isomorphic to $L$. Then if $T(\bar{L})$ is $\bigcap_\beta$—this is in particular the case if $\bar{L} \subset L$—$\bar{L}$ is also $\bigcap_\beta$. Thus, $\bar{L}$ is $\bigcap_\beta$ if and only if $T(\bar{L})$ is $\bigcap_\beta$.

*Proof.* For the proof we may assume $L$ is reduced. By Theorem 2, it suffices to show that $P(G/\bar{L}) \subset P(G)$. Let $M$ and $\bar{M}$ be minimal divisible subgroup of $G$ containing $L$ and $\bar{L}$, respectively, so that $G = \bar{L}$.
We have \( M/L \cong \bar{M}/\bar{L} \) [see Kurosh I, page 168].

Thus, \( P(G/L) = P(\bar{M}/\bar{L}) \cup P(\bar{N}) \)
\[ = P(M/L) \cup P(N) \]
\[ \subset P(G/L) \cup P(G) \]
\[ \subset P(G) \], since \( L \) is \( \cap_{\alpha} \).

**Theorem 3.** Let \( G \) be a divisible group and let \( L \subset G \) be reduced. Then the following statements are equivalent:

(a) \( L \) is \( \cap_{\alpha} \).

(b) \( T(L) \) is \( \cap_{\alpha} \) and for any subgroup \( \bar{R} \) of \( G \) isomorphic to \( R \), either \( L \cap \bar{R} \) is zero or of type \( \geq T_{P(\alpha)} \).

(c) \( S(L_p) < S(G_p) \) if \( p \in P(G) \) and for any subgroup \( \bar{R} \) of \( G \) isomorphic to \( R \), either \( L \cap \bar{R} \) is zero or of type \( \geq T_{P(\alpha)} \).

(d) \( T(L) \) is \( \cap_{\alpha} \) and \( L \subset (\bigoplus_{\alpha \in A} R_{\alpha}) \oplus T(G) \subset G \), where \( R_{\alpha} \cong R \) and each \( L \cap R_{\alpha} \) is either zero or of \( \geq T_{P(\alpha)} \).

**Proof.** By Theorem 2, (a) is equivalent to the conditions \( T(L) \) is \( \cap_{\alpha} \) and \( P(G/L) \subset P(G) \). These conditions imply (b) since \( G/L \) contains a subgroup isomorphic to \( \bar{R}/\bar{R} \cap L \), so that \( P(\bar{R}/\bar{R} \cap L) \subset P(G/L) \subset P(G) \) and therefore \( \bar{R} \cap L \) is zero or of type \( \geq T_{P(\alpha)} \) by N8. Properties (b) and (c) are equivalent by Lemma 2c and Corollary 2. Also (b) implies (d). Finally, suppose (d) holds. Then, \( P(T(G)/(T(G) \cap L)) = P(T(G)/(T(L) \cap L)) = P(T(G)) = P(G) \) by Theorem 2. Let \( G = (\bigoplus_{\alpha \in A} R_{\alpha}) \oplus (\bigoplus_{\alpha \in A} R_{\alpha}) \oplus T(G) \). Then \( R_{\alpha} \cap L \) is 0 for all \( b \in B \), and for each \( a \in A \), \( R_{\alpha} \cap L \) is 0 or has type \( \geq T_{P(\alpha)} \) by hypothesis. Thus, by Lemma 3, \( P(G/L) = \bigcap_{b \in B} P(R_b/(R_b \cap L)) \cup \bigcup_{a \in A} P(R_a/(R_a \cap L)) \cup P(T(G)/(T(G) \cap L)) \subset P(G) \). By Theorem 2, this implies (a).

**Definition.** Define a subset \( (x_{\alpha})_{\alpha \in A} \) of elements of an Abelian group \( H \) to be independent if and only if

\[ n_1x_{\alpha_1} + n_2x_{\alpha_2} + \cdots + n_mx_{\alpha_m} = 0 \]

implies \( n_1 = n_2 = \cdots = n_m = 0 \) where each \( \alpha_i \in A \) and the \( n_i \)'s are integers.

**Corollary 10.** Assume \( L \subset G \) is reduced, then \( L \) is \( \cap_{\alpha} \) if and only if \( T(L) \) is \( \cap_{\alpha} \) and \( L \) contains a subgroup \( H \) which contains a maximal independent subset of \( L \) and which has the form \( \bigoplus_{\alpha \in A} S_{\alpha} \), where each \( S_{\alpha} \) is isomorphic to a subgroup of \( R \) having type \( T_{P(\alpha)} \).

**Proof.** Assume \( L \) is \( \cap_{\alpha} \), then by Lemma 2c \( T(L) \) is \( \cap_{\alpha} \). Let \( M \subset G \) be minimal divisible containing \( L \), and assume that \( M = T(M) \oplus F(M) = T(M) \oplus (\bigoplus_{\alpha \in A} R_{\alpha}) \), where \( R_{\alpha} \cong R \) for all \( \alpha \in A \). Then, by Theorem 3c and Lemma 1a, each \( L \cap R_{\alpha} \) contains a subgroup \( S_{\alpha} \) of
type $T_{\pi(G)}$. Then it is easy to see that $\bigoplus_{a \in A} S_a$ exists; and that, if $x_a \in S_a$, then $(x_a)_{a \in A}$ is a maximal independent subset of $M$ and therefore, of $L$.

Next assume the condition holds and let $M$ be as usual. For each $S_a$, let $R_a \supset S_a$ be a subgroup of $M$ of type $R$. Since any two non-zero subgroups of $R$ have a non-zero intersection, and since $\bigoplus_{a \in A} S_a$ exists, also $\bigoplus_{a \in A} R_a$ exists. Let $x_a \in S_a$; then $(x_a)_{a \in A}$ is a maximal independent subset of $H$; and since $H$ contains a maximal independent subset of $L$, $(x_a)_{a \in A}$ is also a maximal independent subset of $L$. Thus, we must have $M = T(M) \oplus (\bigoplus_{a \in A} R_a)$. Theorem 3d and the fact that $T(L)$ is $\bigcap_{\varrho}$ imply that $L$ is $\bigcap_{\varrho}$.

**Remark.** Concerning the last definition given above, it is well known that $H$ contains a maximal independent subset and that if $(x_a)_{a \in A}$ is independent, then $H/\{(x_a)_{a \in A}\}$ is torsion if and only if $(x_a)_{a \in A}$ is also maximal independent. Thus, Corollary 10 may be worded as follows: Assume $L$ is reduced; then $L$ is $\bigcap_{\varrho}$ if and only if $T(L)$ is $\bigcap_{\varrho}$ and $L$ contains a subgroup $H$ which has the form $\bigoplus_{a \in A} S_a$ where $S_a$ is isomorphic to a subgroup of $R$ of type $T_{\pi(G)}$ and such that $L/H$ is torsion.

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