

Pacific Journal of Mathematics

TRANSFORMATIONS OF SERIES OF *E*-FUNCTIONS

T. M. MACROBERT

TRANSFORMATIONS OF SERIES OF E -FUNCTIONS

T. M. MACROBERT

1. **Introductory.** The transformation [1, p. 25, 2, p. 369]

$$(1) \quad F\left(\alpha, \beta, \gamma, \delta, -l; \frac{1}{\alpha-\beta+1}, \frac{1}{\alpha-\gamma+1}, \frac{1}{\alpha-\delta+1}, \alpha+l+1\right) \\ = \frac{(\alpha+1; l) \left(\frac{1}{2}\alpha-\beta+1; l\right)}{\left(\frac{1}{2}\alpha+1; l\right) (\alpha-\beta+1; l)} F\left(\alpha-\gamma-\delta+1, \frac{1}{2}\alpha, \beta, -l; \frac{1}{\alpha-\gamma+1}, \frac{1}{\alpha-\delta+1}, \beta-\frac{1}{2}\alpha-l\right),$$

where l is a positive integer, is a special case of a formula of Whipple's. It, and other transformations of the same kind, can be employed to obtain transformations of series of E -functions. Two such transformations are:

$$(2) \quad \sum_{n=0}^l \frac{(\alpha; n)(\beta; n)(-l; n)}{n!(\alpha-\beta+l; n)(\alpha+l+1; n)} E\left\{ \frac{p; \alpha_r}{\Delta(m; \rho-n), \Delta(m; \sigma-n), \Delta(m; \alpha+\rho+n), \Delta(m; \alpha+\sigma+n), \rho_1, \rho_2, \dots, \rho_a} : z \right\} \\ = \frac{(\alpha+1; l) \left(\frac{1}{2}\alpha-\beta+1; l\right)}{\left(\frac{1}{2}\alpha+1; l\right) (\alpha-\beta+1; l)} \sum_{n=0}^l \frac{\left(\frac{1}{2}\alpha; n\right)(\beta; n)(-l; n)}{n! \left(\beta-\frac{1}{2}\alpha-l; n\right)} \left(\frac{2}{m}\right)^n \\ \times E\left\{ \frac{\Delta(2m; \alpha+\rho+\sigma+n-1), \alpha_1, \alpha_2, \dots, \alpha_p}{\Delta(2m; \alpha+\rho+\sigma-1), \Delta(m; \rho), \Delta(m; \sigma), \Delta(m; \alpha+\rho+n), \Delta(m; \alpha+\sigma+n), \rho_1, \rho_2, \dots, \rho_a} : z \right\}; \\ (3) \quad \sum_{n=0}^l \frac{(\alpha; n)(\beta; n)(-l; n)}{n!(\alpha-\beta+1; n)(\alpha+l+1; n)} E\left\{ \frac{\Delta(m; \gamma+n), \Delta(m; \gamma-\alpha-n), \alpha_1, \dots, \alpha_p}{\Delta(m; \sigma+n), \Delta(m; \sigma-\alpha-n), \rho_1, \dots, \rho_a} : z \right\} \\ = \frac{(\alpha+1; l) \left(\frac{1}{2}\alpha-\beta+1; l\right)}{\left(\frac{1}{2}\alpha+1; l\right) (\alpha-\beta+1; l)} \sum_{n=0}^l \frac{(\sigma-\gamma; n) \left(\frac{1}{2}\alpha; n\right)(\beta; n)(-l; n)}{n! \left(\beta-\frac{1}{2}\alpha-l; n\right) (-m^2)^n} \\ \times E\left\{ \frac{\Delta(m; \gamma), \Delta(m; \gamma-\alpha-n), \alpha_1, \dots, \alpha_p}{\Delta(m; \sigma-\alpha), \Delta(m; \sigma+n), \rho_1, \dots, \rho_a} : z \right\}.$$

In these formulae m is a positive integer,

$$(4) \quad (\alpha; 0) = 1, (\alpha; m) = \alpha(\alpha+1) \cdots (\alpha+m-1),$$

and $\Delta(m; \alpha)$ denotes the set of parameters

$$\frac{\alpha}{m}, \frac{\alpha+1}{m}, \dots, \frac{\alpha+m-1}{m}.$$

The proofs of (2) and (3) are given in § 2. The following formulae are required.

If m is a positive integer

$$(5) \quad \Gamma(mz) = (2\pi)^{[(1/2)-(1/2)m]} m^{[mz-(1/2)]} \Gamma(z) \Gamma\left(z + \frac{1}{m}\right) \dots \Gamma\left(z + \frac{m-1}{m}\right).$$

From this it follows that, if m and n are positive integers,

$$(6) \quad \Gamma\left(\frac{\alpha+n}{m}\right) \Gamma\left(\frac{\alpha+1+n}{m}\right) \dots \Gamma\left(\frac{\alpha+m-1+n}{m}\right) \\ = \Gamma\left(\frac{\alpha}{m}\right) \Gamma\left(\frac{\alpha+1}{m}\right) \dots \Gamma\left(\frac{\alpha+m-1}{m}\right) m^{-n}(\alpha; n),$$

and

$$(7) \quad \Gamma\left(\frac{\alpha-n}{m}\right) \Gamma\left(\frac{\alpha+1-n}{m}\right) \dots \Gamma\left(\frac{\alpha+m-1-n}{m}\right) \\ = \Gamma\left(\frac{\alpha}{m}\right) \Gamma\left(\frac{\alpha+1}{m}\right) \dots \Gamma\left(\frac{\alpha+m-1}{m}\right) (-m)^n / (1-\alpha; n).$$

The Barnes' integral for the E -function is [2, p. 374]

$$(8) \quad E(p; \alpha_r; q; \rho_s; z) = \frac{1}{2\pi i} \int \frac{\Gamma(\zeta) \prod \Gamma(\alpha_r - \zeta)}{\prod \Gamma(\rho_s - \zeta)} z^\zeta d\zeta$$

where $|\text{amp } z| < \pi$ and the integral is taken up the η -axis with loops, if necessary, to ensure that the origin lies to the left of the contour and the points $\alpha_1, \alpha_2, \dots, \alpha_p$ to the right of the contour. Zero and negative integral values of the parameters are excluded, and the α 's must not differ by integral values. When $p < q + 1$ the contour is bent to the left at both ends. When $p > q + 1$ the formula is valid for $|\text{amp } z| < \frac{1}{2}(p - q + 1)\pi$.

2. Proofs of the transformations. Using (8), (6) and (7), the left-hand side of (2), with $p = q = 0$, can be written

$$\sum_{n=0}^l \frac{(\alpha; n)(\beta; n)(-l; n)}{n!(\alpha - \beta + 1; n)(\alpha + l + 1; n)} \\ \times \frac{1}{2\pi i} \int \frac{\Gamma(\zeta) z^\zeta (1 - \rho + m\zeta; n)(1 - \sigma + m\zeta; n) d\zeta}{\prod_{\rho, \sigma} \left[\prod_{u=0}^{m-1} \left\{ \Gamma\left(\frac{\rho+u}{m} - \zeta\right) \Gamma\left(\frac{\alpha+\rho+u}{m} - \zeta\right) \right\} (\alpha + \rho - m\zeta; n) \right]} \\ = \frac{1}{2\pi i} \int \frac{\Gamma(\zeta) z^\zeta}{\prod_{\rho, \sigma} \left[\prod_{u=0}^{m-1} \left\{ \Gamma\left(\frac{\rho+u}{m} - \zeta\right) \Gamma\left(\frac{\alpha+\rho+u}{m} - \zeta\right) \right\} \right]} Id\zeta,$$

where

$$I = F\left(\alpha, \beta, 1 - \rho + m\zeta, 1 - \sigma + m\zeta, -l; 1 \right. \\ \left. \alpha - \beta + 1, \alpha + \rho - m\zeta, \alpha + \sigma - m\zeta, \alpha + l + 1\right),$$

From (1) it follows that this can be put in the form

$$\frac{(\alpha + 1; l)\left(\frac{1}{2}\alpha - \beta + 1; l\right)}{\left(\frac{1}{2}\alpha + 1; l\right)(\alpha - \beta + 1; 1)} \\ \times \frac{1}{2\pi i} \int \frac{\Gamma(\zeta)z^\zeta}{\prod_{\rho, \sigma} \left[\prod_{u=0}^{m-1} \left\{ \Gamma\left(\frac{\rho + u}{m} - \zeta\right) \Gamma\left(\frac{\alpha + \rho + u}{m} - \zeta\right) \right\} \right]} J d\zeta,$$

where

$$J = F\left(\alpha + \rho + \sigma - 1 - 2m\zeta, \frac{1}{2}\alpha, \beta, -l; 1 \right. \\ \left. \alpha + \rho - m\zeta, \alpha + \sigma - m\zeta, \beta - \frac{1}{2}\alpha - l\right).$$

Now the integral is equal to

$$\sum_{n=0}^l \frac{\left(\frac{1}{2}\alpha; n\right)(\beta; n)(-l; n)}{n!\left(\beta - \frac{1}{2}\alpha - l; n\right)2\pi i} \\ \times \int \frac{\Gamma(\zeta)z^\zeta \Gamma(\alpha + \rho + \sigma - 1 + n - 2m\zeta)}{\prod_{\rho, \sigma} \left[\sum_{u=0}^{m-1} \left\{ \Gamma\left(\frac{\rho + u}{m} - \zeta\right) \Gamma\left(\frac{\alpha + \rho + u}{m} - \zeta\right) \right\} \right]} \\ \left/ \frac{\Gamma(\alpha + \rho - m\zeta)\Gamma(\alpha + \sigma - m\zeta)d\zeta}{\Gamma(\alpha + \rho + n - m\zeta)} \right] \Gamma(\alpha + \rho + \sigma - 1 - 2m\zeta),$$

and, on applying (5) to the gamma functions whose arguments contain $-2m\zeta$ or $-m\zeta$, the right-hand side of (2) with $p = q = 0$ is obtained. Formula (2) can then be derived by generalising.

Formula (3) can be proved in the same way. It should be noted that

$$(\alpha - \gamma + 1 + m\zeta; n) = (-1)^n(\gamma - \alpha - n - m\zeta; n).$$

The restrictions on $amp z$ and on the parameters can be removed by analytical continuation, provided that the functions exist.

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