

# Pacific Journal of Mathematics

**AN INEQUALITY FOR LOGARITHMIC CAPACITIES**

HEINZ RENGGLI

# AN INEQUALITY FOR LOGARITHMIC CAPACITIES

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**1. Introduction.** In his work on capacities, G. Choquet proved that for many capacities the inequality of strong subadditivity holds [1]. It is the purpose of this note to show that a similar inequality holds for logarithmic capacities. More precisely we shall prove the

**THEOREM.** *Let  $A$  and  $B$  be compact sets in the complex  $z$ -plane  $E$ . By  $C(S)$  we denote the logarithmic capacity [2] of a given compact set  $S$ ,  $S \subset E$ , where we agree to put  $C(S) = 0$  whenever  $S = \phi$ . Then*

$$C(A \cup B) \cdot C(A \cap B) \leq C(A) \cdot C(B).$$

**2. Proof of the theorem.** Let  $S, S \subset E$ , be a compact set whose boundary consists of a finite number of analytic arcs. By  $S^*$  we denote that component of  $E - S$  which is unbounded. Then Green's function of  $S^*$  is defined by the properties: it is harmonic in  $S^*$ , vanishes at the finite boundary points of  $S^*$  and has a logarithmic singularity at infinity. We will denote this function by  $g_s(z, \infty)$ .

First we shall deal with the case when the respective boundaries of  $A, B$  and  $A \cap B$  consist of a finite number of non-degenerate analytic arcs. We remark that the difference  $g_{A \cap B}(z, \infty) - g_A(z, \infty)$  is harmonic in  $A^*$ ,  $A^* \subset (A \cap B)^*$ , and at infinity. It is furthermore non-negative on the boundary of  $A^*$  and hence non-negative in  $A^*$  by the maximum principle. Similarly  $g_{A \cup B}(z, \infty) \geq g_B(z, \infty)$  holds in  $B^*$ ,  $B^* \subset (A \cap B)^*$ .

The function

$$h(z) = g_{A \cup B}(z, \infty) + g_{A \cap B}(z, \infty) - g_A(z, \infty) - g_B(z, \infty)$$

is harmonic in  $(A \cup B)^*$  and at infinity. From  $(A \cup B)^* = A^* \cap B^*$  it follows that the boundary points of  $(A \cup B)^*$  belong either to the boundary of  $A^*$  or to the boundary of  $B^*$ . Therefore  $g_{A \cup B}(z, \infty)$  and either  $g_A(z, \infty)$  or  $g_B(z, \infty)$  vanish at these boundary points. With the aid of the remark made above we get the result that  $h(z)$  is non-negative in  $(A \cup B)^*$ .

Therefore

$$g_A(z, \infty) + g_B(z, \infty) \leq g_{A \cup B}(z, \infty) + g_{A \cap B}(z, \infty)$$

holds in  $(A \cup B)^*$ . From this general inequality and using the fact that

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$$\lim_{g \rightarrow \infty} \{g_s(z, \infty) - \log |z|\}$$

is the constant  $\gamma(S)$  of Robin [2] we deduce

$$\gamma(A) + \gamma(B) \leq \gamma(A \cup B) + \gamma(A \cap B).$$

But

$$C(S) = \exp \{-\gamma(S)\}$$

by definition. Hence our theorem is proven for the special case.

The general case follows by the usual approximation techniques [2].

#### REFERENCES

1. G. Choquet, *Theory of capacities*, Ann. Inst. Fourier, Grenoble **5** (1953-54), 131-295.
2. R. Nevanlinna, *Eindeutige analytische Funktionen*, Grundlehren 46, 2. Aufl., Springer, 1953.

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