AN INEQUALITY FOR LOGARITHMIC CAPACITIES

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1. Introduction. In his work on capacities, G. Choquet proved that for many capacities the inequality of strong subadditivity holds [1]. It is the purpose of this note to show that a similar inequality holds for logarithmic capacities. More precisely we shall prove the

**Theorem.** Let $A$ and $B$ be compact sets in the complex $z$-plane $E$. By $C(S)$ we denote the logarithmic capacity [2] of a given compact set $S$, $S \subset E$, where we agree to put $C(S) = 0$ whenever $S = \emptyset$. Then

$$C(A \cup B) \cdot C(A \cap B) \leq C(A) \cdot C(B).$$

2. Proof of the theorem. Let $S$, $S \subset E$, be a compact set whose boundary consists of a finite number of analytic arcs. By $S^*$ we denote that component of $E - S$ which is unbounded. Then Green’s function of $S^*$ is defined by the properties: it is harmonic in $S^*$, vanishes at the finite boundary points of $S^*$ and has a logarithmic singularity at infinity. We will denote this function by $g_{S}(z, \infty)$.

First we shall deal with the case when the respective boundaries of $A$, $B$ and $A \cap B$ consist of a finite number of non-degenerate analytic arcs. We remark that the difference $g_{A \cap B}(z, \infty) - g_{A}(z, \infty)$ is harmonic in $A^*$, $A^* \subset (A \cap B)^*$, and at infinity. It is furthermore non-negative on the boundary of $A^*$ and hence non-negative in $A^*$ by the maximum principle. Similarly $g_{A \cup B}(z, \infty) \geq g_{B}(z, \infty)$ holds in $B^*$, $B^* \subset (A \cap B)^*$.

The function

$$h(z) = g_{A \cup B}(z, \infty) + g_{A \cap B}(z, \infty) - g_{A}(z, \infty) - g_{B}(z, \infty)$$

is harmonic in $(A \cup B)^*$ and at infinity. From $(A \cup B)^* = A^* \cap B^*$ it follows that the boundary points of $(A \cup B)^*$ belong either to the boundary of $A^*$ or to the boundary of $B^*$. Therefore $g_{A \cup B}(z, \infty)$ and either $g_{A}(z, \infty)$ or $g_{B}(z, \infty)$ vanish at these boundary points. With the aid of the remark made above we get the result that $h(z)$ is non-negative in $(A \cup B)^*$.

Therefore

$$g_{A}(z, \infty) + g_{B}(z, \infty) \leq g_{A \cup B}(z, \infty) + g_{A \cap B}(z, \infty)$$

holds in $(A \cup B)^*$. From this general inequality and using the fact that

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\[
\lim_{z \to \infty} \{g_s(z, \infty) - \log |z|\}
\]
is the constant \(\gamma(S)\) of Robin [2] we deduce
\[
\gamma(A) + \gamma(B) \leq \gamma(A \cup B) + \gamma(A \cap B).
\]
But
\[
C(S) = \exp\{-\gamma(S)\}
\]
by definition. Hence our theorem is proven for the special case.

The general case follows by the usual approximation techniques [2].

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