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**ON THE NILPOTENCY CLASS OF A GROUP OF EXPONENT
FOUR**

CHARLES R. B. WRIGHT

ON THE NILPOTENCY CLASS OF A GROUP OF EXPONENT FOUR

C. R. B. WRIGHT

Introduction. If G is a multiplicative group with elements x, y, \dots , we define the commutator (x, y) by $(x, y) = x^{-1}y^{-1}xy$ and, inductively for length n , $(x_1, \dots, x_{n-1}, x_n) = ((x_1, \dots, x_{n-1}), x_n)$. We also use the notation $(x, \dots, y; \dots; z, \dots, w)$ for the commutator $((x, \dots, y), \dots, (z, \dots, w))$. For each positive integer n , let G_n be the subgroup of G generated by all commutators of length n .

A group, G , is of exponent 4 in case $x^4 = 1$ for every x in G but $y^2 \neq 1$ for some y in G . Let F be a free group of rank k , and let F^4 be the subgroup generated by fourth powers of elements of F . Let $B(k) = F/F^4$. Then $B(k)$ is clearly a group of exponent 4 on k generators. Moreover, every group of exponent 4 on k generators is a homomorphic image of $B(k)$.

I. N. Sanov has shown that $B(k)$ is finite. (See [2], pp. 324-325, or [3]). Unfortunately, his proof gives very little additional information about $B(k)$. The present paper is devoted to the study of relations between commutators in the group $B(k)$, a consequence of the relations obtained being that $B(k)_{3k} = 1$.

Preliminaries. Let G be a group of exponent 4, and let a, b, \dots be elements of G . Then

- (1) $(a, b)^2 \equiv (a, b, b, b)(a, b, b, a)(a, b, a, a) \pmod{G_4}$
- (2) $(a, b, a)^2 = (a, b, a, a, a) = (a, b, a; a, b)$
- (3) $(a, b, c) \equiv (b, c, a)(c, a, b) \pmod{G_4}$
- (4) $(a, b; c, d) \equiv (a, c; b, d)(a, d; b, c) \pmod{G_5}$
- (5) $(\mathbf{a}, b; \mathbf{c}, d; f) \equiv (\mathbf{a}, d; \mathbf{c}, f; b)(\mathbf{a}, f; \mathbf{c}, b; d) \pmod{G_6}$

where the bold-face type in (5) has no significance other than to point out which entries are left fixed while the others are cyclicly permuted—whenever bold-face type appears in a computation an application of (5) is about to be made. The relations (1) and (2) can be shown to hold in $B(2)$; hence they certainly hold in any group, G , of exponent 4. Relation (3) is simply the Jacobi identity (which holds in any group) adapted to exponent 4. Relations (4) and (5) were proved in [4] for the case in which the entries are of order 2, but the proofs clearly go through without this restriction, since in proving the relations we are simply

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looking at the first significant terms of $(abcd)^4$ and $(abcdf)^4$ as collected by P. Hall's process. It should be noted that these relations are "identical" in the sense that they hold for every choice of a, b, c, d and f in G . This property gives us the freedom of substitution which we shall use later.

The following result, which appeared in a slightly different form as the Corollary to Lemma 3.2 in [4], is easily proved using (1) and (3).

(A). *Let G be a group of exponent 4. Let*

$$C = (x_1, \dots, x_i, a, x_{i+1}, \dots, x_{n-1})$$

where x_1, \dots, x_{n-1} and a are in G . Then, modulo G_{n+1} , C is a product of commutators of the form $(a, y_1, \dots, y_i, x_{i+1}, \dots, x_{n-1})$, where y_1, \dots, y_i are x_1, \dots, x_i in some order.

Finally, we need to know that if a and b are the generators of $B(2)$, then $B(2)_6$ is generated by $(b, a, a; b, a)$ and $(b, a, b; b, a)$, and $B(2)_6 = 1$. These results may be verified directly or deduced from Burnside's original work in [1].

Throughout this paper we shall be concerned with the relations between commutators in $B(k)$. Our first lemma gives us a method of reducing our problems to a few relatively tractable cases.

LEMMA 1. *Suppose (x_1, \dots, x_n) is a commutator of length n in a group, G , of exponent 4. If one of x_3, \dots, x_n is a and one b , then, modulo G_{n+1} , (x_1, \dots, x_n) is a product of commutators of length n of the following four types:*

- (i) $(x, y, \dots, a, b, \dots)$
- (ii) $(x, y, \dots, b, a, \dots)$
- (iii) (x, y, \dots, a, z, b)
- (iv) $(x, y, \dots, b, z, a).$

Loosely stated, Lemma 1 says that we may bring a and b more or less together and keep them out of the first two positions.

Proof of Lemma 1. Observe first that we can rewrite (3) as

$$(a, b, c) \equiv (a, c, b)(a; b, c) \pmod{G_4}.$$

Using this form and working modulo G_7 we have

$$\begin{aligned} (x, y, a, z, b, w) &\equiv (x, y, a, b, z, w)(x, y, a; b, z; w) \\ &\equiv (x, y, a, b, z, w)(x, y, z; b, w; a)(x, y, w; b, a; z) \\ &\equiv (x, y, a, b, z, w)(x, y, z, b, w, a)(x, y, z, w, b, a) \\ &\quad \cdot (x, y, w, b, a, z)(x, y, w, a, b, z). \end{aligned}$$

Let $G(n, a, b)$ be the (normal) subgroup of G generated by G_{n+1} and all commutators of length n of types (i) and (ii). Let $G^*(n, a, b)$ be the (normal) subgroup of G generated by $G(n, a, b)$ and all commutators of length n of types (iii) and (iv). Then certainly if w is in $G(n, a, b)$ and g is in G , (w, g) is in $G(n + 1, a, b)$, and by the relation just proved, if z is in $G^*(n, a, b)$, then (z, g) is in $G^*(n + 1, a, b)$. Thus it will be sufficient to prove the lemma under the assumption that x_n is either a or b (say b).

We have reduced the problem to showing that if C has length n and if $C = (x_1, x_2, \dots, x_i, a, \dots, b)$, then C is in $G^*(n, a, b)$. If $2 \leq n - i \leq 3$, then C is in $G^*(n, a, b)$. We proceed by induction on $n - i$. Suppose for induction that for some $j \geq 4$ and all $n \geq j + 2$, C is in $G^*(n, a, b)$ whenever $n - i < j$. We shall show that if $n - i = j$, then C is in $G^*(n, a, b)$, so that by finite induction we shall have C in $G^*(n, a, b)$ for all i such that $2 \leq n - i \leq n - 2$, i.e., such that $2 \leq i \leq n - 2$. Thus the lemma will be proved.

Let $i = n - j$. By the inductive assumption and (3) we have, modulo $G^*(n, a, b)$,

$$(x_1, x_2, \dots, x_i, a, \dots, x_{n-3}, x_{n-2}, b) \equiv (X, x_i; A, x_{n-3}; x_{n-2}; b),$$

where $X = (x_1, \dots, x_i)$, and where $A = (a, \dots, x_{n-4})$ if $n - 4 > i$ but $A = a$ if $n - 4 = i$. Now, modulo G_{n+1} , using (4), (3) and (5),

$$\begin{aligned} (X, x_i; A, x_{n-3}; x_{n-2}; b) &\equiv (X, x_{n-3}; A, x_i; x_{n-2}; b)(x_i, x_{n-3}; A, X; x_{n-2}; b) \\ &\equiv (X, x_{n-3}, x_{n-2}; A, x_i; b)(A, \mathbf{x}_i, x_{n-2}; X, \mathbf{x}_{n-3}; b) \\ &\quad \cdot (A, X, x_{n-2}; x_i, \mathbf{x}_{n-3}; b)(A, X; \mathbf{x}_i, \mathbf{x}_{n-3}, x_{n-2}; b) \\ &\equiv (X, x_{n-3}, x_{n-2}; A, x_i; b)(A, x_i, X; b, x_{n-3}; x_{n-2})(A, x_i, b; x_{n-2}, x_{n-3}; X) \\ &\quad \cdot (A, X, x_i; b, x_{n-3}; x_{n-2})(A, X, b; x_{n-2}, x_{n-3}; x_i) \\ &\quad \cdot (A, x_{n-2}; x_i, x_{n-3}, b; X)(A, b; x_i, x_{n-3}, X; x_{n-2}). \end{aligned}$$

But by the inductive assumption $(X, x_{n-3}, x_{n-2}; A, x_i; b)$, $(A, x_i, b; x_{n-2}, x_{n-3}; X)$, $(x_i, x_{n-3}, b; A, x_{n-2}; X)$ and $(A, b; x_{n-3}, x_i, X; x_{n-2})$ are all in $G^*(n, a, b)$. Further,

$$\begin{aligned} (A, x_i, X; b, x_{n-3}; x_{n-2})(A, X, x_i; b, x_{n-3}; x_{n-2}) \\ \equiv (X, x_i, A; b, x_{n-3}; x_{n-2}) \pmod{G_{n+1}}. \end{aligned}$$

Thus, modulo $G^*(n, a, b)$,

$$\begin{aligned} (X, x_i; A, x_{n-3}; x_{n-2}; b) \\ \equiv (X, \mathbf{x}_i, A; \mathbf{b}, x_{n-3}; x_{n-2})(A, X, b; x_{n-2}, x_{n-3}; x_i) \\ \equiv (X, x_i, x_{n-3}; b, x_{n-2}; A)(X, x_i, x_{n-2}; b, A; x_{n-3}) \\ \quad \cdot (A, X; x_{n-2}, x_{n-3}, b; x_i)(x_{n-2}, x_{n-3}; A, X; b; x_i) \end{aligned}$$

$$\begin{aligned}
 &\equiv (A, X; \mathbf{x}_{n-2}, \mathbf{x}_{n-3}, b; x_i)(\mathbf{x}_{n-2}, x_{n-3}; A, X; b; x_i) \\
 &\equiv (A, b; x_{n-2}, x_{n-3}, x_i; X)(A, x_i; x_{n-2}, x_{n-3}, X; b) \\
 &\quad \cdot (x_{n-2}, X; A, b; x_{n-3}; x_i)(x_{n-2}, b; A, x_{n-3}; X; x_i) \\
 &\equiv (x_{n-2}, b; A, x_{n-3}; X; x_i) \\
 &\equiv (\mathbf{x}_{n-2}, b; A, x_{n-3}; x_i; X)(\mathbf{x}_{n-2}, b; A, \mathbf{x}_{n-3}; X, x_i) \\
 &\equiv (x_{n-2}, x_{n-3}; A, x_i; b; X)(x_{n-2}, x_i; A, b; x_{n-3}; X) \\
 &\quad \cdot (x_{n-2}, A; X, x_i, x_{n-3}; b)(X, x_i, x_{n-2}; b, x_{n-3}; A) \\
 &\equiv 1 .
 \end{aligned}$$

Hence, $(x_1, x_2, \dots, x_i, a, \dots, x_{n-3}, x_{n-2}, b)$ is in $G^*(n, a, b)$, as desired. Thus the lemma is proved.

An immediate consequence of Lemma 1 is the following.

COROLLARY. *If $C = (x_1, \dots, x_n)$ and if two of x_3, \dots, x_n are a , then modulo G_{n+1} , C is a product of commutators of length n of the forms:*

- (i) $(x, y, \dots, a, a, \dots)$
- (ii) (x, y, \dots, a, z, a) .

We next observe that, using (1),

$$\begin{aligned}
 (x_1, \dots, x_m, a^2) &= (x_1, \dots, x_m, a)^2(x_1, \dots, x_m, a, a) \\
 &\equiv (x_1, \dots, x_m, a, a) \pmod{G_{m+3}} .
 \end{aligned}$$

Hence,

$$(6) \quad (x_1, \dots, x_i, a, a, x_{i+1}, \dots, x_n) \equiv (x_1, \dots, x_i, a^2, x_{i+1}, \dots, x_n)$$

modulo G_{n+3} .

We may now prove the following useful result.

LEMMA 2. *Let G be a group of exponent 4, and let (x_1, \dots, x_n) be a commutator of length n in elements of G . If some three of x_3, \dots, x_n are a , then modulo G_{n+1} , (x_1, \dots, x_n) is a product of commutators of the forms:*

- (i) $(y_1, y_2, \dots, y_{n-3}, a, a, a)$
- (ii) $(y_1, y_2, \dots, y_{n-4}, a, a, y_{n-3}, a)$.

Proof. We first derive two shifting relations. Using (1) and (3) we obtain modulo G_7 ,

$$\begin{aligned}
 (x, y, a, a, a, z) &\equiv ((x, y, a)^2, z) \equiv (x, y, a, z)^2 \equiv (x, y; a, z)^2(x, y, z, a)^2 \\
 &\equiv (x, y, z, a)^2 \equiv (x, y, z, a, a, a) .
 \end{aligned}$$

Hence,

$$(7) \quad (x, y, a, a, a, z) \equiv (x, y, z, a, a, a) \pmod{G_7}.$$

Thus, modulo longer commutators, a string of three a 's can be shifted to the right.

We also have, modulo G_7 ,

$$(x, y, a, a, z, a) \equiv (x, y, a, z, a, a) \cdot (x, y, a; z, a; a) \equiv (x, y, a, z, a, a).$$

Thus

$$(8) \quad (x, y, a, z, a, a) \equiv (x, y, a, a, z, a) \pmod{G_7}.$$

Further, modulo G_8 ,

$$\begin{aligned} (x, y, a, a, z, a, w) &\equiv (x, y, a, a, a, z, w)(x, y, a, a; a, z; w) \\ &\equiv (x, y, a, a, a, z, w)(x, y, a^2; a, z; w) \\ &\equiv (x, y, a, a, a, z, w)(x, y, z; a, w; a^2) \\ &\equiv (x, y, a, a, a, z, w)(x, y, z, a, w, a, a)(x, y, z, w, a, a, a). \end{aligned}$$

Applying (7) and (8) we get

$$(9) \quad (x, y, a, a, z, a, w) \equiv (x, y, z, a, a, w, a) \pmod{G_8}.$$

Thus, modulo longer commutators, a trio of a 's with one gap may be shifted to the right.

It is clear from (7) and (9) that it is sufficient to prove the lemma under the assumption that $x_n = a$. Considering (x_1, \dots, x_{n-1}) now, it is clear from the Corollary of Lemma 1 that we may restrict ourselves to the consideration of commutators of the following two types:

I $(x_1, x_2, \dots, a, a, \dots, x_{n-1}, a)$

II $(x_1, x_2, \dots, a, x_{n-1}, a, a)$.

By (8), commutators of type II are already of type (ii). Further,

$$(x_1, x_2, \dots, a, a, \dots, x_{n-1}, a) \equiv (x_1, x_2, \dots, a^2, \dots, x_{n-1}, a) \pmod{G_{n+1}}.$$

Now applying Lemma 1 with b replaced by a^2 we find that modulo G_{n+1} , $(x_1, x_2, \dots, a^2, \dots, x_{n-1}, a)$ is a product of commutators of form $(y_1, y_2, \dots, a, a, a, \dots)$ and commutators of form $(y_1, y_2, \dots, a, a, y_{n-1}, a)$. Thus, by (7), any commutator of type I is a product to commutators of types (i) and (ii) modulo G_{n+1} . The lemma follows.

The main theorems.

In this section we derive more consequences of Lemma 1 and find an upper bound on the nilpotency class of $B(k)$. The first theorem is much like Lemma 2.

THEOREM 1. *Let G be a group of exponent 4, and suppose (x_1, \dots, x_n)*

is a commutator of length n with entries from G such that some four (or more) of x_1, \dots, x_n are a . If $n \geq 6$, then (x_1, \dots, x_n) is in G_{n+1} .

Proof. If $(x_1, \dots, x_n) \neq 1$, then since four entries of (x_1, \dots, x_n) are a , it follows that at least three of x_3, \dots, x_n are a . By Lemma 2 and (A) we may restrict attention to commutators of the following types:

$$(i) \quad (a, x_2, \dots, x_{n-3}, a, a, a)$$

$$(ii) \quad (a, x_2, \dots, a, a, x_{n-3}, a).$$

Applying (7) and (9), we may confine our study to commutators of the following types:

$$(i') \quad (a, x_2, a, a, a, x_3, \dots, x_{n-3})$$

$$(ii') \quad (a, x_2, a, a, x_3, a, \dots).$$

But now, modulo G_7 , using (2) and (5),

$$(a, x, a, a, a, y) \equiv (a, x, a; a, x; y) \equiv (a^3, \mathbf{x}; \mathbf{a}, x; y) \equiv 1,$$

and

$$\begin{aligned} (a, x, a, a, y, a) &\equiv (a, x, a^2, y, a) \equiv (a, x, y, a^2, a)(\mathbf{a}, x; \mathbf{a}^2, y; a) \\ &\equiv (a, x, y, a, a, a) \equiv (a, x, a, a, a, y) = 1. \end{aligned}$$

Thus a commutator of type (i') or (ii') is in G_{n+1} . The theorem follows.

Let x_1, \dots, x_k be generators of $B(k)$. Then it is easy to show that x_1, \dots, x_{k-1} generate a group isomorphic to $B(k-1)$. We may thus consider $B(k-1)$ as imbedded in $B(k)$.

If A and B are subgroups of a group, G , we define (A, B) as the subgroup of G generated by all commutators (a, b) with a in A and b in B .

THEOREM 2. For each positive integer k ,

$$(B(k)_{3k-1}, B(k+1)) \subseteq B(k+1)_{3k+1}.$$

Proof. We first show that the theorem holds for $k=2$, then we proceed by induction on k . Thus suppose first that $k=2$. Now as noted above, $B(2)_5$ is generated by $(x_1, x_2, x_1; x_1, x_2)$ and $(x_2, x_1, x_2; x_2, x_1)$. But if y is in $B(3)$, then modulo $B(3)_7$,

$$(x_1, x_2, x_1; x_1, x_2; y) = (x_1^2, \mathbf{x}_2; \mathbf{x}_1, x_2; y) \equiv 1.$$

Similarly, $(x_2, x_1, x_2; x_1, x_2; y) \equiv 1$ modulo $B(3)_7$. Thus the theorem is true if $k=2$.

Now suppose inductively that for some n the theorem is true for all k such that $2 \leq k < n$. We shall show that

$$(B(n)_{3n-1}, B(n + 1)) \subseteq B(n + 1)_{3n+1} .$$

It will be sufficient to show that if y_1, \dots, y_{3n-1} are chosen in any way from x_1, \dots, x_n and if z is in $B(n + 1)$, then $(y_1, \dots, y_{3n-1}, z)$ is in $B(n + 1)_{3n+1}$. Now if four of y_1, \dots, y_{3n-1} are equal, then by Theorem 2 $(y_1, \dots, y_{3n-1}, z)$ is in $B(n + 1)_{3n+1}$. Thus suppose the contrary, i.e., suppose that each of (say) x_2, \dots, x_n appears three times among y_1, \dots, y_{3n-1} and that x_1 appears twice. By (A) we may restrict attention to the case in which $y_1 = x_1$. But in this case, since $n \geq 3$, we must have at least one (say x_n) of x_2, \dots, x_n appearing three times among y_3, \dots, y_n , so that by Lemma 2 we may restrict ourselves to consideration of commutators of the following types:

- (i) $(y_1, y_2, \dots, y_{3n-4}, x_n, x_n, x_n, z)$
- (ii) $(y_1, y_2, \dots, x_n, x_n, y_{3n-4}, x_n, z)$,

where x_1 appears twice among y_1, \dots, y_{3n-4} and each of x_2, \dots, x_{n-1} appears three times. Now by (9),

$$(y_1, y_2, \dots, x_n, x_n, y_{3n-4}, x_n, z) \equiv (y_1, \dots, y_{3n-4}, x_n, x_n, z, x_n)$$

modulo $B(n + 1)_{3n+1}$. But (y_1, \dots, y_{3n-4}) is in $B(n - 1)_{3(n-1)-1}$ so that, by the inductive assumption, a commutator of type (i) or type (ii) is in $B(n + 1)_{3n+1}$. The theorem follows.

Finally, we have the principal goal of this paper.

THEOREM 3. *For each positive integer k , $B(k)_{3k} = 1$.*

Proof. It follows immediately from Theorem 2 that $B(k)_{3k} = B(k)_{3k+1}$ so that, since $B(k)$ is nilpotent, $B(k)_{3k} = 1$.

One may apply the foregoing results to obtain numerical estimates of the derived length and order of $B(k)$. It follows immediately from Theorem 3 that if $B(k)^{(m)} \neq 1$, then $2^m < 3k$, so that the derived length of $B(k)$ is at most $\log_2(3k - 1)$. By means of the Witt formulae (see, for example, [2], p. 169) one can also obtain an upper bound on the order of $B(k)$ using Theorems 2 and 3. Such estimates, both of derived length and order, are easily seen to be imprecise. For example, the Witt formula calculations give the order of $B(3)$ as at most 2^{484} , whereas a little direct computation shows that the order is at most 2^{70} . Also, $\log_2(3 \cdot 3 - 1) = 3$, but one can show that $B(3)''' = 1$.

Finally we would like to point out that it can be shown that $B(k)_k \neq 1$, so that perhaps the upper bound on the class given here is not too far from the true class. Indeed, the bound is precise for $k = 2$, and preliminary work suggests that it may be precise for $k = 3$.

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