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**COLLECTIONS AND SEQUENCES OF CONTINUA IN THE
PLANE. II**

CECIL EDMUND BURGESS

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C. E. BURGESS

1. Introduction. This paper includes a study of some convergence properties of sequences of mutually exclusive continua in E^2 , and these results are used to obtain some restrictions on the types of continua in an uncountable collection of mutually exclusive continua in E^2 . The concept of *width* of a tree-like continuum is introduced, and it is shown that E^2 does not contain uncountably many mutually exclusive tree-like continua with positive widths. This gives a generalization of R. L. Moore's result that E^2 does not contain uncountably many mutually exclusive triodic continua [13]. The author has presented some related results in [7].

Definitions of trees, chains, tree-like continua, and triods can be found in [8].

2. The width of a tree-like continuum. If G is a tree, then a number $\mathscr{W}(G)$ is associated with G as follows. For each chain C in G and each element X of G , there is a distance¹ $\rho(X, C^*)$ from X to C^* . Let

$$\mathscr{W}(G) = \min_{C \text{ in } G} \left[\max_{X \in G} \rho(X, C^*) \right],$$

where each maximum is obtained with C fixed. A number w is called the width of a tree-like continuum M if, for any cofinal sequence G_1, G_2, G_3, \dots of trees defining M , the sequence $\mathscr{W}(G_1), \mathscr{W}(G_2), \mathscr{W}(G_3), \dots$ converges to w .

THEOREM 1. *Every tree-like continuum has a width.*

Proof. Suppose that some tree-like continuum M does not have a width. Then there exist two nonnegative numbers w_1 and w_2 ($w_1 < w_2$) and two cofinal sequences G_1, G_2, G_3, \dots and H_1, H_2, H_3, \dots of trees defining M such that the sequence $\mathscr{W}(G_1), \mathscr{W}(G_2), \mathscr{W}(G_3), \dots$ converges to w_1 and the sequence $\mathscr{W}(H_1), \mathscr{W}(H_2), \mathscr{W}(H_3), \dots$ converges to w_2 . Let ε be a positive number such that

$$(1) \quad 3\varepsilon < w_2 - w_1.$$

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¹ The point set which is the sum of the elements of C is denoted by C^* .

Let i be an integer such that the mesh of G_i is less than $\varepsilon/2$ and

$$(2) \quad |\mathscr{W}(G_i) - w_1| < \varepsilon .$$

There exists an integer j such that H_j is a refinement of G_i and

$$(3) \quad |\mathscr{W}(H_j) - w_2| < \varepsilon .$$

Let C be a chain in G_i such that

$$(4) \quad \mathscr{W}(G_i) = \max_{x \in G_i} \rho(X, C^*) .$$

There is a chain D in H_j such that each link of C contains a link of D . Let Y be an element of H_j such that

$$(5) \quad \rho(Y, D^*) = \max_{x \in H_j} \rho(X, D^*) ,$$

and let Z be an element of G_i that contains Y . Then

$$(6) \quad \rho(Y, D^*) \leq \rho(Z, C^*) + \varepsilon .$$

It follows from (2) and (4) that

$$(7) \quad \rho(Z, C^*) < w_1 + \varepsilon .$$

Now (6) and (7) imply that

$$(8) \quad \rho(Y, D^*) < w_1 + 2\varepsilon .$$

Hence D is a chain in H_j such that

$$(9) \quad \max_{x \in H_j} \rho(X, D^*) < w_1 + 2\varepsilon ,$$

and since

$$(10) \quad \mathscr{W}(H_j) \leq \max_{x \in H_j} \rho(X, D^*) ,$$

it follows from (9) that

$$(11) \quad \mathscr{W}(H_j) < w_1 + 2\varepsilon .$$

Now a combination of (3) and (11) gives

$$(12) \quad w_2 < w_1 + 3\varepsilon ,$$

and this is contrary to (1).

COROLLARY. *Every linearly chainable continuum has width zero.*

REMARK. There exists a tree-like continuum which has width zero and which is not linearly chainable. A continuum which is the sum of a

simple triod T and a ray spiralling around T is such an example. Any tree-like continuum which is almost chainable [9] has width zero.

THEOREM 2. *If the tree-like continuum M has width zero, then every homeomorphic image of M has width zero.*

Proof. In order that a tree-like continuum K should have width zero it is necessary and sufficient that, for every positive number ε , there should exist an ε -tree G covering K and a chain C in G such that every point of K is within a distance ε of some link of C . Hence, Theorem 2 follows from the fact that every homeomorphism of M is uniformly continuous.

THEOREM 3. *Every tree-like triod has a positive width.*

Proof. Suppose that some tree-like triod M has width zero. Then for each positive number ε , there exist an ε -tree G covering M and a chain C in G such that every point of M is within a distance ε of some link of C . A contradiction can be reached by using an argument similar to the proof of Theorem 6 of [9].

3. Convergent sequences of continua in E^2 . A sequence of continua M_1, M_2, M_3, \dots is said to converge homeomorphically to a continuum M if, for each positive number ε , there exists an integer k such that, for $n > k$, there is a homeomorphism of M_n onto M that moves no point more than a distance ε .

THEOREM 4. *If M_1, M_2, M_3, \dots is a sequence of mutually exclusive tree-like continua in E^2 converging to a continuum M and, for each i , w_i is the width of M_i , then the sequence w_1, w_2, w_3, \dots converges to zero.*

The following lemma will be used in the proof of this theorem.

LEMMA. *If n is a positive integer, H is a collection consisting of n mutually exclusive closed disks in E^2 , and K is a collection consisting of n^3 mutually exclusive dendrons in E^2 such that each element of K intersects every element of H , then some element of K contains an arc which intersects every element of H .*

Proof. The case where $n = 1$ is trivial, so suppose that $n > 1$. Each element of K contains a dendron which is irreducible among the elements of H . Hence there exists a collection K' consisting of n^3 mutually exclusive dendrons such that each element of K' is irreducible among the elements of H and is a subset of an element of K . Now

since n^3 is greater than the product of $2n$ and the number of pairs of elements of H , it follows from [7, Theorem 3] that there exist two elements D_1 and D_2 of H and a collection K'' consisting of $2n$ elements of K' such that each element of K'' intersects each element of H and is a subset of $\text{cl}[E^2 - (D_1 + D_2)]$. Now it follows from [6, Theorem 4] that some element of K'' contains an arc which intersects every element of H . Hence some element of K contains such an arc.

Proof of Theorem 4. Suppose that the sequence w_1, w_2, w_3, \dots does not converge to zero, and for convenience suppose that there is a positive number δ such that each w_i is greater than δ . Let ε be a positive number less than $\delta/4$. There exists a finite set B consisting of n points of M such that every point of M is within a distance ε of some point of B . There exist a collection G of open disks of diameter less than ε and a subcollection G' of G such that

- (1) G is an essential covering of M ,
- (2) G' is an essential covering of B , and
- (3) the closures of the elements of G' are mutually exclusive.

Let G'' denote the collection of all closed disks which are the closures of the open disks of G' . There exists an integer k such that, for $i \geq k$, G is an essential covering of M_i . Now for each i ($k \leq i \leq k + n^3$), there exists a tree G_i such that

- (1) G_i is an essential covering of M_i ,
- (2) each element of G_i is an open disk,
- (3) G_i is a refinement of G ,
- (4) no element of G_i intersects an element of G_j for $j \neq i$,
- (5) if C_i is a linear chain in G_i , some element of G_i is a distance greater than δ from C_i^* , and

(6) the nerve of G_i can be realized by a dendron K_i which is covered essentially by G_i and which has a width greater than δ .

It follows from the above lemma that for some integer s ($k \leq s \leq k + n^3$), there is an arc T_s in K_s which intersects every element of G'' . Requirement (6) implies that some point p of K_s is a distance greater than δ from T_s . Let q be a point of M_s such that $\rho(p, q) = \rho(p, M_s)$, let r be a point of M such that $\rho(q, r) = \rho(q, M)$, and let u be a point of B such that $\rho(r, u) = \rho(r, B)$. Now since $\rho(p, M_s) < \varepsilon$, $\rho(q, M) < \varepsilon$, $\rho(r, B) < \varepsilon$, and $\rho(u, T_s) < \varepsilon$, this leads to the contradiction that $\rho(p, T_s) < \delta$. Hence, the sequence w_1, w_2, w_3, \dots converges to zero.

THEOREM 5. *If M_1, M_2, M_3, \dots is a sequence of mutually exclusive tree-like continua in E^2 converging homeomorphically to a continuum M_0 , then the width of each M_i is zero.*

Proof. Let ε be a positive number. It follows from Theorem 4 and

the homeomorphic convergence of the sequence M_1, M_2, M_3, \dots that there exist a positive integer n , a tree G_n covering M_n , and a homeomorphism f of M_n onto M_0 such that $\varepsilon/3$ is greater than each of the width of M_n , the number $\mathscr{W}(G_n)$, the mesh of G_n , and the distance any point of M_n is moved under f . Let C_n be a chain in G_n such that

$$(1) \quad \mathscr{W}(G_n) = \max_{x \in G_n} \rho(x, C_n^*).$$

Now let G denote the tree which is the collection of all images, under f , of elements of G_n , and let C denote the chain in G which consists of all images, under f , of elements of C_n . It follows that the mesh of G is less than ε and that for each element Y in G_n ,

$$(2) \quad \rho(f(Y), C^*) < \rho(Y, C_n^*) + 2\varepsilon/3.$$

A combination of (1) and (2) gives

$$(3) \quad \rho(f(Y), C^*) < \mathscr{W}(G_n) + 2\varepsilon/3.$$

Now since $\mathscr{W}(G_n) < \varepsilon/3$, it follows from (3) that

$$(4) \quad \rho(f(Y), C^*) < \varepsilon.$$

Hence it has been shown that for each positive number ε , there is an ε -tree G covering M_0 such that $\mathscr{W}(G) < \varepsilon$, and from this it follows that M_0 has width zero. That the width of each M_i is zero follows from Theorem 2.

THEOREM 6. *If M_1, M_2, M_3, \dots is a sequence of mutually exclusive continua in E^2 converging homeomorphically to a continuum M_0 , then no M_i has more than two complementary domains.*

Proof. Suppose that M_0 has three complementary domains. Let a, b , and c be three points in the complement of M_0 such that no two of them are in the same complementary domain of M_0 , and let ε be a positive number that is less than the distance from M_0 to $a + b + c$. There exists an integer k such that, for $n > k$, there is a homeomorphism f_n of M_n onto M_0 that moves no point more than a distance $\varepsilon/2$, and hence, for $n > k$, M_n does not contain one of the points a, b , and c . Now let h and j be two integers greater than k . It follows from a theorem proved by Eilenberg [10, Theorem 5] that each of the continua M_h and M_j separates each two of the points a, b , and c in E^2 . On the other hand, M_h and M_j are mutually exclusive so that M_h would lie in some complementary domain of M_j , and hence some two of the points a, b , and c would not be separated by M_h . From this contradiction, it follows that M_0 does not have more than two complementary domains. Consequently, no M_i has more than two complementary domains.

THEOREM 7. *If M_1, M_2, M_3, \dots is a sequence of mutually exclusive continua in E^2 converging homeomorphically to a continuum M_0 that separates E^2 , then each M_i irreducibly separates E^2 into two components.*

Proof. It follows from Theorem 6 that M_0 separates E^2 into two components, so suppose that some proper subcontinuum of M_0 separates E^2 . Then some proper subcontinuum K of M_0 would irreducibly separate E^2 into two components. Let p be a point of $M_0 - K$, let q be a point that is separated in E^2 from p by K , and let ε be a positive number less than the distance from K to $p + q$. Let D be an open circular disk with center at p and with radius $\varepsilon/3$. There exist integers h and j such that the continua M_h and M_j are carried onto M_0 by homeomorphisms f_h and f_j , respectively, that move no point more than a distance $\varepsilon/3$. Let K_h and K_j denote the continua $f_h^{-1}(K)$ and $f_j^{-1}(K)$, respectively. Each of the continua M_h and M_j intersects D and, by [10, Theorem 5], each of the continua K_h and K_j separates q from D in E^2 . Since K_h and K_j are mutually exclusive, it follows that one of them, say K_h , separates the other, K_j , from D in E^2 . This involves the contradiction that M_j intersects both K_j and D but does not intersect K_h . Hence, it follows that each M_i irreducibly separates E^2 into two components.

THEOREM 8. *There does not exist in E^2 a sequence of mutually exclusive triods converging homeomorphically.*

Proof. Let M_1, M_2, M_3, \dots be a sequence of mutually exclusive continua in E^2 converging homeomorphically to a continuum M . It is sufficient to show that M is not a triod.

Case 1. The continuum M separates E^2 . By Theorem 7, M irreducibly separates E^2 into two components so that no proper subcontinuum of M separates M [12]. Hence M is not a triod.

Case 2. The continuum M does not separate E^2 . Then M is tree-like as it contains no open subset of E^2 [2]. By Theorem 5, M has width zero, and hence it follows from Theorem 3 that M is not a triod.

4. Uncountable collections of mutually exclusive continua in E^2 . Roberts [14] has shown that every linearly chainable continuum has uncountable many mutually exclusive homeomorphic images in E^2 . However, this is not the case for tree-like continua with width zero as the continuum described in the remark in § 1 has width zero and contains a simple triod. Anderson [1] has indicated the existence of an uncountable

table collection of mutually exclusive tree-like continua in E^2 such that no one of them is chainable. By Theorem 9, for any uncountable collection G of mutually exclusive homeomorphic continua in E^2 , there exists a sequence of elements of G converging homeomorphically to an element of G . This suggests the following question, which is left unanswered. If M is a tree-like continuum in E^2 such that there exists a sequence of mutually exclusive continua in E^2 converging homeomorphically to M , does M have uncountably many mutually exclusive homeomorphic images in E^2 ?

THEOREM 9. *If G is an uncountable collection of mutually exclusive homeomorphic continua in E^n , then there exists a sequence of elements of G which converges homeomorphically to an element of G .*

By using Borsuk's theorem that, under the metric $d(f, g) = \max_{x \in M} \rho(f(x), g(x))$, the space of all continuous transformations of a compact metric space M into a separable metric space is separable [5, Theorem 2], Theorem 9 can be proved by the method Bing [4] has indicated for the case where G is a collection of arcs in E^2 .

THEOREM 10. *If G is an uncountable collection of mutually exclusive tree-like continua in E^2 , then all except a countable number of continua of G have width zero.*

Proof. It is sufficient to show that some continuum of G has width zero. Suppose that no continuum of G has width zero. It follows from Theorem 1 that there is a positive number δ and an uncountable subcollection G' of G such that each continuum of G' has a width greater than δ . But this is contrary to Theorem 4 since there is a convergent sequence of elements of G' .

5. A remark on homogeneous decomposable plane continua. F. B. Jones [11] has shown that every nondegenerate homogeneous decomposable plane continuum has a continuous decomposition G such that G is a simple closed curve with respect to its elements and each element of G is a homogeneous tree-like continuum. Jones' question as to whether each element of G would be a pseudo-arc has not been answered, but Bing [3] has shown that this would be the case if each element of G were linearly chainable. It follows from Theorem 10 that each element of G has width zero. This suggests the following question. Is a homogeneous tree-like continuum chainable if it has width zero?

Added in proof. The author has recently shown that every homogeneous tree-like continuum in E^2 has width zero hereditarily and that a tree-like continuum has width zero hereditarily if and only if it is

atriodic. These results will be presented in another paper.

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