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In [3] R. H. Bing characterized the homogeneous continua than can be imbedded in the plane and contain arcs by showing that any such continuum is a simple closed curve. The principal result of the present paper pertains to the related problem of characterizing the 1-dimensional homogeneous continua which contain arcs but which are not necessarily imbeddable in the plane. In the paper mentioned above, R. H. Bing also asked if the continua in this larger class might be precisely the simple closed curves, the universal curves, and the solenoids. According to the title there is presented here a construction of another continuum having these properties. The author was motivated in making this construction by conversations with C. E. Burgess and by an abstract [8] of Jack Segal's.

[In this paper a topological space will be called a *continuum* provided that it is connected, bicompact, and metrizable. Also, a topological space will be called *homogeneous* provided that for any two points x and y in the space there exists a homeomorphism h of the space onto itself such that $h(x) = y$. Moreover, according to R. D. Anderson's characterization in [2], a universal curve will denote a 1-dimensional locally connected continuum with no local cut points which has no open subset which can be imbedded in the plane.]

A simple pattern through the somewhat complicated construction which follows may be seen by considering the inverse system

$$(*) \qquad \{X_i, \phi_i\}_{i=0}^{\infty}$$

where for each nonnegative integer i, X_i is a continuum, X_{i+1} is a covering space [5] of X_i relative to the projection ϕ_i, the inverse image in X_{i+1} under ϕ_i of any point in X_i consists of exactly r_i points, and r_i is an integer greater than 1. More briefly, the continuum X_{i+1} is an r_i-fold covering space of the continuum X_i relative to the projection ϕ_i. Let X denote the limit [4; Chapter VIII] of the inverse system (*). It is known [4; Chapter VIII] that X is also a continuum. In the construction to follow each X_i will be a universal curve and the limit space X will be the desired continuum. However, the author was unable to establish the desired properties (in particular the homogenity) of the continuum X by means of the information already given about the system (*). In order to proceed the author constructs particular universal curves

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X_i and mappings ϕ_i in terms of the first universal curve X_0—this is the reason for the complexity of the construction.

The only case treated here is that in which $r_i = 2$, that is, the case in which X_{i+1} is a 2-fold covering space of X_i relative to the projection ϕ_i for all i. It will be evident to the reader how the construction could be modified for arbitrary r_i.

1. Preliminary definitions and conventions. Let S denote Euclidean 3-space, L the z-axis, C the circle in the $xy-$ plane having radius 1 and center the origin, e the point $(1, 0, 0)$, M a universal curve which is a subspace of $S - L$ and contains C, I the closed unit interval $[0, 1]$ of real numbers, and ω the set of all nonnegative integers.

A continuous function σ from I into n space Y is called a *path* in Y. If σ is a path in Y then $\sigma(0)$ is called the *initial point* of σ, $\sigma(1)$ is called the *terminal point* of σ, and σ is said to be a *path from* $\sigma(0)$ to $\sigma(1)$ in Y. If σ is a path in Y such that $\sigma(0) = \sigma(1)$ then σ is called a loop in Y. If α is a path from a to b in Y and β is a path from b to c in Y then the function $\alpha\beta$ defined by

$$(\alpha\beta)(t) = \alpha(2t) \quad \text{for} \quad 0 \leqq t \leqq 1/2$$

and

$$(\alpha\beta)(t) = \beta(2t - 1) \quad \text{for} \quad 1/2 \leqq t \leqq 1$$

is clearly a path from a to c in Y. If σ is a path from a to b in Y then the function σ^{-1} defined by $\sigma^{-1}(t) = \sigma(1 - t)$ for all t in I is clearly a path from b to a in Y. For any continuous function σ from a closed interval of real numbers to $S - L$ let the winding number, $W(\alpha)$, of σ with respect to L be defined by the formula

$$W(\sigma) = \frac{1}{2\pi}\int_\sigma \frac{xdy - ydx}{x^2 + y^2}\,.$$

The fact that W is a well defined real valued function is verified at length by Newman [7]—as are the following facts: If α and β are two paths from a to b in $S - L$ such that one can be deformed into the other in $S - L$ keeping the end points fixed then $W(\alpha) = W(\beta)$. If σ is a loop in $S - L$ then $W(\sigma)$ is an integer. If α is a path from a to b in $S - L$ and β is a path from b to c in $S - L$ then $W(\alpha\beta) = W(\alpha) + W(\beta)$ and $W(\alpha^{-1}) = -W(\alpha)$. If α is a path from a to b in $S - L$, β is a path from b to c in $S - L$, and γ is a path from c to d in $S - L$ then $W((\alpha\beta)\gamma) = W(\alpha(\beta\gamma))$ and therefore the real number $W(\alpha\beta\gamma)$ is well defined. Finally for any loop σ in $S - L$ and any integer n, $W(\sigma^n)$ is well-defined and $W(\sigma^n) = nW(\sigma)$ provided that σ^0 is interpreted as the path having constant value $\sigma(0)$, $\sigma^n = (\sigma^{n-1})\sigma$ for $n > 0$, and $\sigma^n = (\sigma^{n+1})\sigma^{-1}$ for $n < 0$.

2. Construction of the inverse system. The construction of the spaces X_i will be carried out in the same way as the classical construction [6] of the universal covering space except for the fact that a different equivalence relation will be defined on the space of paths. Let the space of paths Ω be the set of all paths in M having initial point e. Define the element τ of Ω by the formula

$$\tau(t) = (\cos 2\pi t,\ \sin 2\pi t,\ 0)$$

for all t in I. Note that $W(\tau^z) = z$ for any integer z. Where $n \in \omega$ and $\alpha \in \Omega$ let $c(\alpha, n)$ be the set of all γ in Ω such that

$$\alpha(1) = \gamma(1)$$

and

$$W(\alpha) = W(\gamma) \bmod (2^n)\ .$$

Where $n \in \omega$ let X_n be the family of all sets $c(\sigma, n)$ such that $\sigma \in \Omega$. Then X_n is a decomposition of Ω into non-empty, mutually disjoint equivalence classes. Also, for any nonnegative integer n define the function ϕ_n from X_{n+1} to X_n and p_n from X_n to M by the formulas

$$\phi_n c(\sigma, n+1) = c(\sigma, n)$$

and

$$p_n c(\sigma, n) = \sigma(1)$$

for all σ in Ω. Note that p_0 is a one-to-one mapping of X_0 onto M.

The definitions of the topologies for the spaces X_i require further construction. Let $\mathfrak{U}$ be the family of all non-empty, connected, open subsets U of M such that for any loop ρ in U, $W(\rho) = 0$. Note that U is a basis for the topology of M and if α and β are paths in the same member of U with the same initial and terminal points then $W(\alpha) = W(\beta)$. Let Q be the set of all ordered triples (U, σ, n) such that $\sigma \in \Omega$, $\sigma(1) \in U \in \mathfrak{U}$, and $n \in \omega$. Where $(U, \sigma, n) \in Q$ let $N(U, \alpha, n)$ be the set of all points in X_n of the form $c(\sigma\delta, n)$ such that δ is a path in U and $\sigma(1) = \delta(0)$. Also, if $q = (U, \sigma, n) \in Q$ then it is assumed that $N(U, \sigma, n)$ may be denoted by $N(q)$ or N_q. Finally, if $n \in \omega$ let $\mathfrak{B}_n$ be the family of all $N(q) \cap X_n$ for $q \in Q$. It will be seen later that the topology assigned to X_n will have $\mathfrak{B}_n$ as a base.

LEMMA 2.1. *If $(U, \sigma, n) \in Q$, ρ is a path in U, and $\sigma(1) = \rho(0)$ then $N(U, \sigma, n) = N(U, \sigma\rho, n)$.*

Proof. After assuming the hypothesis listed above take any path

δ in U such that $\sigma(1) = \delta(0)$. Then $c(\sigma\delta, n)$ is a typical element of $N(U, \sigma, n)$. It follows that

$$c(\sigma\delta, n) = c((\sigma\rho)(\rho^{-1}\delta), n) \in N(U, \sigma\rho, n) .$$

Similarly, take any path δ in U such that $(\sigma\rho)(1) = \delta(0)$. Then $c((\sigma\rho)\delta, n)$ is a typical element of $N(U, \sigma\rho, n)$. It follows that

$$c((\sigma\rho)\delta, n) = c(\sigma(\rho\delta), n) \in N(U, \sigma, n) .$$

Therefore $N(U, \sigma, n) = N(U, \sigma\rho, n)$.

LEMMA 2.2. *If $n \in \omega$ and $x \in A, B \in \mathfrak{B}_n$ then there exists an element C of $\mathfrak{B}_n$ such that $x \in C \subset A \cap B$.*

Proof. Suppose that $n \in \omega$ and $x \in A, B \in \mathfrak{B}_n$, say $A = N(U, \alpha, n)$, $B = N(V, \beta, n)$, $(U, \alpha, n) \in Q$, $(V, \beta, n) \in Q$, $\sigma \in \Omega$, and $x = c(\sigma, n) \in A \cap B$. Take a path γ in U and a path δ in V such that $\alpha(1) = \gamma(0)$, $\beta(1) = \delta(0)$ and $c(\sigma, n) = c(\alpha\gamma, n) = c(\beta\delta, n)$. Then $\sigma(1) = \gamma(1) = \delta(1) \in U \cap V$. Take an element W of $\mathfrak{U}$ such that $\sigma(1) \in W \subset U \cap V$. Let $C = N(W, \sigma, n)$. Let μ be any path in V such that $\sigma(1) = \mu(0)$. Then $c(\sigma\mu, n)$ is an arbitrary element of C and

$$c(\sigma\mu, n) = c(\alpha(\gamma\mu), n) = c(\beta(\delta\mu), n) \in A \cap B .$$

Therefore $x \in C \subset A \cap B$.

This lemma allows us to define a topology on X_n such that $\mathfrak{B}_n$ is a base of open sets.

LEMMA 2.3. *For any $n \in \omega$ the space X_n is arc-wise connected.*

Proof. Let α be that element of Ω which maps I onto $e \in C$. Let $a = c(\alpha, n) \in X_n$. It will suffice to show that any other point in X_n can be connected to a by a path in X_n. Suppose that $\sigma \in \Omega$. Then $c(\sigma, n)$ is a typical point in X_n. For any $s \in I$ define $\rho_s \in \Omega$ by $\rho_s(t) = \sigma(st)$ for all $t \in I$. Now define a function h from I to X_n by the formula $h(s) = c(\rho_s, n)$ for all $s \in I$. Clearly $h(0) = a$ and $h(1) = c(\sigma, n)$. It remains to show that h is continuous. Take any U and σ such that $(U, \sigma, n) \in Q$. Then $N(U, \sigma, n)$ is a typical basic open set in X_n. Take any $s \in I$ such that $h(s) \in N(U, \sigma, n)$. Then $\rho_s(1) = \sigma(s) \in U$ and for some path δ in U from $\sigma(1)$ to $\sigma(s) W(\sigma\delta) = W(\rho_s)$. Let G be that component of $\sigma^{-1}[U]$ which contains s. Then G is an open neighborhood of s in I. It will suffice to show that $h[G] \subset N(U, \sigma, n)$. Take any $r \in G$. Then $\rho_r(1) = \sigma(r) \in U$. Define a path γ in U by

$$\gamma(t) = \sigma((1 - t)s + tr)$$

for all $t \in I$. Then γ is a path in U from $\rho_s(1)$ to $\rho_r(1)$ in U. Take δ as above then $\delta\gamma$ is a path in U from $\sigma(1)$ to $\rho_r(1)$ and

$$W(\sigma(\delta\gamma)) = W((\sigma\delta)\gamma) = W(\rho_s\gamma) = W(\rho_r) .$$

Therefore $h(r) = c(\rho_r, n) \in N(U, \sigma, n)$.

LEMMA 2.4. *If* $(U, \sigma, n) \in Q$ *then*

(A) ϕ_n *maps* $N(U, \sigma, n + 1)$ *onto* $N(U, \sigma, n)$ *in a one-to-one fashion and*

(B) p_n *maps* $N(U, \sigma, n)$ *onto* U *in a one-to-one fashion.*

Proof. Suppose that $(U, \sigma, n) \in Q$. Take any two paths γ and δ in U such that $\gamma(0) = \delta(0) = \sigma(1)$. Then $c(\sigma\gamma, n + 1)$ is a typical element of $N(U, \sigma, n + 1)$ and $c(\sigma\delta, n)$ is a typical element of $N(U, \sigma, n)$. Since σ and n are fixed and γ and δ are paths in $U \in \mathfrak{U}$ then $c(\sigma\gamma, n + 1)$ and $c(\sigma\delta, n)$ depend only upon the choice of terminal point of γ and δ respectively. Moreover, $\phi_n c(\sigma\gamma, n + 1) = c(\sigma\delta, n)$ if and only if $\gamma(1) = \delta(1)$. Therefore ϕ_n maps $N(U, \sigma, n + 1)$ onto $N(U, \sigma, n)$ in a one-to-one fashion. Also $p_n(\sigma\delta, n) = u \in U$ if and only if $\delta(1) = u$. Therefore p_n maps $N(U, \sigma, n)$ onto U in a one-to-one fashion.

LEMMA 2.5. *If* $(U, \sigma, n) \in Q$ *and* $r = 2^n$ *then*

(A) $(\phi_n)^{-1}[N(U, \sigma, n)]$ *is the union of the two disjoint open sets* $N(U, \tau^{ri}\sigma, n + 1)$ *for* $i = 0, 1$
and

(B) $(p_n)^{-1}[U]$ *is the union of the* 2^n *mutually disjoint open sets* $N(U, \tau^i\sigma, n)$ *for* $i = 0, 1, \cdots, 2^n - 1$.

Proof of (A). It follows from Lemma 2.4 that

$$\phi_n[N(U, \tau^{ri}\sigma, n + 1)] = N(U, \tau^{ri}\sigma, n) = N(U, \sigma, n)$$

for $i = 0, 1$. Therefore the union of these two open sets is contained in $(\phi_n)^{-1}[N(U, \sigma, n)]$. In order to establish the opposite inclusion take any $x \in \phi_n^{-1}[N(U, \sigma, n)]$, say $x = c(\rho, n + 1)$ where $\rho \in \Omega$ and $\rho(1) \in U$. Since $\phi_n(x) = c(\rho, n) \in N(U, \sigma, n)$ there is a path δ in U from $\sigma(1)$ to $\rho(1)$ such that $c(\rho, n) = c(\sigma\delta, n)$. Then for such a path, $W(\rho) = W(\sigma\delta)$ mod 2^n and there is an integer k such that $W(\rho) - W(\sigma\delta) = k2^n$. There is yet another integer s such that $k = 2s + i$ where i is either 0 or 1. Then

$$W(\rho) - W(\sigma\delta) = (2s + i)2^n = s2^{n+1} + ri$$

and

$$W(\rho) = [W(\sigma\delta) + ri] \bmod 2^{n+1} .$$

Therefore

$$W(\rho) = W(\tau^{ri}\sigma\delta) \bmod 2^{n+1}$$

and

$$x = c(\rho, n+1) = c(\tau^{ri}\sigma\delta, n+1) \in N(U, \tau^{ri}\sigma, n+1) .$$

In order to complete the proof of part (A) it remains to show that these two open sets are disjoint. Suppose that

$$x \in N(U, \tau^{ri}\sigma, n+1) \cap N(U, \tau^{rj}\sigma, n+1)$$

where $i, j \in \{0, 1\}$. Say $x = c(\rho, n+1)$ where $\rho \in \Omega$ and $\rho(1) \in U$. Take paths δ_i and δ_j in U from $\sigma(1)$ to $\rho(1)$ such that

$$x = c(\tau^{ri}\sigma\delta_i, n+1) = c(\tau^{rj}\sigma\delta_j, n+1) .$$

Since δ_i and δ_j are paths in $U \in \mathfrak{U}$ having the same initial and terminal points then $W(\delta_i) = W(\delta_j)$. Also

$$W(\tau^{ri}\sigma\delta_i) = W(\tau^{rj}\sigma\delta_j) \bmod 2^{n+1} .$$

Therefore

$$W(\tau^{ri}) + W(\sigma) + W(\delta_i) = [W(\tau^{rj}) + W(\sigma) + W(\delta_j)] \bmod 2^{n+1} ;$$
$$W(\tau^{ri}) = W(\tau^{rj}) \bmod 2^{n+1} ; \; ri = rj \bmod 2^{n+1} ; \quad \text{and} \quad i = j \bmod 2 .$$

Since in addition $i, j \in \{0, 1\}$ then $i = j$ and the two given open sets containing x have to be identical.

Proof of (B). Clearly (B) is true for $n = 0$. Since $p_n\phi_n = p_{n+1}$ then $(p_{n+1})^{-1} = (\phi_n)^{-1}(p_n)^{-1}$. Therefore (B) follows from (A) by induction on n.

LEMMA 2.6. *For each $n \in \omega$, X_n is a 2^n-fold covering space of M relative to the projection p_n with coordinate neighborhoods $\mathfrak{U}$ and X_{n+1} is a 2-fold covering space of X_n relative to ϕ_n with coordinate neighborhoods $\mathfrak{B}_n$.*

Proof. Take any $n \in \omega$. Since $\mathfrak{U}$ is a base for the topology of M, in order to show that $(p_n)^{-1}[U]$ is open in X_n for any $U \in \mathfrak{U}$, take any $U \in \mathfrak{U}$. According to Lemma 2.5, $(p_n)^{-1}[U]$ is the union of a finite number of elements of $\mathfrak{B}_n$. Since, in addition, $\mathfrak{B}_n$ is a base for the topology of X_n, $(p_n)^{-1}[U]$ is open in X_n. Therefore p_n is continuous. Applying Lemma 2.4 again, it follows that p_n maps any $B \in \mathfrak{B}_n$ topologically onto $p_n[B] \in U$. Therefore, in view of Lemma 2.5, for any $U \in \mathfrak{U}$ there exists $\mathfrak{E} \subset \mathfrak{B}_n$ such that $\mathfrak{E}$ is a disjoint family and for any $E \in \mathfrak{E}$, p_n maps E topologically onto U. Since in addition M is locally arc-wise connected and X_n is connected then X_n is a covering space of M relative to the projection

p_n with coordinate neighborhoods $\mathfrak{U}$. It follows from Lemma 2.5 that p_n is 2^n-fold.

The proof that X_{n+1} is a 2-fold covering space of X_n relative to the projection ϕ_n with coordinate neighborhoods $\mathfrak{B}_n$ is strictly analogous to the above proof.

LEMMA 2.7. *For each $n \in \omega$, X_n is a universal curve.*

Proof. Take any $n \in \omega$. The preceding Lemma gives that X_n is a covering space of the universal curve M. Therefore X_n is locally homeomorphic to M, since M is locally connected so is X_n, and since M is metrizable so is X_n. From the additional fact that p_n is 2^n-fold it follows that X_n is compact. Therefore X_n is a curve (compact, connected, locally connected, and metrizable) which is locally homeomorphic to M. In [2; § 5] R. D. Anderson characterized the universal curve up to topological equivalence by local properties. Therefore X_n is a universal curve.

3. Construction of the limit space X. Let X be the limit of the inverse system $\{X_i, \phi_i\}_{i\in\omega}$. Since each ϕ_i is onto and each X_i is non-degenerate then X is a non-degenerate continuum [4; Chapter VIII]. It also follows from [4; Chapter VIII] that the dimension of X cannot exceed 1. Therefore, since X is a non-degenerate continuum of dimension less than or equal to 1, it is a 1-dimensional continuum.

Recalling the definition of inverse limit [4; Chapter VIII], X is the set of all sequences $x = \{x_i\}_{i\in\omega}$ such that $x_i \in X_i$ and $\phi(x_{i+1}) = x_i$ for all $i \in \omega$. For each $i \in \omega$ a projection mapping π_i from X to X_i is defined by $\pi_i(x) = x_i$ for all $x \in X$. From [4; Chapter VIII] it follows that each π_i is continuous and onto. Let $\mathfrak{H}$ be the collection of all $(\pi_n)^{-1}[B]$ for $B \in \mathfrak{B}_n$ and $n \in \omega$. Then $\mathfrak{H}$ is a base of open sets for the topology of X. For each $q = (U, \sigma, n) \in Q$ let

$$H_q = H(U, \sigma, n) = (\pi_n)^{-1}[N(U, \sigma, n)]$$

then $\mathfrak{H} = \{H_q \mid q \in Q\}$.

The 1-dimensional continuum X has now been defined but in order to establish its properties some auxiliary machinery is needed.

4. Establishing the properties of X. As in some developments of set theory we will identify any $n \in \omega$ with the set of all nonnegative integers less than n. Then for any $n \in \omega$ consider 2^n as a topological space having the discrete topology and 2^n points. For any $n \in \omega$ define $\psi_n: 2^{n+1} \to 2^n$ by the condition $\psi_n(z) = z \bmod 2^n$ for all $z \in 2^{n+1}$. Then ψ_n maps 2^{n+1} continuously onto 2^n and the inverse image of any point under ψ_n consists of exactly two points. Let $D = \lim_{i\in\omega} \{D_i, \psi_i\}$ where $D_i = 2^i$ for all $i \in \omega$. Then D is topologically equivalent to the Cantor discontinuum. For each $n \in \omega$ let π_n be the projection of D onto 2^n defined by

$$\pi_n(d) = d_n$$

for all $d \in D$. [The projections defined on D and those defined on X should not be confused.] For any $n \in \omega$ and $z \in 2^n \pi_n^{-1}[\{z\}]$ is a basic open set in D and, moreover, it is a closed subset of D which is topologically equivalent to D.

LEMMA 4.1. *If $q = (U, \sigma, n) \in Q$ then H_q is topologically equivalent to $U \times \pi_n^{-1}[\{0\}]$ and hence any basic open set in X is topologically equivalent to the product of a basic open set in M with the Cantor discontinuum. Furthermore, X contains arbitrarily small arcs, even arbitrarily small simple closed curves and universal curves but is not locally connected. Therefore X is neither a solenoid nor a universal curve.*

Proof. Suppose that $q = (U, \sigma, n) \in Q$. For any integer z let $\exp(z)$ denote τ^z and for each $u \in U$ pick a path δ_u in U from $\sigma(1)$ to u. Define the function

$$h : U \times \pi_n^{-1}[\{0\}] \to X$$

by the formula

$$h(u, d)_i = c(\exp(d_i)\sigma\delta_u, i)$$

for all $i \in \omega$, $u \in U$, and $d \in \pi_n^{-1}[\{0\}]$.

Clearly this definition is independent of the choice of path δ_u. Take any $u \in U$, $d \in \pi_n^{-1}[\{0\}]$, and $i \in \omega$. Then

$$\begin{aligned} \phi_i(h(u, d)_{i+1}) &= \phi_i c(\exp(d_{i+1})\sigma\delta_u, i+1) \\ &= c(\exp(d_{i+1})\sigma\delta_u, i) \\ &= c(\exp(d_i)\sigma\delta_u, i) \\ &= h(u, d)_i \end{aligned}$$

and therefore $h(u, d) \in X$.

It is desired that h map $U \times \pi_n^{-1}[\{0\}]$ onto H_q in a one-to-one fashion. Take any $u \in U$, $d \in \pi_n^{-1}[\{0\}]$. Then

$$\begin{aligned} \pi_n h(u, d) = h(u, d)_n &= c(\exp(d_n)\sigma\delta_u, n) \\ &= c(\exp(0)\sigma\delta_u, n) = c(\sigma\delta_u, n) \in N(U, \sigma, n) \end{aligned}$$

and hence

$$h(u, d) \in H_q = \pi_n^{-1}[N(U, \sigma, n)] .$$

Therefore $h[u \times \pi_n^{-1}[\{0\}]] \subset H_q$. In order to establish the opposite inclusion take any $x \in H_q$, say $x_i = c(\rho_i, i)$ for all $i \in \omega$. Let $u = \rho_n(1)$ and let $d \in \pi_n^{-1}[\{0\}]$ be defined by

$$d_i = W(\rho_i \delta_u^{-1} \sigma^{-1}) \bmod 2^i$$

for all $i \in \omega$. Then $h(u, d)_i = c(\exp(d_i)\sigma\delta_u, i)$;

$$\begin{aligned}[\exp(d_i)\sigma\delta_u](1) = \delta_u(1) = u\ ;\ & W(\exp(d_i)\sigma\delta_u) \\ &= d_i + W(\sigma\delta_u) = W(\rho_i\delta_u^{-1}\sigma^{-1}) + W(\sigma\delta_u) = W(\rho_i)\end{aligned}$$

all modulo 2^i; and hence $h(u, d)_i = x_i$ for all $i \in \omega$. Therefore $h(u, d) = x$ and $H_q \subset h[U \times \pi_n^{-1}[\{0\}]$.

In order to show that h is one-to-one take any $u, r \in U$ and $d, f \in \pi_n^{-1}[\{0\}]$ such that $h(u, d) = h(r, f)$. Then for all $i \in \omega$:

$$\begin{aligned} c(\exp(d_i)\sigma\delta_u, i) &= c(\exp(f_i)\sigma\delta_r, i)\ ; \\ u = [\exp(d_i)\sigma\delta_u](1) &= [\exp(f_i)\sigma\delta_r](1) = r\ ; \\ W(\exp(d_i)\sigma\delta_u) &= W(\exp(f_i)\sigma\delta_r) \bmod 2^i\ ; \\ d_i + W(\sigma\delta_u) &= f_i + W(\sigma\delta_r) \bmod 2^i\ ; \\ d_i = f_i \bmod 2^i\ ; &\quad \text{and} \quad d_i = f_i\ . \end{aligned}$$

Therefore $(u, d) = (r, f)$ and hence h is a one-to-one function.

Finally it is desired to show that h is bicontinuous. Take any $m, z \in \omega$ such that $m \geqq n$, $z \in 2^m$, and $z = 0 \bmod 2^n$. It is clear that $\pi_m^{-1}[\{z\}]$ is a typical basic open set in $\pi_n^{-1}[\{0\}]$. Suppose in addition that $V \in \mathfrak{U}$, $V \subset U$, and μ is a path in U from $\sigma(1)$ to a point in V. Let $r = (V, \tau^2\sigma\mu, m)$. It will now be shown that H_r is a typical basic open set in H_q. It is clear that if $s = (V, \rho\mu, m)$ where $\rho(1) = \sigma(1)$ and $W(\rho) = W(\sigma) \bmod 2^n$ then H_s is a typical basic open set in H_q. If $z \in 2^m$ were taken so that $z = W(\rho\sigma^{-1}) \bmod 2^m$ then $z = 0 \bmod 2^n$; $N(U, \tau^2\sigma\mu, m) = N(V, \rho\mu, m)$ and $H_r = H_s$. Therefore H_r is a typical basic open set in H_q.

In order to establish the bicontinuity of h it will be sufficient to show that $h[V \times \pi_m^{-1}[\{z\}] = H_r$ and $h^{-1}[H_r] = V \times \pi_m^{-1}[\{z\}]$. Therefore it will be sufficient to show that for any

$$(u, d) \in U \times \pi_n^{-1}[\{0\}],\ (u, d) \in V \times \pi_m^{-1}[\{z\}]$$

if and only if $h(u, d) \in H_r$. For any $r \in V$ let γ_r be a path in V from $\mu(1)$ to r. If $(u, d) \in V \times \pi_m^{-1}[\{z\}]$ then

$$\begin{aligned} \pi_m h(u, d) = h(u, d)_m &= c(\exp(d_m)\sigma\delta_u, m) \\ &= c(\tau^z\sigma\mu\gamma_u, m) \\ &\in N(V, \tau^z\sigma\mu, m) \end{aligned}$$

and therefore $h(u, d) \in H_r = \pi_m^{-1}[N(V, \tau^2\sigma\mu, m)]$. Now Suppose that

$h(u, d) \in H_r$. Then

$$\pi_m h(u, d) = h(u, d)_m = c(\exp(d_m)\sigma\mu\gamma_u, m) \in N(V, \tau^z\sigma\mu, m)$$

and therefore $[\exp(d_m)\sigma\mu\gamma_u](1) = \gamma_u(1) = u \in V$. Also,

$$\begin{aligned} W(\exp(d_m)\sigma\mu\gamma_u) &= W(\tau^z\sigma\mu) \bmod 2^m ; \\ d_m + W(\sigma\mu\gamma_u) &= z + W(\sigma\mu\gamma_u) \bmod 2^m ; \\ d_m &= z \bmod 2^m ; \end{aligned}$$

and

$$d_m = z .$$

Therefore

$$(u, d) \in V \times \pi_m^{-1}[\{z\}] .$$

LEMMA 4.2. *If U and V are any two elements of $\mathfrak{U}$ such that $U \cap V$ is connected and not empty then $U \cup V \in \mathfrak{U}$.*

Proof. Suppose that $U, V \in \mathfrak{U}$ and $U \cap V$ is connected and not empty. Clearly, $U \cup V$ is connected. Since, in addition, $U \cap V$ is open and M is locally arc-wise connected then $U \cap V$ is arc-wise connected. Let σ be any loop in $U \cup V$. If either $\sigma[I] \subset U$ or $\sigma[I] \subset V$ then $W(\sigma) = 0$. Suppose that neither $\sigma[I] \subset U$ nor $\sigma[I] \subset V$. By a change of parameter adjust σ so that $\sigma(0) = \sigma(1) \in U - V$. Let $\mathfrak{A}$ be the collection of all components A of $\sigma^{-1}[U \cap V]$ such that $\sigma[A^*]$ meets both $U - V$ and $V - V$. From the fact that the distance from $\sigma[I] - V$ to $\sigma[I] - U$ is positive it follows that $\mathfrak{A}$ is finite. Let $\{A_i\}_{i=1}^n$ be an indexing of the elements of $\mathfrak{A}$ such that $A_i < A_{i+1}$ for $i = 1, 2, \cdots, n$. [Each element of A_i is less than each element of A_{i+1}.] Take $\{a_i\}_{i=0}^{n+1} \subset I$ such that $a_0 = 0$, $a_{n+1} = 1$, and $a_i \in A_i$ for $i = 1, 2, \cdots, n$. Then

$$0 = a_0 < a_1 < a_2 < \cdots < a_n < a_{n+1} = 1 .$$

For $i = 0, 1, \cdots, n$ let σ_i be the restriction of σ to the interval $[a_i, a_{i+1}.]$ Note that $\sigma(a_i) \in U \cap V$ for all $i = 1, 2, \cdots, n$. For $i = 1, 2, \cdots, n - 1$ let $\rho_i: [a_i, a_{i+1}] \to U \cap V$ be a continuous function such that $\rho_i(a_i) = \sigma(a_i)$ and $\rho_i(a_{i+1}) = \sigma(a_{i+1})$. Let $\rho_0 = \sigma_0$, $\rho_n = \sigma_n$, and $\rho = \rho_0 \cup \rho_1 \cup \cdots \cup \rho_n$. Obviously $W(\rho_0) = W(\sigma_0)$ and $W(\rho_n) = W(\sigma_n)$. Suppose i is any integer from 1 to $n - 1$. Either $\sigma[a_i, a_{i+1}] \subset U$ or $\sigma[a_i, a_{i+1}] \subset V$ for otherwise there would be $A \in \mathfrak{A}$ between A_i and A_{i+1} contrary to the indexing of $\mathfrak{A}$. If $\sigma[a_i, a_{i+1}] \subset U$ then σ_i and ρ_i are, modulo an order preserving change of parameter, paths in U from $\sigma(a_i)$ to $\sigma(a_{i+1})$ and hence $W(\sigma_i) = W(\rho_i)$. If $\sigma[a_i, a_{i+1}] \subset V$ then σ_i and ρ_i are, modulo an order preserving change of parameter, paths in V from $\sigma(a_i)$ to $\sigma(a_{i+1})$ and hence $W(\sigma_i) =$

$W(\rho_i)$. Therefore

$$W(\sigma) = \sum_{j=0}^{n} W(\sigma_j) = \sum_{j=0}^{n} W(\rho_j) = W(\rho) .$$

It is already clear that $\rho[a_1, a_n] \subset U \cap V \subset U$. Also $\rho[a_0, a_1] \subset U$ for otherwise there would be $A \in \mathfrak{A}$ between 0 and A_1. Analogously, $\rho[a_1, a_{n+1}] \subset U$. Therefore ρ is a loop in U and $W(\sigma) = W(\rho) = 0$.

LEMMA 4.3. *If A is any arc in M then there exists an element U of $\mathfrak{U}$ such that $A \subset U$.*

Proof. Let E be any open cover of A by elements of U. Since A is an arc in the locally connected continuum M there exists a finite chain $\{U_i\}_{i=1}^{n}$ of connected open sets which covers A and refines E. In particular $U_i \cap U_j$ is non-empty if and only if $|i - j| \leqq 1$ for $i, j = 1, 2, \cdots, n$. Moreover, this chain may be taken so that the diameters of its elements are small enough so that any three consecutive members are contained in some one element of E. Take a nonnegative integer k such that $n = 2k + r$ where $r = 0$ or $r = 1$. Let $U_0 = U_1$ and $U_{n+1} = U_n$. Also let

$$V_i = U_{2i-1} \cup U_{2i} \cup U_{2i+1}$$

for all $i = 1, 2, \cdots, k$. Now $\{V_i\}_{i=1}^{k}$ is a new chain of connected open sets which covers A. Since each element of this new chain is the union of three consecutive members of the origin chain then each such element is a connected open subset of a member of $\mathfrak{U}$ and hence is also a member of $\mathfrak{U}$. Moreover the intersection of any two adjacent elements of the new chain is an element of the origin chain and hence is connected. Let $U = V_1 \cup V_2 \cup \cdots \cup V_k$. Then U is a connected open set which contains A and, moreover, by repeated application of the preceding lemma it follows that $U \in \mathfrak{U}$.

LEMMA 4.4. *For any two points x and y in M there exists a homeomorphism h of M onto itself such that $h(x) = y$ and $W(h\sigma) = W(\sigma)$ for any loop σ in M.*

Proof. Suppose that x and y are any two points in M. Let C be an arc in M with ends x and y. Take an element V of $\mathfrak{U}$, according to the preceding lemma, so that $C \subset V$. Take a connected open set U such that $C \subset U \subset U^* \subset V$. Then U is also an element of $\mathfrak{U}$.

According to the proof of Theorem XVII, page 15 of [2] there exists a homeomorphism h of M onto itself such that $h(x) = y$ and h is the identity map on $M - U$.

Let σ be any loop in M. If $\sigma[I] \subset V$ then $W(h\sigma) = W(\sigma) = 0$. In the other case we may reparameterize σ so that $\sigma(0) = \sigma(1) \notin V$. Let $\mathfrak{A}$ be the collection of all components A of $\sigma^{-1}[V - U^*]$ such that $\sigma[A^*]$ meets both $M - V$ and U^*. From the fact that the distance from U^* to $M - V$ is positive it follows that $\mathfrak{A}$ is finite. Let $\{A_i\}_{1=1}^{n}$ be an indexing of the elements of $\mathfrak{A}$ such that $A_i < A_{i+1}$ for $i = 1, 2, \cdots, n-1$. Let $\{a_i\}_{i=0}^{n+1} \subset I$ be such that $a_0 = 0$; $a_{n+1} = 1$; and $a_i \in A_i$ for $i = 1, 2, \cdots, n$, Note that

$$0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = 1 .$$

For $i = 0, 1, 2, \cdots, n$ let σ_i be the restriction of σ to the interval $[a_i, a_{i+1}]$. Clearly $\sigma[a_0, a_1] \subset M - U$ for otherwise there would be an element A of $\mathfrak{A}$ between a_0 and A_1, contrary to the indexing of $\mathfrak{A}$. Analogously it follows that $\sigma[a_n, a_{n+1}] \subset M - U$. Suppose that i is any integer from 1 to $n - 1$. Then either $\sigma[a_i, a_{i+1}] \subset M - U$ or $\sigma[a_i, a_{i+1}] \subset V$ for otherwise there would be an element A of $\mathfrak{A}$ between A_i and A_{i+1} contrary to the indexing of $\mathfrak{A}$. If $\sigma[a_i, a_{i+1}] \subset M - U$ then $h\sigma_i = \sigma_i$ and $W(h\sigma_i) = W(\sigma_i)$. If $\sigma[a_i, a_{i+1}] \subset V$ then $h\sigma_i$ and σ_i are, modulo an order preserving change of parameter, paths in $V \in \mathfrak{U}$ from $\sigma(a_i)$ to $\sigma(a_{i+1})$ and therefore $W(h\sigma_i) = W(\sigma_i)$. Therefore

$$W(h\sigma) = \sum_{j=0}^{n} W(h\sigma_j) = \sum_{j=0}^{n} W(\sigma_j) = W(\sigma) .$$

LEMMA 4.5. *The continuum X is homogeneous.*

Proof. Take any $x, y \in X$. According to the preceding lemma take a homeomorphism h of M onto itself such that $h(p_0(x_0)) = p_0(y_0)$ and $W(h\sigma) = W(\sigma)$ for any loop σ in M.

Note. If α and β are paths in M from a to b then $W(h\alpha) - W(h\beta) = W(\alpha) - W(\beta)$ and therefore

$W(h\alpha) = W(h\beta) \bmod 2^n$ if and only if $W(\alpha) = W(\beta) \bmod 2^n$ for any $n \in \omega$.

For any $n \in \omega$ take $\gamma_n \in x_n$, take $\delta_n \in y_n$, and define a mapping h_n from X_n to itself by the formula

$$h_n c(\sigma, n) = c(\delta_n (h\gamma_n)^{-1} h\sigma, n)$$

for all $\sigma \in \Omega$.

In order to show that h_n is a well defined function take any $\sigma, \sigma^* \in c(\sigma, n)$, any $\gamma_n^* \in x_n$, and any $\delta_n^* \in Y_n$. Then $W(\sigma) = W(\sigma^*) \bmod 2^n$, $W(\gamma_n) = W(\gamma_n^*) \bmod 2^n$, and $W(\delta_n) = W(\delta_n^*) \bmod 2^n$. Also, according to the above note, $W(h\gamma_n) = W(h\gamma_n^*) \bmod 2^n$ and $W(h\sigma) = W(h\sigma^*) \bmod 2^n$. Therefore

$$\begin{aligned}[\delta_n(h\gamma_n)^{-1}h\sigma](1) = [h\sigma](1) = h(\sigma(1)) = h(\sigma^*(1)) \\ = [\delta_n^*(h\gamma_n^*)^{-1}h\sigma^*](1)\end{aligned}$$

and

$$W(\delta_n(h\gamma_n)^{-1}h\sigma) = W(\delta_n^*(h\gamma_n^*)^{-1}h\sigma^*) \bmod 2^n .$$

Therefore

$$c(\delta_n(h\gamma_n)^{-1}h\sigma, n) = c(\delta_n^*(h\gamma_n^*)^{-1}h\sigma^*, n)$$

and h_n is well defined.

Similarly, define the function q_n from X_n to itself by the formula

$$q_n c(\rho, n) = c(h^{-1}((h\gamma_n)\delta_n^{-1}\rho), n)$$

for all $\sigma \in \Omega$. It happens that q_n and h_n are inverse functions. Take any $\sigma \in \Omega$. Then

$$\begin{aligned} q_n h_n c(\sigma, n) &= q_n c(\delta_n(h\gamma_n)^{-1}h\sigma, n) \\ &= c(h^{-1}((h\gamma_n)\delta_n^{-1}\delta_n(h\gamma_n)^{-1}h\sigma), n) \\ &= c(h^{-1}((h\gamma_n)\delta_n^{-1}\delta_n(h\gamma_n)^{-1})h^{-1}h\sigma, n) \\ &= c(\sigma, n) . \end{aligned}$$

Take any $\rho \in \Omega$ then

$$\begin{aligned} h_n q_n c(\rho, n) &= h_n c(h^{-1}((h_n\gamma_n)\delta_n^{-1}\rho), n) \\ &= c(\delta_n(h\gamma_n)^{-1}hh^{-1}((h\gamma_n)\delta_n^{-1}\rho), n) \\ &= c(\delta_n(h\gamma_n)^{-1}(h\gamma_n)\cdot\delta_n^{-1}\rho, n) \\ &= c(\rho, n) . \end{aligned}$$

Therefore q_n is the inverse of h_n and h_n is a one-to-one mapping of X_n onto itself.

If (U, σ, n) is any element of Q then

$$h_n[N(U, \sigma, n)] = N(h[U], \delta_n(h\gamma_n)^{-1}h\sigma, n)$$

and if $(U, \rho, n) \in Q$ then

$$h_n^{-1}[N(U, \rho, n)] = N(h^{-1}[U], h^{-1}((h\gamma_n)\cdot\delta_n^{-1}\rho), n) .$$

Therefore both h_n and h_n^{-1} are continuous and hence h_n is a homeomorphism of X_n onto itself.

In order to apply the theory of [4; Chapter VII] about constructing the mapping on the limit space X from the mappings h_n it is necessary to establish the commutivity relation $h_n\phi_n = \phi_n h_{n+1}$. For any $n \in \omega$ and $\sigma \in \Omega$

$$h_n\phi_n c(\sigma, n + 1) = h_n c(\sigma, n) = c(\delta_n(h\gamma_n)^{-1}h\sigma, n)$$

and

$$\begin{aligned}\phi_n h_{n+1} c(\sigma, n+1) &= \phi_n c(\delta_{n+1}(h\gamma_{n+1})^{-1}h\sigma, n+1) \\ &= c(\delta_{n+1}(h\gamma_{n+1})^{-1}h\sigma, n) \\ &= c(\delta_n(h\gamma_n)^{-1}h\sigma, n)\ .\end{aligned}$$

Therefore $h_n\phi_n = \phi_n h_{n+1}$ for all $n \in \omega$.

Now define the mapping h^* from X to itself by the formula

$$h^*(z)_n = h_n(z_n)$$

for all $n \in \omega$ and $z \in X$. From the commutivity relation established above, the fact that h_n is a homeomorphism onto, and the results in [4; Chapter VIII] it follows that h^* is a homeomorphism of X onto itself. Moreover

$$h^*(x)_n = h_n(x_n) = h_n c(\gamma_n, n) = c(\delta_n(h\gamma_n)^{-1}h\gamma_n, n) = c(\delta_n, n) = y_n$$

for all $n \in \omega$. Therefore $h^*(x) = y$.

5. Additional remarks. If the following proposition were known to have been true this paper could have been materially shortened.

PROPOSITION 5.1. *If Y is the limit of the inverse system*

$$\{Y_i, \theta_i\}_{i=0}^{\infty}$$

where for each nonnegative integer i, Y_i is a homogeneous continuum and Y_{i+1} is a covering space of Y_i relative to the projection θ_i then Y is also a homogeneous continuum.

This is a special case of a theorem abstracted by Jack Segal [8]. However, to the author's knowledge the validity of Proposition 5.1 and of Segal's Theorem are open questions.

Another Question. Are the homogeneous 1-dimensional continua which contain arcs those continua which are inverse limits of simple closed curves or inverse limits of universal curves where in either case the bonding mappings are covering mappings.

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