LEBESGUE DENSITY AS A SET FUNCTION

NATHANIEL F. G. MARTIN
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Lebesgue (or metric) density is usually considered as a point function in the sense that a fixed subset of a space $X$ is given and then the value of the density of this set is obtained at various points of the space. Suppose the density is considered in another sense. That is, let a point $x$ of the space be fixed and consider the class $\mathcal{D}(x)$ of all sets whose density exists at this point. Then to each set $E$ in $\mathcal{D}(x)$ we assign the value of its density at $x$, and denote this number by $D_x(E)$. Thus from this point of view the density is a finite set function. It was shown in [2] that if the space $X$ is the real line then the image of $\mathcal{D}(x)$ under $D_x$ is the closed unit interval.

It is evident from the definition of density of sets of real numbers, which we give below, that $D_x$ is a finitely additive, subtractive, monotone, nonnegative set function and the class $\mathcal{D}(x)$ is closed under the formation of complements, proper differences, and disjoint unions. Therefore, if $\mathcal{D}(x)$ were closed under the formation of intersections, $D_x$ would be a finitely additive measure. This however is not the case for if

$$R_n = \left\{ x: \frac{1}{2} \left( \frac{1}{n} + \frac{1}{n+1} \right) < x < \frac{1}{n} \right\},$$

$$L_n = \left\{ x: -\frac{1}{n} < x < -\frac{1}{2} \left( \frac{1}{n} + \frac{1}{n+1} \right) \right\}$$

and

$$L_n^* = \left\{ x: -\frac{1}{2} \left( \frac{1}{n} + \frac{1}{n+1} \right) < x < -\frac{1}{n+1} \right\},$$

the sets $\bigcup_n (R_n \cup L_n) = E$ and $\bigcup_n (R_n \cup L_n^*) = F$ are members of $D(0)$ but $E \cap F$ is not. In fact $D_\infty(E) = D_\infty(F) = \frac{1}{2}$ and the upper density of $E \cap F$ at zero is not less than $\frac{1}{2}$ while the lower density of $E \cap F$ at zero is zero.

In part 1 of this note we prove a theorem which is somewhat of an analogue of the Lebesgue density theorem [3] in the following respect. As noted above $D_x$ is not a finitely additive measure, but we show that the upper density at $x$, $\bar{D}_x$, is a finitely subadditive outer measure defined on the class of all Lebesgue measurable subsets of $X$ and the class of $\bar{D}_x$-measurable sets is the class of all sets whose density exists at $x$ and has the value zero or one. In part 2 a Lebesgue density of a measurable set $E$ on a fixed $F_x$ set of measure zero is defined and a similar result

Received May 24, 1960. Presented to American Mathematical Society. Part 1 of this note constitutes a portion of the author’s doctoral dissertation written at Iowa State College under the direction of Professor H. P. Thielman.
proven for this function.

1. If $E$ is a measurable subset of the real line $X$ and $I$ is any interval we shall denote the relative Lebesgue measure of $E$ in $I$, $m(E \cap I)/m(I)$, by $\rho(E : I)$.

The upper Lebesgue density of a measurable subset $E$ of $X$ at a point $x \in X$, $\bar{D}_x(E)$, is defined by

$$\bar{D}_x(E) = \limsup_{I \to x} \rho(E : I) = \sup \{ \limsup_{k} \rho(E : I_k) : I_k \to x \}$$

and the lower Lebesgue density of a measurable set $E \subset X$ at a point $x \in X$, $\underline{D}_x(E)$, is defined by

$$\underline{D}_x(E) = \liminf_{I \to x} \rho(E : I) = \inf \{ \liminf_{k} \rho(E : I_k) : I_k \to x \},$$

where $I_k \to x$ means the sequence $\{I_k\}$ of intervals converges to $x$ in the sense that $x \in I_k$ for all $k$ and $m(I_k) \to 0$ as $k \to \infty$. In the case $\bar{D}_x(E) = \underline{D}_x(E)$ the common value is the Lebesgue density of $E$ at $x$ and will be denoted by $D_x(E)$.

**Lemma 1.** A necessary and sufficient condition that a set $E$ be a member of $\mathcal{D}(x)$ is that

$$\bar{D}_x(E) + \bar{D}_x(X - E) = 1.$$

**Proof.** The necessity is immediate. To obtain the sufficiency we note that for any interval $I$ containing $x$, $\rho(E : I) + \rho(X - E : I) = 1$ so that $\bar{D}_x(E) + \bar{D}_x(X - E) \geq 1$. Therefore

$$\bar{D}_x(X - E) \geq 1 - \bar{D}_x(E) = \bar{D}_x(X - E) + \underline{D}_x(E) - \bar{D}_x(E)$$

and it follows that $\bar{D}_x(E) \leq \underline{D}_x(E)$.

**Lemma 2.** The set function $\bar{D}_x$ is a finitely subadditive outer measure defined on the class $\mathcal{M}$ of all Lebesgue measurable subsets of the real line.

**Proof.** It is clear that $\bar{D}_x(\emptyset) = 0$ and $\bar{D}_x \geq 0$. Let $E \subset F$ be two sets from $M$. Then since $\rho(E : I) \leq \rho(F : I)$ for all intervals containing $x$, $\bar{D}_x$ is monotone. Let $E_1, E_2, \ldots, E_n$ be any finite collection of sets from $\mathcal{M}$. Since $\rho(\bigcup_{i=1}^{n} E_i : I) \leq \sum_{i=1}^{n} \rho(E_i : I)$ for all intervals $I$ containing $x$, we have

$$\bar{D}_x\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \limsup_{I \to x} \rho(E_i : I) = \sum_{i=1}^{n} \bar{D}_x(E_i).$$

Thus $\bar{D}_x$ is a finitely subadditive outer measure.

Let $\mathcal{M}(x)$ denote the class of all sets $E$ such that for every $A \in \mathcal{M}$,
\( \bar{D}_x(A) = \bar{D}_x(A \cap E) + \bar{D}_x(A - E) \). Since \( \mathcal{M}(x) \) contains \( X \) and \( \phi \mathcal{M}(x) \) is an algebra (in the sense of Halmos [1]) and the restriction of \( \bar{D}_x \) to \( \mathcal{M}(x) \) is a finitely additive measure.

**Lemma 3.** \( \mathcal{M}(x) \) is a subset of \( \mathcal{D}(x) \).

**Proof.** Let \( E \in \mathcal{M}(x) \). Since the real line \( X \) is a member of \( \mathcal{M} \) and \( \bar{D}_x(X) = 1 \), we have
\[
1 = \bar{D}_x(X) = \bar{D}_x(X \cap E) + \bar{D}_x(X - E) = \bar{D}_x(E) + \bar{D}_x(X - E)
\]
which by Lemma 1 gives \( E \in \mathcal{D}(x) \).

**Lemma 4.** If \( E \in \mathcal{D}(x) \) and \( J \) is any interval with \( x \) as one end point then \( \bar{D}_x(E \cap J) = D_x(E) \).

**Proof.** Let \( D_x(E) = d \). Since \( \bar{D}_x \) is monotone, \( d \geq \bar{D}_x(E \cap J) \) and if \( \{I_k\} \) is any sequence of intervals converging to \( x \), \( \limsup_k \rho((E \cap J) : I_k) \leq d \).

Suppose first that \( J \) is a bounded interval. If \( x \) is the left end point of \( J \), denote the right end point by \( y \) and let
\[
I^*_y = \left\{ z : x \leq z \leq x + \frac{1}{n}(y - x) \right\} ;
\]
if \( x \) is the right end point of \( J \), denote the left end point of \( J \) by \( y \) and let
\[
I^*_y = \left\{ z : x - \frac{1}{n}(x - y) \leq z \leq x \right\} .
\]
In either case \( I^*_y \to x \) and \( \rho(E : I^*_y) = \rho((E \cap J) : I^*_y) \) for all \( n \). Therefore, \( \lim_n \rho((E \cap J) : I^*_y) = d \) and we have \( \bar{D}_x(E \cap J) = D_x(E) \).

Suppose next that \( J \) is unbounded. If \( x \) is the left end point of \( J \) let \( I^*_x = \{ z : x \leq z \leq x + (1/n) \} \) and if \( x \) is the right end point of \( J \) let \( I^*_x = \{ z : x - (1/n) \leq z \leq x \} \). Again we have \( I^*_x \to x \) and \( \rho(E : I^*_x) = \rho((E \cap J) : I^*_x) \) for all \( n \) so that \( \bar{D}_x(E \cap J) = D_x(E) \).

**Lemma 5.** Let \( E \in \mathcal{D}(x) \) and let \( J \) be an interval open on the right with right end point at \( x \) and \( K \) be an interval closed on the left with left end point at \( x \). Define the set \( A \) by \( A = (E \cap K) \cup (J - E) \). Then \( \bar{D}_x(A) = \max \{ D_x(E), D_x(X - E) \} \).

**Proof.** Suppose \( D_x(X - E) \leq D_x(E) = d \). By Lemma 4, \( \bar{D}_x(J - E) = 1 - d \leq d \) and since \( \bar{D}_x \) is monotone, \( \bar{D}_x(A) \geq \bar{D}_x(E \cap K) = d \).

Let \( \varepsilon > 0 \) be given. Then there exists a sequence \( \{I^*_n\} \) converging to \( x \) such that
\[ D_x(A) < \lim \sup_k \rho(A : I_k^*) + \frac{\varepsilon}{2}. \]

For each \( k \), let \( J_k = I_k^* \cap (J \cup K) \). Since \( I_k^* \to x \), \( J_k^* \to x \) and \( \rho(A : I_k^*) = \rho(A : J_k) \) for all but a finite number of \( k \). Therefore

\[ (1) \quad D_x(A) < \lim \sup_k \rho(A : J_k) + \frac{\varepsilon}{2}. \]

For each interval \( J_k \) we have

\[
\rho(A : J_k) - d = \rho(K : J_k)[\rho(E : (K \cap J_k)) - d] \\
+ \rho(J : J_k)[\rho(X - E : (J \cap J_k)) - d].
\]

Since \( E \in \mathcal{E}(x) \) and \( K \cap J_k \to x \), \( \lim_k \rho(E : (K \cap J_k)) = d \). Since \( J \cap J_k \to x \), \( \lim_k \rho(X - E : (J \cap J_k)) = 1 - d \leq d \). Therefore there exist integers \( N_1 \) and \( N_2 \) such that for all \( k > N_1 \), \( \rho(E : (K \cap J_k)) - d < \varepsilon/2 \) and for all \( k > N_2 \), \( \rho(X - E : (J \cap J_k)) - d < \varepsilon/2 \). Thus for all \( k > \max \{N_1, N_2\} \)

\[
\rho(A : J_k) - d < \frac{\varepsilon}{2} \rho(K : J_k) + \frac{\varepsilon}{2} \rho(J : J_k) = \frac{\varepsilon}{2}.
\]

Therefore \( \lim \sup_k \rho(A : J_k) < d + \varepsilon/2 \) and we have by way of equation (1) that \( D_x(A) < d + \varepsilon \). Since \( \varepsilon \) was arbitrary, \( D_x(A) \leq d \) which completes the proof of the lemma.

**Theorem 1.** The class \( \mathcal{M}(x) \) of \( D_x \)-measurable sets is the class of all sets whose density exists at \( x \) and has the value 0 or 1.

**Proof.** First suppose \( E \in \mathcal{M}(x) \) and \( D_x(E) = d \). Let \( J = \{z : x - 1 \leq z < x\} \), \( K = \{z : x \leq z \leq x + 1\} \). Define the set \( A \) by \( A = (E \cap K) \cup (J - E) \). By Lemma 5, \( D_x(A) = \max \{1 - d, d\} \) and by Lemma 4, \( D_x(A \cap E) = D_x(E \cap K) = d \) and \( D_x(A - E) = D_x(J - E) = 1 - d \). Since \( E \in \mathcal{M}(x) \)

\[ 1 = d + 1 - d = D_x(A \cap E) + D_x(A - E) = D_x(A) = \max \{1 - d, d\}. \]

Therefore \( d = 0 \) or 1.

Next let \( E \) be a set whose density at \( x \) is zero or one. Let \( A \) be any Lebesgue measurable set and suppose \( D_x(E) = 0 \). Since \( D_x \) is monotone, \( D_x(A \cap E) \leq D_x(E) = 0 \) and hence \( D_x(A \cap E) = 0 \). Since \( D_x \) is an outer measure

\[ D_x(A - E) \geq D_x(A) - D_x(E) = D_x(A), \]

and since \( D_x \) is monotone \( D_x(A) \geq D_x(A - E) \). Therefore \( D_x(A) = D_x(A \cap E) + D_x(A - E) \) and \( E \) is in \( \mathcal{M}(x) \). In case \( D_x(E) = 1 \) the above argument with \( E \) replaced by \( X - E \) gives the desired result.

2. Suppose that \( Z \) represents an \( F_\sigma \) set of measure zero. Define
the upper Lebesgue density of a measurable set $E$ or $Z$ by
\[ \bar{D}_x(E) = \sup \{ D_x(E) : x \in Z \} \]
and the lower Lebesgue density of $E$ or $Z$ by
\[ D_x(E) = \inf \{ D(E) : x \in Z \} . \]
If $\bar{D}_x(E) = D_x(E)$ we will say that the Lebesgue density of $E$ on $Z$, denoted by $D_x(E)$, exists and has the common value of $\bar{D}_x(E)$ and $D_x(E)$. It is clear that if the density of $E$ exists on $Z$ then the density exists at every point of $Z$ and has the same value at each point. In [2] it was shown that for any number $d$ such that $0 < d < 1$, there exists a set $E$ such that $D_x(E) = d$. Thus if $\mathcal{D}(Z)$ denotes the class of all sets whose density on $Z$ exists, $D_x$ is a set function which maps $\mathcal{D}(Z)$ onto the closed unit interval. It is clear that $D_x$ will have the same properties as $D_x$ where $x$ is any point in $Z$.

**Lemma 7.** $\bar{D}_x$ is a finitely subadditive outer measure defined on the class $\mathcal{M}$.

*Proof.* The lemma follows immediately from the monotonicity and subadditivity of $\bar{D}_x$ and the definition of $\bar{D}_x$.

Let $\mathcal{M}(Z)$ denote the class of all sets $E$ such that $E \in \mathcal{M}$ and for every $A \in \mathcal{M}$, $\bar{D}_x(A) = \bar{D}_x(A \cap E) + \bar{D}_x(A - E)$. Then $\mathcal{M}(Z)$ is an algebra and the restriction of $\bar{D}_x$ to $\mathcal{M}(Z)$ is a finitely additive measure.

**Lemma 8.** $\mathcal{M}(Z)$ is a subset of $\mathcal{D}(Z)$.

*Proof.* Let $E \in \mathcal{M}(Z)$. The real line $X$ is in $\mathcal{M}$ so we have
\[ 1 = \bar{D}_x(X) = \bar{D}_x(E) + \bar{D}_x(X - E) \geq \sup \{ \bar{D}_x(E) + \bar{D}_x(X - E) : x \in Z \} \]
and
\[ \bar{D}_x(E) + \bar{D}_x(X - E) \leq 1 \]
for all $x \in Z$. But for any $x \in Z$, $\bar{D}_x$ is subadditive so that $\bar{D}_x((E) + \bar{D}_x(X - E) \geq 1$. Therefore $\bar{D}_x(E) + \bar{D}_x(X - E) = 1$ for all $x \in Z$ and by Lemma 1, the density of $E$ exists at every point of $Z$. Hence $\bar{D}_x(E) + \bar{D}_x(X - E) = 1$ for all $x$ in $Z$ and
\[ D_x(E) + \bar{D}_x(X - E) = \inf \{ D(E) + \bar{D}_x(E) : x \in Z \} \]
\[ = 1 = \bar{D}_x(E) + \bar{D}_x(X - E) . \]
Since $\bar{D}_x$ if finite, $D_x(E) \geq \bar{D}_x(E)$ and it follows that $E \in \mathcal{D}(Z)$.

**Theorem 2.** The class of all $\bar{D}_x$-measurable sets is the class of
all sets from $\mathcal{D}(Z)$ which are mapped onto 0 or 1 by $D_z$.

**Proof.** Let $\mathcal{H} = \{E : E \in D(Z) \text{ and } D_z(E) = 0 \text{ or } 1\}$. If $E \in \mathcal{H}$ we may show that $E \in \mathcal{M}(Z)$ exactly as was done in Theorem 1.

Suppose $E \in \mathcal{H}(Z)$. By Lemma 8, $E \in \mathcal{D}(Z)$ and hence $D_z(E) = D_z(E) = d$ for all $x \in Z$. Let $x_1$ be any point in $Z$ and let $J = \{z : z < x_1\}$, $K = \{z : z \geq x_1\}$. Define the set $A$ by $A = (J - E) \cup (E \cap K)$. Then by Lemmas 4 and 5, $\bar{D}_{x_1}(A) = \max\{d, 1 - d\}$, $\bar{D}_{x_1}(A \cap E) = d$, and $\bar{D}_{x_1}(A - E) = 1 - d$. Since $A \in \mathcal{M}$ and $E \in \mathcal{M}(Z)$, $\sup \{\bar{D}_x(A) : x \in Z\} = \sup \{\bar{D}_x(A \cap E) + \bar{D}_x(A - E) : x \in Z\}.$

Let $\varepsilon > 0$ be given. Then there exists an $x_2 \in Z$ such that

$$\bar{D}_{x_2}(A) + \varepsilon > \sup \{\bar{D}_x(A \cap E) + \bar{D}_x(A - E) : x \in Z\} \geq \bar{D}_{x_1}(A \cap E) + \bar{D}_{x_1}(A - E) = 1.$$ 

Suppose $x_2 < x_1$. Then $\bar{D}_{x_2}(A) = D_{x_2}(X - E)$ and $1 - d + \varepsilon > 1$. Since $\varepsilon$ was arbitrary and $1 - d \leq 1$ we have $1 - d = 1$ and hence $d = 0$.

Suppose $x_2 > x_1$. Then $\bar{D}_{x_2}(A) = D_{x_2}(E)$ and $d + \varepsilon > 1$. Since $\varepsilon$ was arbitrary and $d \leq 1$ we have $d = 1$.

Suppose $x_2 = x_1$. Then $\bar{D}_{x_2}(A) = \max\{d, 1 - d\}$, and $\max\{d, 1 - d\} + \varepsilon > 1$. Since $\varepsilon$ was arbitrary $\max\{d, 1 - d\} \geq 1$. But both $d$ and $1 - d$ do not exceed 1 so that $d = 0$ or 1.

Therefore $E$ is in $\mathcal{H}$ and we have $\mathcal{M}(Z) = \mathcal{H}$.

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The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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