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LEBESGUE DENSITY AS A SET FUNCTION

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Lebesgue (or metric) density is usually considered as a point function in the sense that a fixed subset of a space X is given and then the value of the density of this set is obtained at various points of the space. Suppose the density is considered in another sense. That is, let a point x of the space be fixed and consider the class $\mathcal{D}(x)$ of all sets whose density exists at this point. Then to each set E in $\mathcal{D}(x)$ we assign the value of its density at x , and denote this number by $D_x(E)$. Thus from this point of view the density is a finite set function. It was shown in [2] that if the space X is the real line then the image of $\mathcal{D}(x)$ under D_x is the closed unit interval.

It is evident from the definition of density of sets of real numbers, which we give below, that D_x is a finitely additive, subtractive, monotone, nonnegative set function and the class $\mathcal{D}(x)$ is closed under the formation of complements, proper differences, and disjoint unions. Therefore, if $\mathcal{D}(x)$ were closed under the formation of intersections, D_x would be a finitely additive measure. This however is not the case for if

$$R_n = \left\{ x: \frac{1}{2} \left(\frac{1}{n} + \frac{1}{n+1} \right) < x < \frac{1}{n} \right\},$$

$$L_n = \left\{ x: -\frac{1}{n} < x < -\frac{1}{2} \left(\frac{1}{n} + \frac{1}{n+1} \right) \right\}$$

and

$$L_n^* = \left\{ x: -\frac{1}{2} \left(\frac{1}{n} + \frac{1}{n+1} \right) < x < -\frac{1}{n+1} \right\},$$

the sets $\bigcup_n (R_n \cup L_n) = E$ and $\bigcup_n (R_n \cup L_n^*) = F$ are members of $D(0)$ but $E \cap F$ is not. In fact $D_0(E) = D_0(F) = \frac{1}{2}$ and the upper density of $E \cap F$ at zero is not less than $\frac{1}{2}$ while the lower density of $E \cap F$ at zero is zero.

In part 1 of this note we prove a theorem which is somewhat of an analogue of the Lebesgue density theorem [3] in the following respect. As noted above D_x is not a finitely additive measure, but we show that the upper density at x , \bar{D}_x , is a finitely subadditive outer measure defined on the class of all Lebesgue measurable subsets of X and the class of \bar{D}_x -measurable sets is the class of all sets whose density exists at x and has the value zero or one. In part 2 a Lebesgue density of a measurable set E on a fixed F_x set of measure zero is defined and a similar result

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proven for this function.

1. If E is a measurable subset of the real line X and I is any interval we shall denote the relative Lebesgue measure of E in I , $m(E \cap I)/m(I)$, by $\rho(E : I)$.

The upper Lebesgue density of a measurable subset E of X at a point $x \in X$, $\bar{D}_x(E)$, is defined by

$\bar{D}_x(E) = \limsup_{I \rightarrow x} \rho(E : I) = \sup \{ \limsup_k \rho(E : I_k) : I_k \rightarrow x \}$ and the lower Lebesgue density of a measurable set $E \subset X$ at a point $x \in X$, $\underline{D}_x(E)$, is defined by

$$\underline{D}_x(E) = \liminf_{I \rightarrow x} \rho(E : I) = \inf \{ \lim_k \rho(E : I_k) : I_k \rightarrow x \},$$

where $I_k \rightarrow x$ means the sequence $\{I_k\}$ of intervals converges to x in the sense that $x \in \bar{I}_k$ for all k and $m(I_k) \rightarrow 0$ as $k \rightarrow \infty$. In the case $\underline{D}_x(E) = \bar{D}_x(E)$ the common value is the Lebesgue density of E at x and will be denoted by $D_x(E)$.

LEMMA 1. *A necessary and sufficient condition that a set E be a member of $\mathcal{D}(x)$ is that*

$$\bar{D}_x(E) + \bar{D}_x(X - E) = 1.$$

Proof. The necessity is immediate. To obtain the sufficiency we note that for any interval I containing x , $\rho(E : I) + \rho(X - E : I) = 1$ so that $\underline{D}_x(E) + \bar{D}_x(X - E) \geq 1$. Therefore

$$\bar{D}_x(X - E) \geq 1 - \underline{D}_x(E) = \bar{D}_x(X - E) + \bar{D}_x(E) - \underline{D}_x(E)$$

and it follows that $\bar{D}_x(E) \leq \underline{D}_x(E)$.

LEMMA 2. *The set function \bar{D}_x is a finitely subadditive outer measure defined on the class \mathcal{M} of all Lebesgue measurable subsets of the real line.*

Proof. It is clear that $\bar{D}_x(\phi) = 0$ and $\bar{D}_x \geq 0$. Let $E \subset F$ be two sets from \mathcal{M} . Then since $\rho(E : I) \leq \rho(F : I)$ for all intervals containing x , \bar{D}_x is monotone. Let E_1, E_2, \dots, E_n be any finite collection of sets from \mathcal{M} . Since $\rho(\bigcup_{i=1}^n E_i : I) \leq \sum_{i=1}^n \rho(E_i : I)$ for all intervals I containing x , we have

$$\bar{D}_x\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n \limsup_{I \rightarrow x} \rho(E_i : I) = \sum_{i=1}^n \bar{D}_x(E_i).$$

Thus \bar{D}_x is a finitely subadditive outer measure.

Let $\mathcal{M}(x)$ denote the class of all sets E such that for every $A \in \mathcal{M}$,

$\bar{D}_x(A) = \bar{D}_x(A \cap E) + \bar{D}_x(A - E)$. Since $\mathcal{M}(x)$ contains X and $\phi \mathcal{M}(x)$ is an algebra (in the sense of Halmos [1]) and the restriction of \bar{D}_x to $\mathcal{M}(x)$ is a finitely additive measure.

LEMMA 3. $\mathcal{M}(x)$ is a subset of $\mathcal{D}(x)$.

Proof. Let $E \in \mathcal{M}(x)$. Since the real line X is a member of \mathcal{M} and $\bar{D}_x(X) = 1$, we have

$$1 = \bar{D}_x(X) = \bar{D}_x(X \cap E) + \bar{D}_x(X - E) = \bar{D}_x(E) + \bar{D}_x(X - E)$$

which by Lemma 1 gives $E \in \mathcal{D}(x)$.

LEMMA 4. If $E \in \mathcal{D}(x)$ and J is any interval with x as one end point then $\bar{D}_x(E \cap J) = D_x(E)$.

Proof. Let $D_x(E) = d$. Since \bar{D}_x is monotone, $d \geq \bar{D}_x(E \cap J)$ and if $\{I_k\}$ is any sequence of intervals converging to x , $\limsup_k \rho((E \cap J) : I_k) \leq d$.

Suppose first that J is a bounded interval. If x is the left end point of J , denote the right end point by y and let

$$I_n^* = \left\{ z : x \leq z \leq x + \frac{1}{n}(y - x) \right\};$$

if x is the right end point of J , denote the left end point of J by y and let

$$I_n^* = \left\{ z : x - \frac{1}{n}(x - y) \leq z \leq x \right\}.$$

In either case $I_n^* \rightarrow x$ and $\rho(E : I_n^*) = \rho((E \cap J) : I_n^*)$ for all n . Therefore, $\lim_n \rho((E \cap J) : I_n^*) = d$ and we have $\bar{D}_x(E \cap J) = D_x(E)$.

Suppose next that J is unbounded. If x is the left end point of J let $I_n^* = \{z : x \leq z \leq z + (1/n)\}$ and if x is the right end point of J let $I_n^* = \{z : x - (1/n) \leq z \leq x\}$. Again we have $I_n^* \rightarrow x$ and $\rho(E : I_n^*) = \rho((E \cap J) : I_n^*)$ for all n so that $\bar{D}_x(E \cap J) = D_x(E)$.

LEMMA 5. Let $E \in \mathcal{D}(x)$ and let J be an interval open on the right with right end point at x and K be an interval closed on the left with left end point at x . Define the set A by $A = (E \cap K) \cup (J - E)$. Then $\bar{D}_x(A) = \max \{D_x(E), D_x(X - E)\}$.

Proof. Suppose $D_x(X - E) \leq D_x(E) = d$. By Lemma 4, $\bar{D}_x(J - E) = 1 - d \leq d$ and since \bar{D}_x is monotone, $\bar{D}_x(A) \geq \bar{D}_x(E \cap K) = d$.

Let $\epsilon > 0$ be given. Then there exists a sequence $\{I_k^*\}$ converging to x such that

$$\bar{D}_x(A) < \limsup_k \rho(A : I_k^*) + \frac{\varepsilon}{2} .$$

For each k , let $J_k = I_k^* \cap (J \cup K)$. Since $I_k^* \rightarrow x, J_k^* \rightarrow x$ and $\rho(A : I_k^*) = \rho(A : J_k)$ for all but a finite number of k . Therefore

$$(1) \quad \bar{D}_x(A) < \limsup_k \rho(A : J_k) + \varepsilon/2 .$$

For each interval J_k we have

$$\begin{aligned} \rho(A : J_k) - d &= \rho(K : J_k)[\rho(E : (K \cap J_k)) - d] \\ &\quad + \rho(J : J_k)[\rho(X - E : (J \cap J_k)) - d] . \end{aligned}$$

Since $E \in \mathcal{D}(x)$ and $K \cap J_k \rightarrow x, \lim_k \rho(E : (K \cap J_k)) = d$. Since $J \cap J_k \rightarrow x, \lim_k \rho(X - E : (J \cap J_k)) = 1 - d \leqq d$. Therefore there exist integers N_1 and N_2 such that for all $k > N_1, \rho(E : (K \cap J_k)) - d < \varepsilon/2$ and for all $k > N_2, \rho(X - E : (J \cap J_k)) - d < \varepsilon/2$. Thus for all $k > \max \{N_1, N_2\}$

$$\rho(A : J_k) - d < \frac{\varepsilon}{2} \rho(K : J_k) + \frac{\varepsilon}{2} \rho(J : J_k) = \frac{\varepsilon}{2} .$$

Therefore $\limsup_k \rho(A : J_k) < d + \varepsilon/2$ and we have by way of equation (1) that $\bar{D}_x(A) < d + \varepsilon$. Since ε was arbitrary, $\bar{D}_x(A) \leqq d$ which completes the proof of the lemma.

THEOREM 1. *The class $\mathcal{M}(x)$ of \bar{D}_x -measurable sets is the class of all sets whose density exists at x and has the value 0 or 1.*

Proof. First suppose $E \in \mathcal{M}(x)$ and $D_x(E) = d$. Let $J = \{z : x - 1 \leqq z < x\}, K = \{z : x \leqq z \leqq x + 1\}$. Define the set A by $A = (E \cap K) \cup (J - E)$. By Lemma 5, $\bar{D}_x(A) = \max \{1 - d, d\}$ and by Lemma 4, $\bar{D}_x(A \cap E) = \bar{D}_x(E \cap K) = d$ and $\bar{D}_x(A - E) = \bar{D}_x(J - E) = 1 - d$. Since $E \in \mathcal{M}(x)$

$$1 = d + 1 - d = \bar{D}_x(A \cap E) + \bar{D}_x(A - E) = \bar{D}_x(A) = \max \{1 - d, d\} .$$

Therefore $d = 0$ or 1 .

Next let E be a set whose density at x is zero or one. Let A be any Lebesgue measurable set and suppose $D_x(E) = 0$. Since \bar{D}_x is monotone, $\bar{D}_x(A \cap E) \leqq D_x(E) = 0$ and hence $\bar{D}_x(A \cap E) = 0$. Since \bar{D}_x is an outer measure

$$\bar{D}_x(A - E) \geqq \bar{D}_x(A) - \bar{D}_x(E) = \bar{D}_x(A) ,$$

and since \bar{D}_x is monotone $\bar{D}_x(A) \geqq \bar{D}_x(A - E)$. Therefore $\bar{D}_x(A) = \bar{D}_x(A \cap E) + \bar{D}_x(A - E)$ and E is in $\mathcal{M}(x)$. In case $D_x(E) = 1$ the above argument with E replaced by $X - E$ gives the desired result.

2. Suppose that Z represents an F_σ set of measure zero. Define

the upper Lebesgue density of a measurable set E or Z by

$$\bar{D}_z(E) = \sup \{\bar{D}_x(E) : x \in Z\}$$

and the lower Lebesgue density of E or Z by

$$D_z(E) = \inf \{D_x(E) : x \in Z\} .$$

If $\bar{D}_z(E) = D_z(E)$ we will say that the Lebesgue density of E on Z , denoted by $D_z(E)$, exists and has the common value of $\bar{D}_z(E)$ and $D_z(E)$. It is clear that if the density of E exists on Z then the density exists at every point of Z and has the same value at each point. In [2] it was shown that for any number d such that $0 < d < 1$, there exists a set E such that $D_z(E) = d$. Thus if $\mathcal{D}(Z)$ denotes the class of all sets whose density on Z exists, D_z is a set function which maps $\mathcal{D}(Z)$ onto the closed unit interval. It is clear that D_z will have the same properties as D_x where x is any point in Z .

LEMMA 7. \bar{D}_z is a finitely subadditive outer measure defined on the class \mathcal{M} .

Proof. The lemma follows immediately from the monotonicity and subadditivity of \bar{D}_x and the definition of \bar{D}_z .

Let $\mathcal{M}(Z)$ denote the class of all sets E such that $E \in \mathcal{M}$ and for every $A \in \mathcal{M}$, $\bar{D}_z(A) = \bar{D}_z(A \cap E) + \bar{D}_z(A - E)$. Then $\mathcal{M}(Z)$ is an algebra and the restriction of \bar{D}_z to $\mathcal{M}(Z)$ is a finitely additive measure.

LEMMA 8. $\mathcal{M}(Z)$ is a subset of $\mathcal{D}(Z)$.

Proof. Let $E \in \mathcal{M}(Z)$. The real line X is in \mathcal{M} so we have

$$1 = \bar{D}_z(X) = \bar{D}_z(E) + \bar{D}_z(X - E) \geq \sup \{\bar{D}_x(E) + \bar{D}_x(X - E) : x \in Z\}$$

and

$$\bar{D}_x(E) + \bar{D}_x(X - E) \leq 1$$

for all $x \in Z$. But for any $x \in Z$, \bar{D}_x is subadditive so that $\bar{D}_x(E) + \bar{D}_x(X - E) \geq 1$. Therefore $\bar{D}_x(E) + \bar{D}_x(X - E) = 1$ for all $x \in Z$ and by Lemma 1, the density of E exists at every point of Z . Hence $D_x(E) + D_x(X - E) = 1$ for all x in Z and

$$\begin{aligned} D_z(E) + \bar{D}_z(X - E) &\geq \inf \{D_x(E) + \bar{D}_x(X - E) : x \in Z\} \\ &= 1 = \bar{D}_z(E) + \bar{D}_z(X - E) . \end{aligned}$$

Since \bar{D}_z is finite, $D_z(E) \geq \bar{D}_z(E)$ and it follows that $E \in \mathcal{D}(Z)$.

THEOREM 2. The class of all \bar{D}_z -measurable sets is the class of

all sets from $\mathcal{D}(Z)$ which are mapped onto 0 or 1 by D_z .

Proof. Let $\mathcal{K} = \{E : E \in D(Z) \text{ and } D_z(E) = 0 \text{ or } 1\}$. If $E \in \mathcal{K}$ we may show that $E \in \mathcal{M}(Z)$ exactly as was done in Theorem 1.

Suppose $E \in \mathcal{M}(Z)$. By Lemma 8, $E \in \mathcal{D}(Z)$ and hence $D_z(E) = D_x(E) = d$ for all $x \in Z$. Let x_1 be any point in Z and let $J = \{z : z < x_1\}$, $K = \{z : z \geq x_1\}$. Define the set A by $A = (J - E) \cup (E \cap K)$. Then by Lemmas 4 and 5, $\bar{D}_{x_1}(A) = \max\{d, 1 - d\}$, $\bar{D}_{x_1}(A \cap E) = d$, and $\bar{D}_{x_1}(A - E) = 1 - d$. Since $A \in \mathcal{M}$ and $E \in \mathcal{M}(Z)$,

$$\sup\{\bar{D}_x(A) : x \in Z\} = \sup\{\bar{D}_x(A \cap E) + \bar{D}_x(A - E) : x \in Z\}.$$

Let $\varepsilon > 0$ be given. Then there exists an $x_2 \in Z$ such that

$$\begin{aligned} \bar{D}_{x_2}(A) + \varepsilon &> \sup\{\bar{D}_x(A \cap E) + \bar{D}_x(A - E) : x \in Z\} \\ &\geq \bar{D}_{x_1}(A \cap E) + \bar{D}_{x_1}(A - E) = 1. \end{aligned}$$

Suppose $x_2 < x_1$. Then $\bar{D}_{x_2}(A) = D_{x_2}(X - E)$ and $1 - d + \varepsilon > 1$. Since ε was arbitrary and $1 - d \leq 1$ we have $1 - d = 1$ and $d = 0$.

Suppose $x_2 > x_1$. Then $\bar{D}_{x_2}(A) = D_{x_2}(E)$ and $d + \varepsilon > 1$. Since ε was arbitrary and $d \leq 1$ we have $d = 1$.

Suppose $x_2 = x_1$. Then $\bar{D}_{x_2}(A) = \max\{d, 1 - d\}$, and $\max\{d, 1 - d\} + \varepsilon > 1$. Since ε was arbitrary $\max\{d, 1 - d\} \geq 1$. But both d and $1 - d$ do not exceed 1 so that $d = 0$ or 1.

Therefore E is in \mathcal{K} and we have $\mathcal{M}(Z) = \mathcal{K}$.

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| | |
|---|-----|
| Tsuyoshi Andô, <i>Convergent sequences of finitely additive measures</i> | 395 |
| Richard Arens, <i>The analytic-functional calculus in commutative topological algebras</i> | 405 |
| Michel L. Balinski, <i>On the graph structure of convex polyhedra in n-space</i> | 431 |
| R. H. Bing, <i>Tame Cantor sets in E^3</i> | 435 |
| Cecil Edmund Burgess, <i>Collections and sequences of continua in the plane. II</i> | 447 |
| J. H. Case, <i>Another 1-dimensional homogeneous continuum which contains an arc</i> | 455 |
| Lester Eli Dubins, <i>On plane curves with curvature</i> | 471 |
| A. M. Duguid, <i>Feasible flows and possible connections</i> | 483 |
| Lincoln Kearney Durst, <i>Exceptional real Lucas sequences</i> | 489 |
| Gertrude I. Heller, <i>On certain non-linear operators and partial differential equations</i> | 495 |
| Calvin Virgil Holmes, <i>Automorphisms of monomial groups</i> | 531 |
| Wu-Chung Hsiang and Wu-Yi Hsiang, <i>Those abelian groups characterized by their completely decomposable subgroups of finite rank</i> | 547 |
| Bert Hubbard, <i>Bounds for eigenvalues of the free and fixed membrane by finite difference methods</i> | 559 |
| D. H. Hyers, <i>Transformations with bounded mth differences</i> | 591 |
| Richard Eugene Isaac, <i>Some generalizations of Doeblin's decomposition</i> | 603 |
| John Rolfe Isbell, <i>Uniform neighborhood retracts</i> | 609 |
| Jack Carl Kiefer, <i>On large deviations of the empiric $D. F.$ of vector chance variables and a law of the iterated logarithm</i> | 649 |
| Marvin Isadore Knopp, <i>Construction of a class of modular functions and forms. II</i> | 661 |
| Gunter Lumer and R. S. Phillips, <i>Dissipative operators in a Banach space</i> | 679 |
| Nathaniel F. G. Martin, <i>Lebesgue density as a set function</i> | 699 |
| Shu-Teh Chen Moy, <i>Generalizations of Shannon-McMillan theorem</i> | 705 |
| Lucien W. Neustadt, <i>The moment problem and weak convergence in L^2</i> | 715 |
| Kenneth Allen Ross, <i>The structure of certain measure algebras</i> | 723 |
| James F. Smith and P. P. Saworotnow, <i>On some classes of scalar-product algebras</i> | 739 |
| Dale E. Varberg, <i>On equivalence of Gaussian measures</i> | 751 |
| Avrum Israel Weinzweig, <i>The fundamental group of a union of spaces</i> | 763 |