1. Introduction. Let $X$ be a non-empty set and $\mathcal{F}$ be a $\sigma$-algebra of subsets of $X$. Consider the infinite product space $\Omega = \prod_{n=-\infty}^{\infty} X_n$ where $X_n = X$ for $n = 0, \pm 1, \pm 2, \cdots$ and the infinite product $\sigma$-algebra $\mathcal{F} = \prod_{n=-\infty}^{\infty} \mathcal{F}_n$ where $\mathcal{F}_n = \mathcal{F}$ for $n = 0, \pm 1, \pm 2, \cdots$. Elements of $\Omega$ are bilateral infinite sequences $\{\cdots, x_{-1}, x_0, x_1, \cdots\}$ with $x_n \in X$. Let us denote the elements of $\Omega$ by $\omega$. If $\omega = \{\cdots, x_{-1}, x_0, x_1, \cdots\}$ $x_n$ is called the $n$th coordinate of $\omega$ and shall be considered as a function on $\Omega$ to $X$. Let $T$ be the shift transformation on $\Omega$ to $\Omega$: the $n$th coordinate of $T\omega$ is equal to the $(n + 1)$th coordinate of $\omega$. For any function $g$ on $\Omega$, $Tg$ is the function defined by $Tg(\omega) = g(T\omega)$ so that $Tx_n = x_{n+1}$. We shall consider two probability measures $\mu, \nu$ defined on $\mathcal{F}$. Let $\Omega_n = \prod_{i=1}^{n} X_i$ where $X_i = X$, $i = 1, 2, \cdots, n$ and $\mathcal{F}_n = \prod_{i=1}^{n} \mathcal{F}_i$ where $\mathcal{F}_i = \mathcal{F}$, $i = 1, 2, \cdots, n$. Then $\Omega_i = X$ and $\mathcal{F}_i = \mathcal{F}$. Let $\mathcal{F}_m,n$, $m \leq n$, $n = 0, \pm 1, \pm 2, \cdots$, be the $\sigma$-algebra of subsets of $\Omega$ consisting of sets of the form

$$[\omega = \{\cdots, x_{-1}, x_0, x_1, \cdots\} : (x_m, x_{m+1}, \cdots, x_n) \in E]$$

where $E \in \mathcal{F}_{n-m+1}$. Let $\mathcal{F}_{-\infty,n}$ be the $\sigma$-algebra generated by $\bigcup_{n=-\infty}^{\infty} \mathcal{F}_{m,n}$. Let $\mu_{m,n}, \nu_{m,n}$ be the contractions of $\mu, \nu$, respectively, to $\mathcal{F}_{m,n}$ and $\mathcal{F}_{-\infty,n}$. Throughout this paper $\nu_{m,n}$ is assumed to be absolutely continuous with respect to $\mu_{m,n}, \nu_{m,n} \ll \mu_{m,n}$, for $m < n$, $n = 0, \pm 1, \pm 2, \cdots$. Let $f_{m,n}$ be the derivative of $\nu_{m,n}$ with respect to $\mu_{m,n}, f_{m,n} = d\nu_{m,n}/d\mu_{m,n}$. $f_{m,n}$ is $\mathcal{F}_{m,n}$ measurable and nonnegative. $f_{m,n}$ is also positive with $\nu$ probability one. Hence $1/f_{m,n}$ is well defined with $\nu$ probability one. A fundamental theorem of Information Theory by Shannon and McMillan may be considered as a theorem concerning the asymptotic properties of $f_{m,n}$ as $n \to \infty$. The theorem may be stated as follows: Let $X$ be a finite set of $K$ points and $\mathcal{F}$ be the $\sigma$-algebra of all subsets of $X$. Let $\nu$ be any stationary ($T$ invariant) probability measure on $\mathcal{F}$ and $\mu$ be the equally distributed independent (product) measure. Then $n^{-1} \log f_{1,n}$ converges in $L_1(\nu)$. In particular, if $\nu$ is ergodic, the limit function is equal to $\log K - H$ with $\nu$ probability one where $H$ is the entropy of $\nu$ measure [3] [8]. Generalizations to arbitrary $X, \mathcal{F}$ were first studied by A. Pérez. He introduced an $A_\mu$ condition on $\nu$ as follows. $\nu$ is said to satisfy $A_\mu$ condition if $\nu_{-\infty,n}$ is absolutely continuous with respect to $\nu_{-\infty,0}, \mu_{1,n}$ for $n = 1, 2, \cdots$. He proved the following theorem. If $\nu, \mu$ are stationary and $\mu$ is the product (independent) measure on $\mathcal{F}$ and if

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(a) $\lim_{n\to\infty} n^{-1} \int \log f_{1,n} d\nu$ exists and is finite,
(b) $\nu$ satisfies condition $A_\mu$,
then $\{n^{-1} \log f_{1,n}\}$ converges in $L_\nu$ [6]. Later Pérez announced that the theorem remains to be true for any stationary measures $\mu, \nu$ [8].
The present writer proved that for Markovian $\mu, \nu$ with $\nu$ being stationary and $\mu$ having stationary transition probabilities the $\nu$-integrability of $\log f_{1,2}$ implies the $L_\nu$ convergence of $\{n^{-1} \log f_{1,n}\}$. The proof is based on an iteration formula for $f_{1,n}[4]$. In this paper we shall study the case that $\nu$ is stationary and $\mu$ is Markovian with stationary transition probabilities. It shall be proved that the condition

(c) $\int (\log f_{1,n} - \log f_{1,n-1}) d\nu \leq M < \infty$ for $n = 1, 2, 3, \ldots$ implies the $L_\nu$ convergence of $\{n^{-1} \log f_{1,n}\}$. In fact the conditions (c) and (a) are equivalent for this case, so that the theorem is a generalization of the theorem of Pérez given in [6]. The proof is conducted along similar lines used by McMillan. The crucial step is proving the $L_\nu$ convergence of $\{\log f_{-n,0} - \log f_{-n,-1}\}$. The condition (c) is shown to be necessary and sufficient for this convergence.

2. Generalizations of Shannon-McMillan theorem. Let $x, \mathcal{X}, \Omega,$ $\mathcal{F}, \Omega_n, \mathcal{F}_n, \mathcal{F}_{m,n}, \mu_{m,n}, \nu_{m.n}, f_{m,n}$ be as in I. Notations for conditional probabilities and conditional expectations relative to one or several random variables will be as in [2], Chapter 1, § 7. A probability measure on $\mathcal{F}$ is Markovian if, for any $A \in \mathcal{X}$, $m < n$, $n = 0, \pm 1, \pm 2, \cdots$

$$P[x_n \in A \mid x_{n-m}, \cdots, x_{n-1}] = P[x_n \in A \mid x_{n-1}]$$

with probability one. A Markovian measure is said to have stationary transition probabilities if for any $A \in \mathcal{X}$ and any integer $n$

$$P[x_n \in A \mid x_{n-1}] = T^n P[x_0 \in A \mid x_{-1}]$$

with probability one. In this paper, since we have two probability measures $\mu, \nu$, we need to use subscripts $\mu, \nu$ to indicate conditional probabilities and conditional expectations taken under $\mu, \nu$ respectively. For any $E \subset \Omega, I_E$, the indicator of $E$, is the real valued function on $\Omega$ defined by

$$I_E(\omega) = 1 \quad \text{if} \quad \omega \in E$$
$$= 0 \quad \text{if} \quad \omega \notin E .$$

The log in this paper is the logarithm with base 2.

**Lemma 1.** Define $\nu'_{m,n}$ on $\mathcal{F}_{m,n}$ by

(1) $\nu'_{m,n}(E) = \int P_{\mu}[E \mid x_m, \cdots, x_{n-1}] d\nu$, 

then \( \nu'_{m,n} \) is a probability measure on \( \mathcal{F}_{m,n} \) with \( \nu'_{m,n}(E) = \nu_{m,n}(E) \) for \( E \in \mathcal{F}_{m,n-1} \). Furthermore \( \nu_{m,n} \ll \nu'_{m,n} \) with

\[
d\nu_{m,n} / d\nu'_{m,n} = f_{m,n} / f_{m,n-1}.
\]

**Proof.**

\[
\nu'_{m,n}(E) = \int P_{\mu}[E | x_m, \ldots, x_{n-1}] d\nu
= \int P_{\mu}[E | x_m, \ldots, x_{n-1}] f_{m,n} d\mu
= \int E_{\mu}[E_{|f_{m,n-1}} | x_m, \ldots, x_{n-1}] d\mu
= \int f_{m,n} d\mu.
\]

Hence \( \nu_{m,n} \) is a probability measure on \( \mathcal{F}_{m,n} \). Furthermore, for \( E \in \mathcal{F}_{m,n} \)

\[
\nu_{m,n}(E) = \int f_{m,n} d\mu = \int (f_{m,n} / f_{m,n-1}) f_{m,n-1} d\mu
= \int (f_{m,n} / f_{m,n-1}) d\nu'_{m,n}.
\]

Hence \( \nu_{m,n} \) is absolutely continuous with respect to \( \nu'_{m,n} \) and \( d\nu_{m,n} / d\nu'_{m,n} = f_{m,n} / f_{m,n-1} \).

**THEOREM 1.** If \( \nu \) is stationary and \( \mu \) is Markovian with stationary transition probabilities then

\[
f_{m,n} / f_{m,n-1} = T^s(f_{m-n,0} / f_{m-n,-1})
\]

with \( \nu \) probability one for all \( m < n, n = 0, \pm 1, \pm 2, \ldots \).

**Proof.** If \( \mu \) is Markovian and has stationary transition probabilities then for any \( A \in \mathcal{A} \)

\[
P_{\mu}[x_n \in A | x_m, \ldots, x_{n-1}] = P_{\mu}[x_n \in A | x_{n-1}]
= T^s P_{\mu}[x_0 \in A | x_{-1}]
\]

with \( \mu \) probability one and, therefore, also with \( \nu \) probability one. Hence for any \( A \in \mathcal{A}, B \in \mathcal{F}_{n-m} \)

\[
\nu'_{m,n}[x_n \in A, (x_m, \ldots, x_{n-1}) \in B]
= \int_{[x_m, \ldots, x_{n-1}] \in B} P_{\mu}[x_n \in A | x_m, \ldots, x_{n-1}] d\nu
= \int_{[x_m, \ldots, x_{n-1}] \in B} P_{\mu}[x_n \in A | x_{n-1}] d\nu
\]
It follows that
\[ \nu_m,0([x_m, \ldots, x_n) \in C] = \nu_{m-1,0}([x_{m-n}, \ldots, x_0) \in C] \]
for every \( C \in \mathcal{F}_{n-m+1} \). Since by Lemma 1
\[
d\nu_{m,n}/d\nu_{m,n} = f_{m,n}/f_{m,n-1}, \quad d\nu_{m-n,0}/d\nu'_{m-n,0} = f_{m-n,0}/f_{m-n-1}
\]
(2) follows easily.

**Lemma 2.** If \( \mu \) is Markovian and \( m_1 < m_2 < 0 \) then \( \nu_{m,0} \) is an extension of \( \nu_{m,0} \) to \( \mathcal{F}_{m,0} \).

**Proof.** For any \( A \in \mathcal{F}, \beta \in \mathcal{F}_{m,2} \)
\[
\nu_{m,0}([x_0 \in A, (x_{m_2}, \ldots, x_{-1}) \in B]) = \nu_{m-1,0}([x_{m_1}, \ldots, x_{-1}) \in B])
\]
It follows that
\[ \nu_{m_1,0}(E) = \nu_{m_2,0}(E) \]
for every \( E \in \mathcal{F}_{m_2,0} \).

**Theorem 2.** If \( \mu \) is Markovian and \( m_1 < m_2 < 0 \) then
\[
\int (\log f_{m_1,0} - \log f_{m_1,-1}) d\nu
\]
(3)
\[ \geq \int (\log f_{m_2,0} - \log f_{m_2,-1}) d\nu \geq 0. \]

**Proof.** By Lemma 2 \( \nu_{m,0} \) is an extension of \( \nu_{m,0} \) to \( \mathcal{F}_{m,0} \). Since \( \nu_{m,0} \ll \nu_{m,0}, \nu_{m_2,0} \ll \nu_{m,0} \) by Lemma 1, \( d\nu_{m,0}/d\nu'_{m,0} \) is the conditional expectation of \( d\nu_{m,0}/d\nu'_{m,0} \) relative to \( \mathcal{F}_{m,0} \) under the measure \( \nu_{m,0} \). Jensen's
inequality for conditional expectation implies that

\[ 0 \leq \int \left( d\nu_{m_2,0} / d\nu_{m_2,0} \right) \log \left( d\nu_{m_2,0} / d\nu_{m_2,0} \right) d\nu_{m_1,0} \]

\[ \leq \int \left( d\nu_{m_1,0} / d\nu_{m_1,0} \right) \log \left( d\nu_{m_1,0} / d\nu_{m_1,0} \right) d\nu_{m_1,0} . \]

Hence

(4) \[ 0 \leq \int \log \left( d\nu_{m_2,0} / d\nu_{m_2,0} \right) d\nu \leq \int \log \left( d\nu_{m_1,0} / d\nu_{m_1,0} \right) d\nu \]

and (3) follows from (4) and Lemma 1.

**Theorem 3.** If \( \mu \) is Markovian then \( \{ \log f_{m,0} - \log f_{m,-1} \} \) converges with \( \nu \) probability one as \( m \to -\infty \). The limit function may take \( \pm \infty \) as its values.

**Proof.** It is sufficient to prove that \( \{ f_{m,-1}/f_{m,0} \} \) converges with \( \nu \) probability one as \( m \to -\infty \). Since \( \nu_{m,0} \) is absolutely continuous with respect to \( \nu_{m,0}' \) and \( d\nu_{m,0} / d\nu_{m,0}' = f_{m,0}/f_{m,-1} \) by Lemma 1, \( f_{m,-1}/f_{m,0} \) is the derivative of \( \nu_{m,0} \) continuous part of \( \nu_{m,0} \) with respect to \( \nu_{m,0} \). Since, by Lemma 2, \( \nu_{m,0}' \) is an extension of \( \nu_{m,0}' \) if \( m_1 < m_2 \), \( \{ -f_{-k,-1}/f_{-k,0}, \nu_{-k,0}, k \geq 1 \} \) is a \( \nu \) semimartingale ([2] pp. 632). Since

\[ \int | -f_{-k,-1}/f_{-k,0} | d\nu = \int f_{-k,-1}/f_{-k,0} d\nu \leq 1 \]

the semimartingale convergence theorem implies that \( \{ f_{-k,-1}/f_{-k,0} \} \) converges with \( \nu \) probability one as \( k \to -\infty \).

The following lemma may be considered as an improvement of a theorem by A. Pérez ([6] Theorem 7; pp. 194).

**Lemma 3.** Let \( \beta_1 \subset \beta_2 \subset \cdots \) be a sequence of \( \sigma \)-algebras of subsets of \( \Omega \) and \( \beta \) be the \( \sigma \)-algebra generated by \( \bigcup \beta_k \). Let \( \phi, \lambda \) be two probability measures defined on \( \beta \) and \( \phi_k, \lambda_k \) be the contractions of \( \phi, \lambda \), respectively, to \( \beta_k \). If \( \phi_k \) is absolutely continuous with respect to \( \lambda_k \) for \( k = 1, 2, \cdots \) and if there is a finite number \( M \) such that

\[ \int \log (d\phi_k/d\lambda_k) d\phi \leq M \]

for \( k = 1, 2, \cdots \) then

(i) \( \phi \) is absolutely continuous with respect to \( \lambda \),

(ii) \( \log (d\phi/d\lambda) \) is \( \phi \) integrable and there exists

\[ \lim_{k \to \infty} \int \log (d\phi_k/d\lambda_k) d\phi = \int \log (d\phi/d\lambda) d\phi , \]
(iii) \{\log (d\phi_k/d\lambda_k)\} converges in \(L_1(\phi)\) to \(\log (d\phi/d\lambda)\).

Proof.

(i) Let \(h_k = d\phi_k/d\lambda_k\). Then \(\{h_k, \beta_k, k \geq 1\}\) is a martingale under \(\lambda\) measure. Now

\[
M \equiv \int \log (d\phi_k/d\lambda_k) d\phi = \int (\log h_k)h_k d\lambda .
\]

and

\[
M + \frac{1}{n} \geq \int (h_k \log h_k + \frac{1}{n})d\lambda \geq (\log n) \int (h_k \leq n) h_k d\lambda .
\]

Hence

\[
\int (h_k \leq n) h_k d\lambda \leq (\log n)^{-1}(M + \frac{1}{n})
\]

so that \(\int (h_k \leq n) h_k d\lambda \to 0\) as \(n \to \infty\), uniformly in \(k\). Hence \(\{h_k\}\) converges with \(\lambda\) probability one and also in \(L_1(\lambda)\) ([2] Theorem 4.1, pp. 319). Let the limit function be \(h\). Then \(\int_A h d\lambda = \phi(A)\) for all \(A \in \bigcup_k \beta_k\) and so for all \(A \in \beta\). This proves that \(\phi\) is absolutely continuous and that \(h = (d\phi/d\lambda)\).

(ii) The sequence \(\{h_k \log h_k\}\) converges with \(\lambda\) probability one to \(h \log h\). Since the functions \(h_k \log h_k\) are bounded below uniformly by the number \(\frac{1}{4}\),

\[
\int h \log h d\lambda \leq \lim \int h_k \log h_k d\lambda = \lim \int \log h_k d\phi \leq M .
\]

Hence \(h \log h\) is \(\lambda\) integrable. Since the real valued function \(\xi \log \xi\) is continuous and convex, \(h_1 \log h_1, h_2 \log h_2, \ldots, h \log h\) constitute a semi-martingale under the measure \(\lambda([2], \text{Theorem 1.1, pp. 295})\). Hence

\[
\int h_1 \log h_1 d\lambda \leq \int h_2 \log h_2 d\lambda \leq \cdots \leq h \log h d\lambda ,
\]

so that \(\lim_{k \to \infty} h_k \log h_k d\lambda\) exists and is equal to \(\int h \log h d\lambda\). Now

\[
\int |h \log h| d\phi = \int h | \log h | d\lambda = \int h \log h | d\lambda ,
\]

hence \(\log h\) is \(\phi\) integrable and

\[
(6) \quad \int h d\phi = \int h \log h d\lambda = \lim_{k \to \infty} \int h_k \log h_k d\lambda = \lim_{k \to \infty} \int \log h_k d\phi .
\]

1 Inequality (5) was pointed out by the referee. The proof of Lemma 3 was much shortened by following his suggestions.
(iii) Since $h, \log h, h \log h, \cdots, h \log h$ constitute a semimartingale under the measure $\lambda$, we have, for $E \in \beta_h$,

$$\int_E h_k \log h_k d\lambda \leq \int_E h_{k+1} \log h_{k+1} d\lambda \leq \int_E h \log h d\lambda.$$ 

Hence

$$\int_E \log h_k d\phi \leq \int_E \log h_{k+1} d\phi \leq \int_E \log h d\phi,$$

so that $\log h_1, \log h_2, \cdots, \log h$ constitute a semimartingale under the measure $\phi$. Hence (ii) implies that $\log h_k$ are uniformly $\phi$ integrable and $\{\log h_k\}$ converges to $\log h$ in $L_1(\phi)$ ([2], Theorem 4.1s, pp. 324).

**Theorem 4.** If $\mu$ is Markovian and there is a finite number $M$ such that

$$\int [\log f_{m,0} - \log f_{m,-1}] d\nu \leq M$$

for $m = -1, -2, \cdots$ then $\{\log f_{m,0} - \log f_{m,-1}\}$ converges in $L_1(\nu)$ as $m \to -\infty$.

**Proof.** By Lemma 2 $\nu_{m,0}$ is an extension of $\nu'_{m,0}$ if $m_1 < m_2 < 0$ and

$$d\nu_{m,0} / d\nu'_{m,0} = f_{m,0} / f_{m,-1}.$$ 

If there is a probability measure $\nu'$ defined on the $\sigma$-algebra generated by $\bigcup_{m=-1}^{m_0} \mathcal{F}_{m,0}$ which is an extension of $\nu'_{m,0}$ for $m = -1, -2, \cdots$, then the conclusion of the theorem follows easily from Lemma 3. If $X$ is the real line and if $\mathcal{F}$ is the $\sigma$-algebra of Borel sets then the existence of $\nu'$ follows from the Consistency Theorem of Kolmogorov. For the general case we shall proceed by using the usual representation by space $\Omega'$ of sequences of real numbers as follows:

Let $g_k = f_{-k,0} / f_{-k,-1}$.

Let $G$ be the map of $\Omega$ into the space $\Omega'$ of real sequences $\{\xi_1, \xi_2, \cdots\}$ defined by

$$G(\omega) = \{g_1(\omega), g_2(\omega), \cdots\}.$$ 

Considering $\xi_k$ as functions on $\Omega'$ we have

$$\xi_k(G(\omega)) = g_k(\omega).$$

Let $\beta_k$ be the collection of Borel subsets of $\Omega'$ which are determined by conditions on $\xi_1, \xi_2, \cdots, \xi_k$ and $\beta$ be the collection of all Borel subsets
of $Q'$. Let $\phi$ be the probability measure on $\beta$ and $\phi_k, \lambda_k$ be the probability measures on $\beta_k$ defined by

$$
\phi(E) = \nu(G^{-1}E), \\
\phi_k(E) = \nu_{\beta_k}(G^{-1}E), \\
\lambda_k(E) = \nu_{\beta_k}(G^{-1}E).
$$

$\{g_k\}$ converges in $L_1(\nu)$ if and only if $\{\xi_k\}$ converges in $L_1(\phi)$. Now $\lambda_k$ are consistent; Kolmogorov's Consistency Theorem implies the existence of a probability measure $\lambda$ on $\beta$ which is an extension of every $\lambda_k$ and $d\phi_k/d\lambda_k = \xi_k$. Hence Lemma 3 is applicable and the $L_1(\phi)$ convergence of $\{\xi_k\}$ is obtained.

**Theorem 5.** If $\nu$ is stationary and $\mu$ is Markovian with stationary transition probabilities and if

$$
\int \log f_{0,0} d\nu < \infty
$$

and if there is a finite number $M$ such that

$$
\int (\log f_{0,n} - \log f_{0,n-1}) d\nu \leq M
$$

for $n = 1, 2, \cdots$ then $n^{-1}\log f_{0,n}$ converges in $L_1(\nu)$ as $n \to \infty$. In particular, if $\nu$ is ergodic, the limit is equal to a nonnegative constant with $\nu$ probability one.

**Proof.** By Theorem 4 $\{\log f_{m,0} - \log f_{m-1,0}\}$ converges in $L_1(\nu)$ as $m \to -\infty$. Let $\overline{h}$ be the $L_1(\nu)$ limit of the sequence. Let $\overline{h}$ be the $L_1(\nu)$ limit of the sequence $\{n^{-1}\sum_{i=1}^{n} T^i h\}$. By Theorem 1 $f_{0,n}/f_{0,n-1} = T^n(f_{i,0}/f_{i-1,0})$, hence

$$
n^{-1} \log f_{0,n} = n^{-1} \log f_{0,0} + n^{-1} \sum_{i=1}^{n} T^i \log (f_{i,0}/f_{i-1,0})
$$

$$
\int n^{-1} \sum_{i=1}^{n} T^i \log (f_{i,0}/f_{i-1,0}) - \overline{h} | d\nu
$$

$$
\leq n^{-1} \sum_{i=1}^{n} \int | T^i \log (f_{i,0}/f_{i-1,0}) - T^i h | d\nu
$$

$$
+ \int n^{-1} \sum T^i h - \overline{h} | d\nu
$$

$$
= n^{-1} \sum_{i=1}^{n} \int | \log (f_{i,0}/f_{i-1,0}) - h | d\nu
$$

$$
+ \int n^{-1} \sum T^i h - \overline{h} | d\nu \to 0 \text{ as } n \to \infty.
$$

Thus the $L_1(\nu)$ convergence of $\{n^{-1}\log f_{0,n}\}$ is proved. The limit is $\overline{h}$.
which is the $L_1(\nu)$ limit of $\{n^{-1} \sum_{i=1}^n T^i h\}$. If $\nu$ is ergodic

$$\overline{h} = \int h \, d\nu$$

with $\nu$ probability one and

$$\int h d\nu = \lim_{m \to \infty} \int [\log f_{m,0} - \log f_{m,-1}] d\nu \geq 0 .$$

**Corollary 1.** Under the hypothesis of Theorem 5 if $\nu$ is stationary and ergodic but not Markovian then $\nu$ is singular to $\mu$.

**Proof.** If $\mu$ is Markovian but $\nu$ is not Markovian then there is a positive integer $n_0$ such that

$$\mu[f_{0,n_0-1} \neq f_{0,n_0}] > 0 .$$

For, if for every positive integer $n$

$$\mu[f_{0,n-1} \neq f_{0,n}] = 0$$

then

$$P_\nu[x_n \in A \mid x_0, \ldots, x_{n-1}] = P_\mu[x_n \in A \mid x_{n-1}]$$

with $\nu$ probability one for every $A \in \mathcal{F}$ and $\nu$ is Markovian instead. Now since

$$f_{0,n_0-1} = E_\mu[f_{0,n_0} \mid x_0, \ldots, x_{n_0-1}]$$

and the function $\xi \log \xi$ is strictly convex, hence

$$\int f_{0,n_0} \log f_{0,n_0} d\mu - \int f_{0,n_0-1} \log f_{0,n_0-1} d\mu > 0$$

so that

$$\int [\log f_{0,n_0} - \log f_{0,n_0-1}] d\nu > 0 .$$

Since $\int [\log f_{0,n} - \log f_{0,n-1}] d\nu$ is non-decreasing in $n$,

$$\lim_{n \to \infty} \int [\log f_{0,n} - \log f_{0,n-1}] d\nu = a > 0 .$$

Now $\nu$ is ergodic; the $L_1(\nu)$ limit $\overline{h}$ of $\{n^{-1} \log f_{0,n}\}$ is equal to $a$ with $\nu$ probability one. Let $n_1, n_2, \ldots$ be a sequence of positive integers for which $\{n_k^{-1} \log f_{0,n_k}\}$ converges with $\nu$ probability one to $a$ so that $\{1/f_{0,n_k}\}$ converges to 0 as $n_k \to \infty$. Let $\mathcal{F}'$ be the $\sigma$-algebra generated by $\bigcup_n \mathcal{F}_n$ and let $\mu_{\mathcal{F}'}$, $\nu_{\mathcal{F}'}$ be the contractions of $\mu$, $\nu$, respectively, to $\mathcal{F}'$. Since $1/f_{0,n}$ is the derivative of $\nu$-continuous part of $\mu_{0,n}$ with respect
to \( \nu_{0,n} \), \( \{1/f_{0,n}\} \) converges with \( \nu \) probability one to the derivative of \( \nu \)-continuous part of \( \mu' \) with respect to \( \nu' \) by a theorem of Anderson and Jessen [1]. Now we have

\[
\lim_{n \to \infty} 1/f_{1,n} = 0
\]

with \( \nu \) probability one and \( \mu' \) is singular to \( \nu' \). Hence \( \mu, \nu \) are singular to each other.

Extensions of Theorem 5 and Corollary 1 to \( K \)-Markovian \( \mu \) are immediate.

3. Discussion. As was mentioned in the introduction the crucial step in establishing Theorem 5 is to prove the \( L_1(\nu) \) convergence of \( \{\log f_{-n} - \log f_{-n,-1}\} \). If \( \mu \) is the product (independent) measure on \( \mathcal{F} \) the measure \( \nu' \) in the proof of Theorem 4 is actually \( \nu_{-\infty,1} \times \mu_{0,0} \). Thus condition (c) or, equivalently, condition (a) implies condition (b) in the introduction. In [7] it is stated that the condition (b) is necessary for the \( L_1(\nu) \) convergence of \( \{\log f_{-n,0} - \log f_{-n,-1}\} \) ([7] Theorem 2(b)). A simple is as follows. Let \( X \) be the real line and \( \mathcal{A} \) be the collection of all Borel sets. Let \( \nu = \mu \) and distribution of \( x_0 \) be Gaussian. Let 

\[
\nu(x_0 = x) = \mu(x_0 = x) = 1.
\]

Then \( \nu_{-1,0} \) is singular to \( \nu_{-1,-1} \times \nu_{0,0} \), however the \( L_1(\nu) \) convergence of \( \{\log f_{-n,0} - \log f_{-n,-1}\} \) is trivially true since 

\[
f_{0,n} \equiv 1.
\]

REFERENCES

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