WIRTINGER-TYPE INTEGRAL INEQUALITIES

W. J. COLES

1. Introduction. The following inequalities (and other similar ones) are known:

(i) if \( u'(x) \in L^2 \) and \( u(0) = 0 \), then
\[
\int_0^{\pi/2} u^2 dx \leq \int_0^{\pi/2} u''^2 dx \tag{4}
\]

(ii) if \( u''(x) \in L^2 \) and \( u(0) = u(\pi) = 0 \), then
\[
\int_0^\pi u^2 dx \leq \int_0^\pi u''^2 dx \tag{3}
\]

in each case, equality occurs if and only if \( u(x) = A \sin x \). P. R. Beesack [1] has generalized these two types of inequalities by considering the underlying differential equations \( y'' + py = 0 \) and \( y^{(4)} - py = 0 \) respectively, together with the equations satisfied by \( y'/y \). In [2], a relation was obtained between the equation \( y^{(2m)} - py = 0 \) and the inequality
\[
(-1)^n \int_a^b pu^2 dx \leq \int_a^b u^{(m)} dx .
\]

In this paper we let \( Ly \) be the general self-adjoint linear operator of even order
\[
\sum_{i=0}^{n} (f_i y^{(i)})^{(4)}
\]

and extend the methods of [2] to relate the equation
\[
L y = 0
\]
and the inequalities
\[
0 \leq \sum_{i=0}^{n} (-1)^{n+i} \int_a^b f_i u^{(i+2)} dx
\]

and
\[
0 \geq \int_a^b \frac{1}{f_n} \cdot u^2 dx + (-1)^n \int_a^b \frac{1}{f_0} \cdot u^{(m)} dx .
\]

2. Notation and lemmas. Let \( y_i = f_i y^{(i)} \), \( v_i = \sum_{i=0}^{k} y_{n-k}^{(i-k)} \), \( u_{i,j} = v_{n-i}/y^{(j)} \), and \( y_{i,j} = y^{(i)}/y^{(j)} \) \((i = 0, \cdots, n)\).
Then
\[(4) \quad v_i = v_{i-1} + y_{n-i} \quad (i = 1, \ldots, n).\]

Let \((k_0 \cdots k_n)\) be an \((n + 1)\)-tuple consisting of 0’s and 1’s, such that \(\sum_{i=1}^{n} k_i\) is even. Let
\[
(5) \quad c_i = \begin{cases} a, & k_i = 0 \\ b, & k_i = 1 \end{cases}; \quad d_i = \begin{cases} a, & k_{i+1} = 1 \\ b, & k_{i+1} = 1 \end{cases};
\]
\[c_i^* = a + b - c_i; \quad d_i^* = a + b - d_i; \quad p_i = (-1)^{\sum_{j=0}^{i} k_j}; \quad q_i = (-1)^{i} p_i; \quad (i = 0, \ldots, n).\]

We now and henceforth assume that (1) has a solution on \([a, b]\) such that
\[(6) \quad p_i y^{(n-i)}(x) > 0 \quad \text{on } (a, b) \quad \text{and at } c_i^*; \quad p_i y^{(n-i)}(c_i) \geq 0 \quad (i = 2, \ldots, n); \quad q_i v_i(d_i) \geq 0 \quad (i = 0, \ldots, n - 1);\]
and that the \(f_i(x) \in L[a, b],\) with \(\int_a^b f_i(x) dx \neq 0,\) and
\[(7) \quad (-1)^{n+i} f_i(x) \leq 0 \quad \text{on } [a, b] \quad (i = 0, \ldots, n - 1); \quad f_n(x) \geq 0 \quad \text{on } [a, b].\]

**Lemma 1.** We have
\[(8) \quad p_i y^{(n-i)}(x) > 0 \quad \text{on } (a, b) \quad \text{and at } c_i^* \quad (i = 1, \ldots, n).\]

**Proof.** By hypothesis the lemma is true for \(i = 1.\) Suppose that, for some \(i\) such that \(1 \leq i \leq n - 1,\) the statement holds. Integrating and multiplying by \((-1)^{k_{i+1}}\) we have
\[p_{i+1} y^{(n-i-1)}(x) = p_{i+1} y^{(n-i-1)}(c_{i+1}) + (-1)^{k_{i+1}} \int_{c_{i+1}}^{x} p_i y^{(n-i)}(t) dt > 0\]
on \((a, b)\) and at \(c_{i+1}^*.\) This completes Lemma 1.

**Lemma 2.** We have
\[(9) \quad q_i v_i(x) \geq 0 \quad \text{on } [a, b], > 0 \quad \text{at } d_i^* \quad (i = 0, \ldots, n - 1).\]

**Proof.** We proceed by induction on \(i \quad (i = n-1, \ldots, 1, 0).\) Now \(v'_{n-1}(x) = v_n(x) - y_0 = - y_0,\) so
\[q_{n-1} v_{n-1}(x) = q_{n-1} v_{n-1}(d_{n-1}) - (-1)^{n+k} \int_{a_{n-1}}^{x} (-1)^{n} f_{n} p_{n} y dt \geq 0;\]
since $|y| > 0$ and $\int_a^b f_\psi(x)dx \neq 0$, the inequality is strict at $d^*_{n-1}$.

Now suppose that, for some $i (n - 1 \geq i \geq 1)$, the statement holds. Then, integrating (4) and multiplying by $q_{i-1}$,

$$q_{i-1}v_{i-1}(x) = q_{i-1}v_{i-1}(d_{i-1}) + (-1)^{i+1}k_i \int_{d_{i-1}}^x q_{i}v_i dt$$

$$- (-1)^{i+1}k_i \int_{d_{i-1}}^x (-1)^i f_{n-i}p_i y^{(n-i)} dt$$

so $q_{i-1}v_{i-1}(x) \geq 0$ on $(a, b)$ and $> 0$ at $d^*_{i-1}$. This completes Lemma 2.

3. The formal identity. Since (at least formally)

$$u_{ii} = v'_{n-i-1}/y^{(i)} + f_i$$

we have

(10) $u_{ii} = u_{i+1,i} + u_{i+1,i+1}y_{i+1,i} + f_i$.

Now we use (10) and induction to derive the formal identity

(11) $0 = \sum_{i=0}^{n-1} (-1)^{n+i} \left\{ u_{i+1,i}y^{(i)} \right\}^b_a$ 

$$+ \int_a^b u_{i+1,i+1}(u^{(i+1)} - y_{i+1,i}u^{(i)})^2 dx$$

$$+ \sum_{i=0}^{n} (-1)^{i} \int_a^b f_i u^{(i)}^2 dx ;$$

then we will justify the formal steps.

First,

$$\int_a^b u_{i+1,i}u^{(i)}^2 dx = \int_a^b 2u_{i+1,i}u^{(i)}u^{(i+1)} dx$$

$$= \int_a^b 2u_{i+1,i}u^{(i)}^2 dx - \int_a^b u_{i+1,i}u^{(i)} dx ,$$

so

(12) $\int_a^b (u_{i+1,i} + u_{i+1,i+1}y_{i+1,i})u^{(i)}^2 dx$

$$= \int_a^b (u_{i+1,i} + u_{i+1,i+1}y_{i+1,i})u^{(i)}^2 dx$

Since $v_n(x) \equiv Ly \equiv 0$, $u_{n0}(x) \equiv 0$; using (10) and (12) with $i = 0$,

$$0 = \int_a^b u_{10}^2 dx + \int_a^b u_{11}(u' - y_{10}u)^2 dx + \int_a^b f_0 u^2 dx - \int_a^b u_{11}u^2 dx .$$
Suppose that, for some $k$ such that $1 \leq k \leq n - 1$,

\begin{equation}
0 = \sum_{i=0}^{k-1} (-1)^i \left\{ u_{i+1,i} u^{(i)^2} \right\}_a^b \\
+ \int_a^b u_{i+1,i+1}(u^{(i+1)} - y_{i+1,i}u^{(i)^2}) dx \\
+ \sum_{i=0}^{k-1} (-1)^i \int_a^b f_i u^{(i)^2} dx + (-1)^k \int_a^b u_{k+1} u^{(k)^2} dx \, .
\end{equation}

Using (10) and (12) with $i = k$, and substituting for the last term in (13), we obtain (13) with $k$ replaced by $k + 1$. Hence (13) holds for $k = 1, \ldots, n$; with $k = n$, using the fact that $u_{nn} = f_n$, and multiplying by $(-1)^n$, we have (11).

**Lemma 3.** Let $u(x)$ be a function such that

\begin{equation}
\nu^{(n)} \in L_2[a, b]; u^{(i)}(c_{n-i}) = 0 \quad (i = 0, \ldots, n - 1) .
\end{equation}

(Note that (14) implies that the zero of $u^{(i)}$ at $c_{n-i}$ is of order $\geq 1$ ($i = 0, \ldots, n - 2$) and $> \frac{1}{2}$ ($i = n - 1$). Then (11) is valid.

**Proof.** Our concern is with possible zeros of $y^{(i)}$ ($i = 0, \ldots, n - 1$) on $[a, b]$; by Lemma 1, the only possible zero of $y^{(i)}$ is at $c_{n-i}$. Let $i$ be such that $0 \leq i \leq n - 1$, and suppose that $y^{(i)}$ has a zero of order $r$ at $c_{n-i}$. Then $r \leq n - i$. For if $r > n - i$ then $y^{(i+k)}(c_{n-i}) = 0$ ($k = 1, \ldots, n - i$), and so $c_{n-i} = c_{n-i-1} = \cdots c_i$; thus $y^{(n)}(c_i) = 0$. But, by Lemma 2, $v_i(c_i) \neq 0$ (since $c_i = d_{i}^{*}$), and $v_i(x) = f_n(x)y^{(n)}(x)$. Thus $r \leq n - i$.

Now, since $c_{n-i} = \cdots = c_i$, $w^{(i)}$ has a zero of order $\geq r$ at $c_{n-i}$ ($i = 0, \ldots, n - 2$), and of order $> \frac{1}{2}$ ($i = n - 1$). The lemma now follows, as does the fact (to be used in the proof of Lemma 5) that $u_{i+1,i}(c_{n-i})u^{(i)^2}(c_{n-i}) = 0$ ($i = 0, \ldots, n - 1$).

**Lemma 4.** On $[a, b]$, $(-1)^{n+i-1}u_{i,i}(x) \leq 0$ ($i = 1, \ldots, n$).

**Proof.** By Lemmas 1 and 2,

\begin{align*}
(-1)^{n+i-1}u_{i,i} &= (-1)^{n+i-1}(-1)^{n-i}q_{n-i}v_{n-i}/p_{n-i}y^{(i)} \\
&= -q_{n-i}v_{n-i}/p_{n-i}y^{(i)} \leq 0 
\end{align*} 

**Lemma 5.**

\begin{equation}
(-1)^{n+i}u_{i+1,i}u^{(i)^2} \mid_a^b \leq 0 \quad (i = 0, \ldots, n - 1) .
\end{equation}

**Proof.** Since $c_j = d_{j-1}^{*}$,

\begin{equation}
(-1)^{n+i}u_{i+1,i}u^{(i)^2} \mid_a^b = (-1)^{n+i+k-n-i}u_{i+1,i}u^{(i)^2} \mid_{d_{n-i-1}}^{d_{k-n-i-1}} .
\end{equation}

Evaluation at $c_{n-i}$ gives zero, and
on \([a, b]\) and so at \(d_{n-1}\).

4. The inequality. We now state

**Theorem 1.** Let \(f_i(x) \in L[a, b] \ (i = 0, \ldots, n)\), with \(\int_a^b f_0(x)dx \neq 0\).

Let \(f_i(x) \ (i = 0, \ldots, n)\) satisfy (7), and let \(y(x)\) be a solution of (1) which satisfies (6). Let \(u(x)\) satisfy (14). Then

\[
(2) \quad 0 \leq \sum_{i=0}^{n} (-1)^{n+i} \int_a^b f_i(x)u^{(i)}(x)dx.
\]

Further, equality obtains if and only if \(u(x) = cy(x)\) and (6) is modified to make \(q \cdot v_i(d_i) = 0 \ (i = 0, \ldots, n-1)\).

**Proof.** The Theorem follows immediately from the lemmas, except for the last statement, which follows from the fact that equality obtains if and only if \(u^{(i+1)}(x) = y_{i+1}(x)u^{(i)}(x) \ (i = 0, \ldots, n - 1)\) and \(v_i(d_i) = 0 \ (i = 1, \ldots, n)\).

5. The reciprocal inequality. We now derive a set of inequalities which includes (3); we prove

**Theorem 2.** Let the \(f_i(x) \ (i = 0, \ldots, n)\) and \(y(x)\) satisfy the hypothesis of Theorem 1; in addition, let \(f_i(x) = 0\) or \(f_i(x) \neq 0\) on \([a, b]\) \((i = 0, \ldots, n)\). Let \(u(x)\) satisfy

\[
(15) \quad u^{(n)} \in L[a, b]; \ u^{(i)}(d_i) = 0 \quad (i = 0, \ldots, n - 1).
\]

Then, for each \(k \ (1 \leq k \leq n)\) such that \(f_{n-k}(x) \neq 0\),

\[
(16) \quad 0 \geq \int_a^b \frac{1}{f_n(x)} u^2(x)dx + (-1)^k \int_a^b \frac{1}{f_{n-k}(x)} u^{(k)}dx.
\]

**Proof.** The proof is similar to that of Theorem 1, so we present it here in less detail. Let \(r_{ij} = y^{(n-i)}/v_j\); then, formally,

\[
(17) \quad r_{ij} = r_{i+1,j} + r_{i+1,i} v_{i+1}/v_i - r_{i+1,i} f_{n-i-1}.
\]

Thus

\[
(18) \quad \int_a^b r_{ij} u^{(i)} dx = \int_a^b u^{(i)} dx + \int_a^b r_{i+1,i} (u^{(i+1)} - \frac{v_{i+1} u^{(i)}}{v_i})^2 dx
\]

\[
- \int_a^b f_{n-i-1} r_{i+1,i} u^{(i)} dx - \int_a^b r_{i+1,i+1} u^{(i+1)} dx \quad (i = 0, \ldots, n-2),
\]
and

\begin{equation}
\left[ \frac{1}{a} \frac{\partial}{\partial x} \right] f_n = \frac{1}{a} \left( f_{n+1} - f_n \right)
\end{equation}

\[ + \int_a^b r_{i+1,i} w_i^2 \, dx + \int_a^b \frac{1}{a} w_i \, dx \]

\[ \alpha = \sum (- D_m r_{i+1,i+1} + r_{i+1,i+1}) \]

\[ \text{(i = 0, \ldots, n - 1).} \]

Repeated application of (18) to \( \int_a^b r_n w_i \, dx \) gives

\[ \int_a^b \frac{1}{a} f_n \, dx = \sum_{i=0}^{k-1} (-1)^i \left\{ r_{i+1,i} w_i^2 \right\} \]

\[ + \sum_{i=0}^{k-1} (-1)^i \left\{ r_{i+1,i+1} \left( w_i^{(i+1)} - \frac{v_{i+1} w_i^i}{v_i} \right) \right\} \]

\[ - \int_a^b f_{n-i-1} r_{i+1,i} w_i \, dx \]

\[ + (-1)^{k-1} \left\{ \int_a^b r_{k,k-1} \frac{v_k}{v_{k-1}} \, dx \right\} \]

\[ - \int_a^b \frac{1}{a} f_{n-k} (w_i^{(k)} - r_{k,k-1} f_{n-k} w_i^{(k-1)}) \, dx \]

\[ + \left\{ \int_a^b \frac{1}{a} w_i^{(k)} \, dx \right\} \quad (k = 1, \ldots, n). \]

We now show that, if \( f_{n-k}(x) \neq 0 \), (20) is valid. Let a \( v_i \) have a zero of order \( r \); such a zero must be at \( d_i \). Now, \( r \leq n - i \). For we have

\[ v_i' = q_{j+1} q_{j+1} v_{j+1} + (-1)^{j+1} f_{n-j-1} b_{j+1} y_{j(n-j-1)} \]

since \( y_{j(n-j-1)}(d_j) \neq 0 \), if \( v_i'(d_i) = 0 \) then \( f_{n-j-1} \equiv 0 \), and \( v_i' \equiv v_{j+1} \). Thus, if \( r > n - i \), \( v_i^{(n-i-1)} = v_{n-1} \) and also \( v_i^{(n-i-1)} = v_n \equiv 0 \). The first of these implies that \( v_i^{(n-i)} = v_{i-1} = v_n - y_0 = -y_0 \neq 0 \), a contradiction. Further, we have \( d_i = \ldots = d_{i+r-1} \), so \( w_i^{(i)} \) has a zero of order greater than \( r - \frac{1}{2} \) at \( d_i \). This suffices to justify (20). We note in addition that \( r_{i+1,i} u_{i+1}^{(i+1)}(d_i) = 0 \) (i = 0, \ldots, n - 1).

Now by hypothesis \((-1)^{i+1} f_{n-i-1} \leq 0 \) (i = 0, \ldots, n - 1). Lemma 4 implies that \((-1)^{i} r_{i+1,i+1} \leq 0 \) (i = 0, \ldots, n - 2). Finally,
$$(-1)^i r_{i+1,i} u^{(n)}|^b_a = - p_{i+1} y^{(n-i-1)} u^{(i)}|^b_a;$$
evaluation at $d_i^*$ gives a non-positive quantity; evaluation at $d_i$ gives zero. Hence the inequality (16) follows from (20).

6. Concluding remarks. If we want (16) for only one particular value of $k$ ($k < n$), we need correspondingly less hypotheses on $y(x)$ and its derivatives, $u(x)$ and its derivatives, and $f_i(x)$ ($i = 0, \ldots, n$), since only $k + 1$ of the functions in each of these sets are actually involved in any of the proofs.

Since $(-1)^{n-i} f_i(x) \leq 0$, from (2) we may delete any combination of terms excluding the last, and to the right-hand side of (16) we may add any terms of the form

$$(-1)^i \int_a^b \frac{1}{f_{n-j}} u^{(i)} dx$$

Thus, e.g., (2) implies

$$0 \leq (-1)^k \int_a^b f_{n-k} u^{(k)} dx + \int_a^b f_n u^{(n)} dx,$$

which perhaps corresponds more obviously to (16) than does (2).

Finally, the set of allowed values of $(k_0 \ldots k_n)$ can be split into halves such that one half, together with the inequality $Ly \geq 0$, and also the other half, together with $Ly \leq 0$, will produce the inequalities.

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