STRONGLY CONTINUOUS MARKOV PROCESSES

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Introduction. This paper is a continuation of [3]. We deal here with Markov processes with continuous parameter, while in [3] the discrete parameter case was studied. The notion of a "Markov Process" (here and in [3]) is different from the standard one: A stationary probability measure is assumed to exist, but the Chapman-Kolmogoroff Equation is replaced by a weaker condition. The exact definitions are given in § 1.

All problems are discussed from a Hilbert space point of view and convergence will mean, always, either strong or weak convergence.

1. Notation and background. We shall repeat here, for completeness, the notation of [3] and some of the results.

Let $(\Omega, \Sigma, \mu)$ be a given measure space where $\mu(\Omega) = 1$, and $\mu \geq 0$. The measure will be called the probability measure. The space of real square integrable functions is denoted by $L^2$.

Let $X_t(\omega)$ be a family of measurable real functions where $0 \leq t < \infty$ and $\omega \in \Omega$. This will be called the Markov process and we assume:

If $A$ is a Borel set on the real line and $t_1 < t_2 < t_3$, then the conditional probability that $X_{t_3} \in A$ given $X_{t_1}$ and $X_{t_2}$ is equal to the conditional probability that $X_{t_3} \in A$ given $X_{t_2}$.

Also we assume that the process is stationary. Namely:

$$\mu(X_{t_1+s} \in A_1 \cap X_{t_2+s} \in A_2) = \mu(X_{t_1} \in A_1 \cap X_{t_2} \in A_2)$$

for all $t_1, t_2, s$ positive real numbers and $A_1, A_2$ Borel sets.

For any set $\sigma \subset \Omega$, $\chi_\sigma$ denotes the characteristic function of this set. Let $B_t$ be the closed subspace of $L^2$ generated by the functions $\chi_{x,t}$. The self adjoint projection on $B_t$ is denoted by $E_t$. Finally, let $T_t$ be the transformation from $B_{t+s}$ to $B_t$, defined by

$$T_t \chi_{x,t} = \chi_{x,t+s}$$

where we used additivity to extend it to whole of $B_0$. In [3] the following equations are proved:

1.1 $E_{t_1}E_{t_2}E_{t_3} = E_{t_1}E_{t_3}$ if $t_1 < t_2 < t_3$.

1.2 a. $\| T_t x \| = \| x \|$, for $x \in B_0$.

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b. \( T_t B_0 = B_t \).
c. \( (T_{t_1 + s} x, T_{t_1 + s} y) = (T_{t_1} x, T_{t_1} y) \), for \( x \in B_0 \) \( y \in B_0 \).

See Theorem 2.1 and Lemma 2.4.

Let \( P_t \) be the operator on \( B_0 \) defined by \( P_t = E_0 T_t \).

**Theorem 1.1.** The operators \( P_t \) form a semi group of contractions on \( B_0 \). The adjoint semi group is given by \( P_t^* = T_t^{-1} E_t \).

**Proof.** It is clear that \( ||P_t|| \leq 1 \). Let \( x \) and \( y \) be vectors of \( B_0 \) and choose \( z \in B_0 \) so that \( T_t z = E_t y \). Thus \( z = T_t^{-1} E_t y \). Then

\[
(P_t P_s x, y) = (E_0 T_s E_0 T_t x, y) = (T_s E_0 T_t x, y)
\]

\[
= (T_s E_0 T_t x, E_s y) = (E_0 T_t x, z) = (T_t x, z)
\]

Where we used Equation 1.2c. On the other hand

\[
(P_{s+t} x, y) = (E_0 T_{s+t} x, y) = (E_s E_t T_{s+t} x, y) = (E_t T_{s+t} x, y)
\]

\[
= (T_{s+t} x, E_s y) = (T_{s+t} x, T_s z) = (T_t x, z)
\]

Here we used Equations 1.1 and 1.2c. Now

\[
(P_t x, y) = (T_t x, y) = (T_t x, E_s y) = (x, z) = (x, T_t^{-1} E_t y)
\]

The fact that \( P_t \) is a semi group is our version of the Chapman-Kolmogoroff Equation.

In most of this paper it will be assumed that the semi group \( P_t \) is strongly continuous. We shall say, in this case that the Markov process is strongly continuous.

**Theorem 2.1.** The Markov process is strongly continuous if and only if

\[
\lim_{t \to 0} \mu(X_0 \in A \cap T_t x \in A) = \mu(X_0 \in A)
\]

**Proof.** Note that

\[
\mu(X_0 \in A) = || \chi_{x_0 \in A} ||^2
\]

\[
\mu(X_0 \in A \cap T_t x \in A) = (T_t \chi_{x_0 \in A}, \chi_{x_0 \in A}) = (P_t \chi_{x_0 \in A}, \chi_{x_0 \in A})
\]

Thus

\[
\mu(X_0 \in A) - \mu(X_0 \in A \cap T_t x \in A) = (\chi_{x_0 \in A} - P_t \chi_{x_0 \in A}, \chi_{x_0 \in A})
\]

and this converges to zero if \( P_t \) converges to the identity operator strongly. On the other hand
Thus the condition of the Theorem implies that $P_t x$ converges to $x$ for a set of functions, $x$, that span $B_0$ and because $\|P_t\| \leq 1$ this must hold for every $x$ in $B_0$.

2. Limit of transition probabilities as $t \to \infty$. This section is an extension of § 3 of [3]. Throughout this section we assume:

CONDITION D. There exist a finite a measure $\varphi$, on the real line, and an $\varepsilon > 0$ such that if $A$ is a Borel set and $\varphi(A) < \varepsilon$ then

$$E_0 \chi_{x \in A} \neq \chi_{x \in A}.$$

This condition was given in [3] and is similar to Doeblin’s condition as given in [1] page 192. Another form of the condition is: if $\varphi(A) < \varepsilon$ then

$$\|T_t \chi_{x \in A}\|^2 = \|\chi_{x \in A}\|^2 > \|P_t \chi_{x \in A}\|^2.$$

In this form it is seen immediately that $t$ can be replaced by any larger number. Thus one can choose $t$ to be of the form $n\delta$ for any fixed $\delta > 0$. ($n$ a positive integer). For a fixed $\delta > 0$ $X_{n\delta}$ form a discreet Markov process for which a Doeblin condition holds. Let $H_\delta$ be the space of all functions in $B_0$ such that

$$x \in \bigcap_{n=0}^{\infty} B_{n\delta}, \, T_{k\delta}x \in \bigcap_{n=0}^{\infty} B_{n\delta} \quad k = 1, 2, \cdots.$$

In [3] Theorem 3.7 it was proved that if $x$ is orthogonal to $H_\delta$ then $T_k x$ tends weakly to zero as $k$ tends to infinity ($k$ integer).

**Theorem 1.2.** $x \in H_\delta$ if and only if $T_t x = x$ for some $t > 0$. Thus $H_\delta$ is the same for all $\delta$ and will be denoted by $H$. The space $H$ is generated by a finite number of disjoint characteristic functions and is invariant under $T_t$ for all $t > 0$.

**Proof.** It is enough to prove first statement for the rest follows from Theorem 3.8 and Corollary 2 of Theorem 3.11 of [3].

In Corollary 2 of Theorem 3.11 of [3] it was shown that if $x \in H_\delta$ then $T_{k\delta}x = x$ for some $x$. Thus it is enough to show that if $T_t x = x$ for some $t > 0$, then $x \in H_\delta$. Now if $T_t x = x$ then

$$(T_{t+a} x, T_a x) = (T_t x, x) = \|x\|^2 = \|T_a x\|^2.$$
Thus
\[ T_{t+1}x = T_t x. \]
In particular
\[ x = T_1 x = T_2 x = \cdots. \]
Thus
\[ x \in \bigcap_k B_{tk}. \]
But by Theorem 2.2 of [3]
\[ \bigcap_{k=0}^\infty B_{tk} = \bigcap_{n=0}^\infty B_{tn}. \]
Again by Theorem 2.2 of [3]. Thus it suffices to show that \( T_{m^\delta}x \in B_0 \)
for then \( T_{m^\delta}x \in \bigcap_{n=0}^\infty B_{n^\delta} \) by the same Theorem. Now
\[
\sup_{z \in B_\delta, ||z||=1} (T_{m^\delta}x, z) = \sup_{z^1 \in B_{k^\delta}, ||z^1||=1} (T_{m^\delta+k^\delta}x, z^1)
\]
\[
= \sup_{z^1 \in B_{k^\delta}, ||z^1||=1} (T_{m^\delta}x, z^1) = || T_{m^\delta}x ||
\]
for
\[ T_{m^\delta}x \in \bigcap_{n=0}^\infty B_{n^\delta} \subset B_{kt} \quad \text{if} \quad kt > m^\delta. \]
Thus
\[ T_{m^\delta}x \in B_0 \quad \text{and} \quad x \in H. \]
Notice that on \( H \) \( P_t = T_t \), and \( P_t \) is a unitary operator.

In the rest of the paper we shall assume that the process \( \{X_t\} \), is strongly continuous.

**Lemma 2.2.** On the space \( H \) \( T_t \) is the identity operator for all \( t \).

**Proof.** Let \( \chi \) be one of the atoms generating \( H \). Thus \( \chi \) is a characteristic function that is not the sum of two characteristic functions
in $H$. Let $t$ be so small that $(T_t \chi, \chi) \neq 0$. Now $T_t \chi$ is also a characteristic function in $H$ and $\| T_t \chi \| = \| \chi \|$. Thus $T_t \chi = \chi$ because $\chi$ is an atom. Also for every $n T_n \chi = P_n \chi = (P_t)^n \chi = \chi$, hence $T_t \chi = P_t \chi = \chi$ for all $t$.

**Theorem 3.2.** Let $x \in B_o$ and let $y$ be the projection of $x$ on $H$, then

$$\text{weak limit } P_t x = \text{weak limit } T_t x = y .$$

**Proof.** By the previous lemma it suffices to show that if $x$ is orthogonal to $H$ then $T_t x$ tends weakly to zero. Let $z \in B_o$, $\| z \| = 1$ be a given vector and let $\varepsilon > 0$. Choose $\delta_0$ so that $\| T_\delta x - x \| \leq \varepsilon / 2$ if $\delta \leq \delta_0$. By Theorem 3.7 of [3] if $n$ is large enough then

$$\| (T_{n\delta_0} x, z) \| \leq \varepsilon / 2 .$$

Thus

$$\| (T_t x, z) \| = \| ((T_t - T_{n\delta_0}) x, z) + (T_{n\delta_0} x, z) \| \leq \varepsilon / 2 + \| (T_t - T_{n\delta_0}) x \| .$$

Now

$$\| (T_t - T_{n\delta_0}) x \| ^2 = 2 \| x \| ^2 - 2 (T_t x, T_{n\delta_0} x) = 2 \| x \| ^2 - 2 (T_{t-n\delta_0} x, x) = \| T_{t-n\delta_0} x - x \| ^2$$

by Equation 1.2.c. If $n$ is so chosen that

$$t - n\delta_0 < \delta_0 \text{ then } \| (T_t - T_{n\delta_0}) x \| \leq \varepsilon / 2 .$$

3. **Differentiability.** In this section we do not assume Condition D. The process $\{X_t\}$ is assumed to be strongly continuous. It is known that in this case the function $P_t x$ is differentiable at the origin for $x$ in a dense subset of $B_o$. The derivative, $Q$, of $P_t$ is an unbounded closed operator. Let $D(Q)$ be the domain of $Q$. The simplest case is when $Q$ is bounded. A necessary and sufficient condition for this is that the semi group $P_t$ is continuous in the uniform topology. (See 2 Theorem VIII, 2)

**Theorem 1.3.** The operator $Q$ is everywhere defined if and only if the expression

$$1 - \frac{\mu(X_0 \in A \cap X_t \in A)}{\mu(X_0 \in A)}$$

tends to zero uniformly, for all Borel sets $A$. 
**Proof.** If \( \| I - P_i \| \to 0 \) then

\[
1 - \frac{\mu(X_0 \in A \cap X_i \in A)}{\mu(X_0 \in A)} = \frac{(\chi_{x_0 \in A_i} - P_i \chi_{x_0 \in A}, \chi_{x_0 \in A_i})}{\| \chi_{x_0 \in A_i} \|^2} \leq \| I - P_i \| .
\]

Thus the condition is necessary. Conversely let

\[
x = \sum a_i \chi_i \quad \text{where} \quad \sum a_i^2 \| \chi_i \|^2 = 1 \quad \text{and} \quad \chi_i = \chi_{x_0 \in A_i}, A_i \cap A_j = \phi .
\]

Then

\[
1 - (P_i x, x) = \sum a_i a_j ((\chi_i, \chi_j) - (P_i \chi_i, \chi_j))
\]

\[
\leq \left( \sum a_i^2 \| (\chi_i, \chi_j) - (P_i \chi_i, \chi_j) \|^2 \right)^{1/2} \left( \sum a_j^2 \| (\chi_i, \chi_j) - (P_i \chi_i, \chi_j) \|^2 \right)^{1/2}.
\]

By Schwarz's inequality. Let us consider each term separately.

\[
\sum a_i^2 \| (\chi_i, \chi_j) - (P_i \chi_i, \chi_j) \| = \sum a_i^2 \sum_j \| (\chi_i, \chi_j) - (P_i \chi_i, \chi_j) \| .
\]

For a fixed \( i \) we have

\[
\sum_j \| (\chi_i, \chi_j) - (P_i \chi_i, \chi_j) \| = \sum_{j \neq i} (P_i \chi_i, \chi_j) + \| \chi_i \|^2 - (P_i \chi_i, \chi_i)
\]

\[
= \sum_j (P_i \chi_i, \chi_j) - (P_i \chi_i, \chi_i) + \| \chi_i \|^2 - (P_i \chi_i, \chi_i)
\]

\[
= (P_i \chi_i, 1) - (P_i \chi_i, \chi_i) + \| \chi_i \|^2 - (P_i \chi_i, \chi_i)
\]

where 1 is the identity function. Now

\[
(P_i \chi_i, 1) = (T_i \chi_i, 1) = (T_i \chi_i, T_i 1) = (\chi_i, 1) = \| \chi_i \|^2 .
\]

Thus the sum over \( j \) is equal to

\[
2 \| \chi_i \|^2 \left( 1 - \frac{(P_i \chi_i, \chi_i)}{\| \chi_i \|^2} \right)
\]

and

\[
\sum a_i^2 \| (\chi_i, \chi_j) - (P_i \chi_i, \chi_j) \| \leq 2 \mathbf{s} \left( 1 - \frac{(P_i \chi_i, \chi_i)}{\| \chi_i \|^2} \right) .
\]

\[
\sum a_i^2 \| \chi_i \|^2 = 2 \mathbf{s} \left( 1 - \frac{(P_i \chi_i, \chi_i)}{\| \chi_i \|^2} \right) .
\]

For the second term we get

\[
\sum a_j^2 \| (\chi_i, \chi_j) - (P_i \chi_i, \chi_j) \| = \sum a_j^2 \sum_i \| (\chi_i, \chi_j) - (P_i \chi_i, \chi_j) \|
\]

and
\[
\sum_i \left( \langle \chi_i, \chi_i \rangle - (P_t \chi_i, \chi_i) \right) = \| \chi_i \|^2 - (P_t \chi_i, \chi_i) + \sum_i (P_t \chi_i, \chi_i) \\
\quad = \| \chi_i \|^2 - (P_t \chi_i, \chi_i) + \sum_i (P_t \chi_i, \chi_i) - (P_t \chi_i, \chi_i) \\
\quad = \| \chi_i \|^2 - (P_t \chi_i, \chi_i) + (P_t 1, \chi_i) - (P_t \chi_i, \chi_i) \\
\quad = 2(\| \chi_i \|^2 - (P_t \chi_i, \chi_i)).
\]

And the second term has the same bound. Thus
\[
1 - (P_t x, x) \leq 2 \sup_i \left( 1 - \frac{(P_t \chi_i, \chi_i)}{\| \chi_i \|^2} \right).
\]

Now
\[
\| P_t x - x \|^2 = \| P_t x \|^2 + \| x \|^2 - 2(P_t x, x) \\
\quad \leq 2((I - P_t)x, x) \leq 4 \sup_i \left( 1 - \frac{(P_t \chi_i, \chi_i)}{\| \chi_i \|^2} \right).
\]

By assumption this tends to zero uniformly. Hence \( \| P_t x - x \| \) tends to zero uniformly, for \( x \) in a dense subset of \( B_n \), and hence everywhere because \( \| P_t \| \leq 1 \).

**Remarks.** It is enough to assume the condition of the Theorem for a family of Borel sets, \( A \), such that the functions \( \chi_A \) generate \( B_o \). It follows, from the fact that \( Q \) is bounded, that
\[
1 - \frac{\mu(X \in A \cap X_t \in A)}{\mu(X_t \in A)} \leq (\text{const})t.
\]

Theorem 1.3 is well known for processes with countable state space. A brief discussion of this case is given in [1] page 265.

The function \( P_t x \) is differentiable for many \( x \)’s even if \( Q \) is unbounded. In order to study this we will need:

**Lemma 2.3.** Let \( R_t \) be strongly continuous semi group of operators, defined on a reflexive space \( X \). If \( x \in X \) then \( R_t x \) is differentiable if the expression \((1/t) \| R_t x - x \|\) is bounded for all \( t \).

This is included in Theorem 10.7.2 of [4]

Let \( y \in L_2 \) and \( \Omega_{t} \) be a subset of \( \Omega \) such that \( \chi_{\Omega_t \in \Omega_0} \). Then
\[
\| \chi_{\Omega_t \in \Omega_0} \cdot E_t y \|^2 = \| \chi_{\Omega_1} \cdot E_t y \|^2 + \| \chi_{\Omega_2} \cdot E_t y \|^2
\]
where \( \Omega_2 = \Omega - \Omega_1 \). Now \( \chi_{\Omega_1} \cdot E_t y \) is the projection of \( y \) on the subspace generated by characteristic function, in \( B_n \), of subsets of \( \Omega_i \). Thus
\[
\| \chi_{\Omega_1} \cdot E_t y \| = \sup \{ \Sigma(y, \chi_{\Omega_1}) \alpha_i \mid \chi_{\Omega_1} = \chi_{\Omega_{t} \in A_t} \in B_n \text{ and } A_i \text{ are disjoint Borel sets, such that } X_{\Omega} \in A_i \subset \Omega_i \text{, and } \Sigma \alpha_i \| \chi_{\Omega} \|^2 = 1 \}.
\]
But

\[ | \sum (y, \chi_i)a_i | \leq \sum \frac{|(y, \chi_i)|}{||\chi_i||}|a_i||\chi_i| \leq \left( \sum \frac{(y, \chi_i)^2}{||\chi_i||^2} \right)^{1/2}. \]

Hence

\[ ||\chi_{a_1} \cdot E_0 y||^2 = \sup \left\{ \sum \frac{(y, \chi_i)^2}{||\chi_i||^2} : \chi_i = \chi_{x_0 \in A_i} \in B_0, \right\} \]

where \( A_i \) disjoint Borel sets and \( x_0 \in A_i \subset \Omega_1 \).

A similar expression holds for \( ||\chi_{a_2} \cdot E_0 y||^2 \).

**Theorem 3.3.** Let \( A \) be a Borel set. The function \( P_t \chi_{x_0 \in A} \) is differentiable at zero if and only if the two expressions below, are bounded:

1. \( \frac{1}{t^2} \sup \left\{ \sum \frac{\mu(X_i \in A \cap X_0 \in A_i)^2}{\mu(X_0 \in A_i)} : A_i \text{ disjoint} \right\} \)

Borel sets and \( A_i \cap A = \emptyset \).

2. \( \frac{1}{t^2} \sup \left\{ \sum \frac{(\mu(X_i \in A \cap X_0 \in A_i)^2 - \mu(x_0 \in A_i)^2)}{\mu(x_0 \in A_i)} : A_i \text{ disjoint} \right\} \)

Borel sets and \( A_i \subset A \).

**Proof.** By Lemma 2.3 and the above discussion it is enough to show that

\[ \frac{1}{t^2} \sup \left\{ \sum \frac{(P_t \chi_{x_0 \in A} - \chi_{x_0 \in A})^2}{||\chi_{x_0 \in A_i}||^2} : A_i \text{ disjoint and } A_i \cap A = \emptyset \right\} \]

and

\[ \frac{1}{t^2} \sup \left\{ \sum \frac{(P_t \chi_{x_0 \in A} - \chi_{x_0 \in A}, \chi_{x_0 \in A_i})^2}{||\chi_{x_0 \in A_i}||^2} : A_i \text{ disjoint and } A_i \subset A \right\} \]

are both bounded. But these expressions are equal to 1 and 2 respectively.

**Remark.** If \( A \) is an atom for \( B_0 \) then the second expression is

\[ \frac{1}{t^2} \left( \chi(X_i \in A \cap X_0 \in A) - \frac{\mu(X_0 \in A)}{\mu(X_0 \in A)} \right)^2 \frac{\mu(X_0 \in A)}{\mu(X_0 \in A)} \]

\[ = \left( \frac{1}{t^2} \left( 1 - \frac{\mu(X_i \in A \cap X_0 \in A)}{\mu(X_0 \in A)} \right) \right)^2 \frac{\mu(X_0 \in A)}{\mu(X_0 \in A)}. \]
A more precise information is available in the following special case.

Theorem 4.3. Let \( x \in B_0 \). Then \( x \in D(Q) \) and \( (Qx, x) = 0 \) if and only if \((1/t^2)(||x||^2 - (P_t x, x))\) is bounded. In this case \( Q^* x \) exists and is equal to \( -Qx \).

Proof. If \( y \in B_0 \) then

\[
|| y - P_t y ||^2 = || y ||^2 + || P_t y ||^2 - 2(P_t y, y) \\
\leq 2(|| y ||^2 - (T_t y, y)) = || y - T_t y ||^2
\]

thus

\[
\frac{|| T_t y - y ||}{\sqrt{t}} = \sqrt{2}\frac{(y - P_t y, y)}{t} \geq \frac{|| P_t y - y ||}{\sqrt{t}}.
\]

Also if \( y \) and \( z \) are any two vectors in \( B_0 \) then

b. \[
\left( \frac{1}{t} (P_t - 1)z, y \right) = \frac{1}{t} (T_t z - z, y) = \frac{1}{t} (T_t z, y - T_t y) \\
= \frac{1}{t} (T_t z - z, y - T_t y) + \frac{1}{t} (z, y - P_t y)
\]

where we used Equation 1.2.c for the third equality.

Let \( x \) be such that \((1/t^2)(||x||^2 - (P_t x, x))\) is bounded. Then from (a) we get

\[
|| \frac{1}{t^2} (P_t x - x) ||^2 \leq 2\frac{(x - P_t x, x)}{t^2}
\]

and is bounded by assumption. Thus we know from Lemma 2.3 that \( x \in D(Q) \). Moreover

\[
(Qx, x) = -\lim_{t \to 0} t \frac{(x - P_t x)}{t^2} = 0.
\]

Conversely let \( x \in D(Q) \) and \( (Qx, x) = 0 \). If \( y \in D(Q) \) then it follows from (b) that

\[
(Qx, y) = \lim_{t \to 0} \frac{1}{t} ((P_t - 1)x, y) \\
= \lim_{t \to 0} \frac{1}{t} (T_t x - x, y - T_t y) + \frac{1}{t} (x, y - P_t y)
\]

the second term tends to \(- (x, Qy)\) while the first is bounded by
\[
\left| \frac{1}{t} (T_t x - x, y - T_t y) \right| \leq \frac{\| T_t x - x \| \| y - T_t y \|}{\sqrt{t}} = \left( \frac{2(x - P_t x, x)}{t} \cdot \frac{2(y - P_t y, y)}{t} \right)^{1/2}
\]

as \( t \to 0 \) this tends to

\[(4(Qx, x)(Qy, y))^{1/2} = 0 .
\]
Thus

\[(Qx, y) = - (x, Qy)
\]
or

\[x \in D(Q^*) \text{ and } Q^* x = - Qx .
\]

Now

\[
(x - P_t x, x) = \int_0^t (Q P_u x, x) du \leq t \max_{u \leq t} |(Q P_u x, x)|
\]

\[= t \max_{u \leq t} |(P_u x, Qx)| = t \max_{u \leq t} |(P_u x - x, Qx)|
\]

\[\leq \text{const. } t^2
\]

because \( ||P_u x - x|| \leq \text{const. } u .\)

**Remark.** If \( x \) is a characteristic function then it is easy to see that \( Qx = 0 \) if \( (Qx, x) = 0 .\)

The referee called my attention to the fact that this theorem generalizes to arbitrary semi groups of contraction operators, when \( T_t \) is replaced by the group of unitary operators which project down to \( P_t \) as in s. Nagy theorem (See Riesz Nagy appendix to the third edition). Some simple changes have to be done to take care of the complex case.

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