

Pacific Journal of Mathematics

STRONGLY CONTINUOUS MARKOV PROCESSES

SHAUL FOGUEL

STRONGLY CONTINUOUS MARKOV PROCESSES

S. R. FOGUEL

Introduction. This paper is a continuation of [3]. We deal here with Markov processes with continuous parameter, while in [3] the discrete parameter case was studied. The notion of a "Markov Process" (here and in [3]) is different from the standard one: A stationary probability measure is assumed to exist, but the Chapman-Kolmogoroff Equation is replaced by a weaker condition. The exact definitions are given in § 1.

All problems are discussed from a Hilbert space point of view and convergence will mean, always, either strong or weak convergence.

1. Notation and background. We shall repeat here, for completeness, the notation of [3] and some of the results.

Let (Ω, Σ, μ) be a given measure space where $\mu(\Omega) = 1$, and $\mu \geq 0$. The measure will be called the probability measure. The space of real square integrable functions is denoted by L_2 .

Let $X_t(\omega)$ be a family of measurable real functions where $0 \leq t < \infty$ and $\omega \in \Omega$. This will be called the Markov process and we assume:

If A is a Borel set on the real line and $t_1 < t_2 < t_3$ then the conditional probability that $X_{t_3} \in A$ given X_{t_1} and X_{t_2} is equal to the conditional probability that $X_{t_3} \in A$ given X_{t_2} .

Also we assume that the process is stationary. Namely:

$$\mu(X_{t_1+s} \in A_1 \cap X_{t_2+s} \in A_2) = \mu(X_{t_1} \in A_1 \cap X_{t_2} \in A_2)$$

for all t_1, t_2, s positive real numbers and A_1, A_2 Borel sets.

For any set $\sigma \subset \Omega$, χ_σ denotes the characteristic function of this set. Let B_t be the closed subspace of L_2 generated by the functions $\chi_{X_t \in A}$. The self adjoint projection on B_t is denoted by E_t . Finally, let T_t be the transformation from B_0 to B_t defined by

$$T_t \chi_{X_0 \in A} = \chi_{X_t \in A}$$

where we used additivity to extend it to whole of B_0 . In [3] the following equations are proved:

$$\begin{array}{ll} 1.1 & E_{t_1} E_{t_2} E_{t_3} = E_{t_1} E_{t_3} & \text{if } t_1 < t_2 < t_3 . \\ 1.2 \text{ a.} & \| T_t x \| = \| x \| , & \text{for } x \in B_0 . \end{array}$$

Received August 3, 1960. This paper was supported by a contract from the National Science Foundation.

- b. $T_t B_0 = B_t .$
- c. $(T_{t_1+s}x, T_{t_2+s}y) = (T_{t_1}x, T_{t_2}y) , \quad \text{for } x \in B_0 \ y \in B_0 .$

See Theorem 2.1 and Lemma 2.4.

Let P_t be the operator on B_0 defined by $P_t = E_0 T_t$.

THEOREM 1.1. *The operators P_t form a semi group of contractions on B_0 . The adjoint semi group is given by $P_t^* = T_t^{-1} E_t$.*

Proof. It is clear that $\| P_t \| \leq 1$. Let x and y be vectors of B_0 and choose $z \in B_0$ so that $T_s z = E_s y$. Thus $z = T_s^{-1} E_s y$. Then

$$\begin{aligned} (P_s P_t x, y) &= (E_0 T_s E_0 T_t x, y) = (T_s E_0 T_t x, y) \\ &= (T_s E_0 T_t x, E_s y) = (E_0 T_t x, z) = (T_t x, z) . \end{aligned}$$

Where we used Equation 1.2c. On the other hand

$$\begin{aligned} (P_{s+t} x, y) &= (E_0 T_{s+t} x, y) = (E_0 E_s T_{s+t} x, y) = (E_s T_{s+t} x, y) \\ &= (T_{s+t} x, E_s y) = (T_{s+t} x, T_s z) = (T_t x, z) . \end{aligned}$$

Here we used Equations 1.1 and 1.2c. Now

$$(P_s x, y) = (T_s x, y) = (T_s x, E_s y) = (x, z) = (x, T_s^{-1} E_s y) .$$

The fact that P_t is a semi group is our version of the Chapman-Kolmogoroff Equation.

In most of this paper it will be assumed that the semi group P_t is strongly continuous. We shall say, in this case that the Markov process is strongly continuous.

THEOREM 2.1. *The Markov process is strongly continuous if and only if*

$$\lim_{t \rightarrow 0} \mu(X_0 \in A \cap X_t \in A) = \mu(X_0 \in A) .$$

Proof. Note that

$$\begin{aligned} \mu(X_0 \in A) &= \| \chi_{X_0 \in A} \|^2 \\ \mu(X_0 \in A \cap X_t \in A) &= (T_t \chi_{X_0 \in A}, \chi_{X_0 \in A}) = (P_t \chi_{X_0 \in A}, \chi_{X_0 \in A}) . \end{aligned}$$

Thus

$$\mu(X_0 \in A) - \mu(X_0 \in A \cap X_t \in A) = (\chi_{X_0 \in A} - P_t \chi_{X_0 \in A}, \chi_{X_0 \in A})$$

and this converges to zero if P_t converges to the identity operator strongly. On the other hand

$$\begin{aligned} \| P_t \chi_{x_0 \in A} - \chi_{x_0 \in A} \|^2 &= \| P_t \chi_{x_0 \in A} \|^2 + \| \chi_{x_0 \in A} \|^2 - 2(P_t \chi_{x_0 \in A}, \chi_{x_0 \in A}) \\ &\leq 2(\| \chi_{x_0 \in A} \|^2 - (P_t \chi_{x_0 \in A}, \chi_{x_0 \in A})) \\ &= 2(\mu(X_0 \in A) - \mu(X_0 \in A \cap X_t \in A)) . \end{aligned}$$

Thus the condition of the Theorem implies that $P_t x$ converges to x for a set of functions, x , that span B_0 and because $\| P_t \| \leq 1$ this must hold for every x in B_0 .

2. Limit of transition probabilities as $t \rightarrow \infty$. This section is an extension of § 3 of [3]. Throughout this section we assume:

CONDITION D. *There exist a finite a measure φ , on the real line, and an $\varepsilon > 0$ such that if A is a Borel set and $\varphi(A) < \varepsilon$ then*

$$E_0 \chi_{x_t \in A} \neq \chi_{x_t \in A} .$$

This condition was given in [3] and is similar to Doeblin's condition as given in [1] page 192. Another form of the condition is: if $\varphi(A) < \varepsilon$ then

$$\| T_t \chi_{x_0 \in A} \|^2 = \| \chi_{x_0 \in A} \|^2 > \| P_t \chi_{x_0 \in A} \|^2 .$$

In this form it is seen immediately that t can be replaced by any larger number. Thus one can choose t to be of the form $n\delta$ for any fixed $\delta > 0$. (n a positive integer). For a fixed $\delta > 0$ $X_{n\delta}$ form a discreet Markov process for which a Doeblin condition holds. Let H_δ be the space of all functions in B_0 such that

$$x \in \bigcap_{n=0}^{\infty} B_{n\delta}, T_{k\delta} x \in \bigcap_{n=0}^{\infty} B_{n\delta} \quad k = 1, 2, \dots .$$

In [3] Theorem 3.7 it was proved that if x is orthogonal to H_δ then $T_{k\delta} x$ tends weakly to zero as k tends to infinity (k integer).

THEOREM 1.2. *$x \in H_\delta$ if and only if $T_t x = x$ for some $t > 0$. Thus H_δ is the same for all δ and will be denoted by H . The space H is generated by a finite number of disjoint characteristic functions and is invariant under T_t for all $t > 0$.*

Proof. It is enough to prove first statement for the rest follows from Theorem 3.8 and Corollary 2 of Theorem 3.11 of [3].

In Corollary 2 of Theorem 3.11 of [3] it was shown that if $x \in H_\delta$ then $T_{k\delta} x = x$ for some x . Thus it is enough to show that if $T_t x = x$ for some $t > 0$, then $x \in H_\delta$. Now if $T_t x = x$ then

$$(T_{t+a} x, T_a x) = (T_t x, x) = \| x \|^2 = \| T_a x \|^2$$

Thus

$$T_{t+a}x = T_a x .$$

In particular

$$x = T_t x = T_{2t} x = \dots .$$

Thus

$$x \in \bigcap_{k=0}^{\infty} B_{tk} .$$

But by Theorem 2.2 of [3]

$$\bigcap_{k=0}^{\infty} B_{tk} = \bigcap_{n=0}^{\infty} B_{\delta n} .$$

Now

$$T_{m\delta} x = T_{m\delta+t} x = T_{m\delta+2t} x = \dots$$

or

$$T_{m\delta} x \in \bigcap_{k=0}^{\infty} B_{m\delta+kt} = \bigcap_{n=m}^{\infty} B_{n\delta} .$$

Again by Theorem 2.2 of [3]. Thus it suffices to show that $T_{m\delta} x \in B_0$ for then $T_{m\delta} x \in \bigcap_{n=0}^{\infty} B_{n\delta}$ by the same Theorem. Now

$$\begin{aligned} \sup_{z \in B_0, \|z\|=1} (T_{m\delta} x, z) &= \sup_{z^1 \in B_{kt}, \|z^1\|=1} (T_{m\delta+kt} x, z^1) \\ &= \sup_{z^1 \in B_{kt}, \|z^1\|=1} (T_{m\delta} x, z^1) = \| T_{m\delta} x \| \end{aligned}$$

for

$$T_{m\delta} x \in \bigcap_{n=m}^{\infty} B_{n\delta} \subset B_{kt} \quad \text{if } kt > m\delta .$$

Thus

$$T_{m\delta} x \in B_0 \quad \text{and} \quad x \in H_{\delta} .$$

Notice that on H $P_t = T_t$, and P_t is a unitary operator.

In the rest of the paper we shall assume that the process $\{X_i\}$, is strongly continuous.

LEMMA 2.2. *On the space H T_t is the identity operator for all t .*

Proof. Let χ be one of the atoms generating H . Thus χ is a characteristic function that is not the sum of two characteristic functions

in H . Let t be so small that $(T_t\chi, \chi) \neq 0$. Now $T_t\chi$ is also a characteristic function in H and $\|T_t\chi\| = \|\chi\|$. Thus $T_t\chi = \chi$ because χ is an atom. Also for every n $T_{nt}\chi = P_{nt}\chi = (P_t)^n\chi = \chi$, hence $T_t\chi = P_t\chi = \chi$ for all t .

THEOREM 3.2. *Let $x \in B_0$ and let y be the projection of x on H , then*

$$\text{weak limit}_{t \rightarrow \infty} P_t x = \text{weak limit}_{t \rightarrow \infty} T_t x = y .$$

Proof. By the previous lemma it suffices to show that if x is orthogonal to H then $T_t x$ tends weakly to zero. Let $z \in B_0, \|z\| = 1$ be a given vector and let $\varepsilon > 0$. Choose δ_0 so that $\|T_{\delta_0} x - x\| \leq \varepsilon/2$ if $\delta \leq \delta_0$. By Theorem 3.7 of [3] if n is large enough then

$$|(T_{n\delta_0} x, z)| \leq \varepsilon/2 .$$

Thus

$$\begin{aligned} |(T_t x, z)| &= |((T_t - T_{n\delta_0})x, z) + (T_{n\delta_0} x, z)| \\ &\leq \varepsilon/2 + \|(T_t - T_{n\delta_0})x\| . \end{aligned}$$

Now

$$\begin{aligned} \|(T_t - T_{n\delta_0})x\|^2 &= 2\|x\|^2 - 2(T_t x, T_{n\delta_0} x) \\ &= 2\|x\|^2 - 2(T_{t-n\delta_0} x, x) = \|T_{t-n\delta_0} x - x\|^2 \end{aligned}$$

by Equation 1.2.c. If n is so chosen that

$$t - n\delta_0 < \delta_0 \quad \text{then} \quad \|(T_t - T_{n\delta_0})x\| \leq \varepsilon/2 .$$

3. Differentiability. In this section we do not assume Condition D. The process $\{X_t\}$ is assumed to be strongly continuous. It is known that in this case the function $P_t x$ is differentiable at the origin for x in a dense subset of B_0 . The derivative, Q , of P_t is an unbounded closed operator. Let $D(Q)$ be the domain of Q . The simplest case is when Q is bounded. A necessary and sufficient condition for this is that the semi group P_t is continuous in the uniform topology. (See 2 Theorem VIII. 2)

THEOREM 1.3. *The operator Q is everywhere defined if and only if the expression*

$$1 - \frac{\mu(X_0 \in A \cap X_t \in A)}{\mu(X_0 \in A)}$$

tends to zero uniformly, for all Borel sets A .

Proof. If $\|I - P_t\| \rightarrow 0$ then

$$1 - \frac{\mu(X_0 \in A \cap X_t \in A)}{\mu(X_0 \in A)} = \frac{(\chi_{X_0 \in A} - P_t \chi_{X_0 \in A}, \chi_{X_0 \in A})}{\|\chi_{X_0 \in A}\|^2} \leq \|I - P_t\|.$$

Thus the condition is necessary. Conversely let

$$x = \sum a_i \chi_i \text{ where } \sum a_i^2 \|\chi_i\|^2 = 1 \text{ and } \chi_i = \chi_{X_0 \in A_i}, A_i \cap A_j = \phi.$$

Then

$$\begin{aligned} 1 - (P_t x, x) &= \sum_{i,j} a_i a_j ((\chi_i, \chi_j) - (P_t \chi_i, \chi_j)) \\ &\leq \left(\sum_{i,j} a_i^2 |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| \right)^{1/2} \left(\sum_{i,j} a_j^2 |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| \right)^{1/2}. \end{aligned}$$

By Schwarz's inequality. Let us consider each term separately.

$$\sum_{i,j} a_i^2 |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| = \sum_i a_i^2 \sum_j |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)|.$$

For a fixed i we have

$$\begin{aligned} \sum_j |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| &= \sum_{j \neq i} (P_t \chi_i, \chi_j) + \|\chi_i\|^2 - (P_t \chi_i, \chi_i) \\ &= \sum_j (P_t \chi_i, \chi_j) - (P_t \chi_i, \chi_i) + \|\chi_i\|^2 - (P_t \chi_i, \chi_i) \\ &= (P_t \chi_i, 1) - (P_t \chi_i, \chi_i) + \|\chi_i\|^2 - (P_t \chi_i, \chi_i) \end{aligned}$$

where 1 is the identity function. Now

$$(P_t \chi_i, 1) = (T_t \chi_i, 1) = (T_t \chi_i, T_t 1) = (\chi_i, 1) = \|\chi_i\|^2.$$

Thus the sum over j is equal to

$$2 \|\chi_i\|^2 \left(1 - \frac{(P_t \chi_i, \chi_i)}{\|\chi_i\|^2} \right)$$

and

$$\sum_{i,j} a_i^2 |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| \leq 2 \sup_i \left(1 - \frac{(P_t \chi_i, \chi_i)}{\|\chi_i\|^2} \right).$$

$$\sum a_i^2 \|\chi_i\|^2 = 2 \sup_i \left(1 - \frac{(P_t \chi_i, \chi_i)}{\|\chi_i\|^2} \right).$$

For the second term we get

$$\sum a_j^2 |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| = \sum_j a_j^2 \sum_i |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)|$$

and

$$\begin{aligned} \sum_i |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| &= \|\chi_j\|^2 - (P_t \chi_j, \chi_j) + \sum_{i \neq j} (P_t \chi_i, \chi_j) \\ &= \|\chi_j\|^2 - (P_t \chi_j, \chi_j) + \sum_i (P_t \chi_i, \chi_j) - (P_t \chi_j, \chi_j) \\ &= \|\chi_j\|^2 - (P_t \chi_j, \chi_j) + (P_t \mathbf{1}, \chi_j) - (P_t \chi_j, \chi_j) \\ &= 2(\|\chi_j\|^2 - (P_t \chi_j, \chi_j)). \end{aligned}$$

And the second term has the same bound. Thus

$$1 - (P_t x, x) \leq 2 \sup \left(1 - \frac{(P_t \chi_i, \chi_i)}{\|\chi_i\|^2} \right).$$

Now

$$\begin{aligned} \|P_t x - x\|^2 &= \|P_t x\|^2 + \|x\|^2 - 2(P_t x, x) \\ &\leq 2((I - P_t)x, x) \leq 4 \sup_i \left(1 - \frac{(P_t \chi_i, \chi_i)}{\|\chi_i\|^2} \right). \end{aligned}$$

By assumption this tends to zero uniformly. Hence $\|P_t x - x\|$ tends to zero uniformly, for x in a dense subset of B_0 , and hence everywhere because $\|P_t\| \leq 1$.

REMARKS. It is enough to assume the condition of the Theorem for a family of Borel sets, A , such that the functions χ_A generate B_0 . It follows, from the fact that Q is bounded, that

$$1 - \frac{\mu(X_0 \in A \cap X_t \in A)}{\mu(X_0 \in A)} \leq (\text{const})t.$$

Theorem 1.3 is well known for processes with countable state space. A brief discussion of this case is given in [1] page 265.

The function $P_t x$ is differentiable for many x 's even if Q is unbounded. In order to study this we will need:

LEMMA 2.3. *Let R_t be strongly continuous semi group of operators, defined on a reflexive space X . If $x \in X$ then $R_t x$ is differentiable if the expression $(1/t) \|R_t x - x\|$ is bounded for all t .*

This is included in Theorem 10.7.2 of [4]

Let $y \in L_2$ and Ω_1 be a subset of Ω such that $\chi_{\Omega_1 \in B_0}$. Then

$$\|E_0 y\|^2 = \|\chi_{\Omega_1} \cdot E_0 y\|^2 + \|\chi_{\Omega_2} \cdot E_0 y\|^2$$

where $\Omega_2 = \Omega - \Omega_1$. Now $\chi_{\Omega_1} \cdot E_0 y$ is the projection of y on the subspace generated by characteristic function, in B_0 , of subsets of Ω_1 . Thus

$$\begin{aligned} \|\chi_{\Omega_1} \cdot E_0 y\| &= \sup \{ \sum (\chi_i, y) a_i \mid \chi_i = \chi_{X_0 \in A_i} \in B_0 \text{ and } A_i \text{ are disjoint} \\ &\text{Borel sets, such that } X_0 \in A_i \subset \Omega_1, \text{ and } \sum a_i^2 \|\chi_i\|^2 = 1 \}. \end{aligned}$$

But

$$|\sum (y, \chi_i) a_i| \leq \sum \frac{|(y, \chi_i)|}{\|\chi_i\|} |a_i| \|\chi_i\| \leq \left(\sum \frac{(y, \chi_i)^2}{\|\chi_i\|^2} \right)^{1/2}.$$

Hence

$$\|\chi_{\Omega_1} \cdot E_0 y\|^2 = \sup \left\{ \sum \frac{(y, \chi_i)^2}{\|\chi_i\|^2} \mid \chi_i = \chi_{X_0 \in A_i} \in B_0, \right. \\ \left. A_i \text{ disjoint Borel sets and } X_0 \in A_i \subset \Omega_1 \right\}$$

A similar expression holds for $\|\chi_{\Omega_2} \cdot E_0 y\|^2$.

THEOREM 3.3. *Let A be a Borel set. The function $P_t \chi_{X_0 \in A}$ is differentiable at zero if and only if the two expressions below, are bounded:*

1. $\frac{1}{t^2} \sup \left\{ \sum \frac{\mu(X_t \in A \cap X_0 \in A_i)^2}{\mu(X_0 \in A_i)} \mid A_i \text{ disjoint Borel sets and } A_i \cap A = \phi \right\}.$
2. $\frac{1}{t^2} \sup \left\{ \sum \frac{(\mu(X_t \in A \cap X_0 \in A_i) - \mu(X_0 \in A_i))^2}{\mu(X_0 \in A_i)} \mid A_i \text{ disjoint Borel sets and } A_i \subset A \right\}.$

Proof. By Lemma 2.3 and the above discussion it is enough to show that

$$\frac{1}{t^2} \sup \left\{ \sum \frac{(P_t \chi_{X_0 \in A} - \chi_{X_0 \in A}, \chi_{X_0 \in A_i})^2}{\|\chi_{X_0 \in A_i}\|^2} \mid A_i \text{ disjoint and } A_i \cap A = \phi \right\}$$

and

$$\frac{1}{t^2} \sup \left\{ \sum \frac{(P_t \chi_{X_0 \in A} - \chi_{X_0 \in A}, \chi_{X_0 \in A_i})^2}{\|\chi_{X_0 \in A_i}\|^2} \mid A_i \text{ disjoint and } A_i \subset A \right\}$$

are both bounded. But these expressions are equal to 1 and 2 respectively.

REMARK. If A is an atom for B_0 then the second expression is

$$\frac{1}{t^2} \left(\frac{\mu(X_t \in A \cap X_0 \in A) - \mu(X_0 \in A)}{\mu(X_0 \in A)} \right)^2 \mu(X_0 \in A) \\ = \left(\frac{1}{t} \left(1 - \frac{\mu(X_t \in A \cap X_0 \in A)}{\mu(X_0 \in A)} \right) \right)^2 \mu(X_0 \in A).$$

A more precise information is available in the following special case.

THEOREM 4.3. *Let $x \in B_0$. Then $x \in D(Q)$ and $(Qx, x) = 0$ if and only if $(1/t^2)(\|x\|^2 - (P_t x, x))$ is bounded. In this case Q^*x exists and is equal to $-Qx$.*

Proof. If $y \in B_0$ then

$$\begin{aligned} \|y - P_t y\|^2 &= \|y\|^2 + \|P_t y\|^2 - 2(P_t y, y) \\ &\leq 2(\|y\|^2 - (T_t y, y)) = \|y - T_t y\|^2 \end{aligned}$$

thus

a.
$$\frac{\|T_t y - y\|}{\sqrt{t}} = \sqrt{\frac{2(y - P_t y, y)}{t}} \geq \frac{\|P_t y - y\|}{\sqrt{t}}.$$

Also if y and z are any two vectors in B_0 then

b.
$$\begin{aligned} \left(\frac{1}{t}(P_t - 1)z, y\right) &= \frac{1}{t}(T_t z - z, y) = \frac{1}{t}(T_t z, y - T_t y) \\ &= \frac{1}{t}(T_t z - z, y - T_t y) + \frac{1}{t}(z, y - P_t y) \end{aligned}$$

where we used Equation 1.2.c for the third equality.

Let x be such that $(1/t^2)(\|x\|^2 - (P_t x, x))$ is bounded. Then from (a) we get

$$\left\| \frac{1}{t^2}(P_t x - x) \right\|^2 \leq 2 \frac{(x - P_t x, x)}{t^2}$$

and is bounded by assumption. Thus we know from Lemma 2.3 that $x \in D(Q)$. Moreover

$$(Qx, x) = -\lim_t t \frac{(x - P_t x, x)}{t^2} = 0.$$

Conversely let $x \in D(Q)$ and $(Qx, x) = 0$. If $y \in D(Q)$ then it follows from (b) that

$$\begin{aligned} (Qx, y) &= \lim_{t \rightarrow 0} \frac{1}{t} ((P_t - 1)x, y) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (T_t x - x, y - T_t y) + \frac{1}{t} (x, y - P_t y) \end{aligned}$$

the second term tends to $-(x, Qy)$ while the first is bounded by

$$\begin{aligned} \left| \frac{1}{t} (T_t x - x, y - T_t y) \right| &\leq \frac{\|T_t x - x\|}{\sqrt{t}} \frac{\|y - T_t y\|}{\sqrt{t}} \\ &= \left(\frac{2(x - P_t x, x)}{t} \cdot \frac{2(y - P_t y, y)}{t} \right)^{1/2} \end{aligned}$$

as $t \rightarrow 0$ this tends to

$$(4(Qx, x)(Qy, y))^{1/2} = 0.$$

Thus

$$(Qx, y) = -(x, Qy)$$

or

$$x \in D(Q^*) \quad \text{and} \quad Q^*x = -Qx.$$

Now

$$\begin{aligned} (x - P_t x, x) &= \int_0^t (QP_u x, x) du \leq t \max_{u \leq t} |(QP_u x, x)| \\ &= t \max_{u \leq t} |(P_u x, Qx)| = t \max_{u \leq t} |(P_u x - x, Qx)| \\ &\leq \text{const. } t^2 \end{aligned}$$

because $\|P_u x - x\| \leq \text{const. } u$.

REMARK. If x is a characteristic function then it is easy to see that $Qx = 0$ if $(Qx, x) = 0$.

The referee called my attention to the fact that this theorem generalizes to arbitrary semi groups of contraction operators, when T_t is replaced by the group of unitary operators which project down to P_t as in s_z Nagy theorem (See Riesz Nagy appendix to the third edition). Some simple changes have to be done to take care of the complex case.

BIBLIOGRAPHY

1. J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953.
2. J. T. Schwartz and N. Dunford, *Linear Operators*. Interscience, New York, 1958.
3. S. R. Foguel, *Weak and strong convergence of Markov Processes*, Pacific J. Math., **10** (1960), 1221-1234.
4. E. Hille and R. S. Phillips, *Functional analysis and semi groups*, Amer. Math. Soc. New York, 1957.

UNIVERSITY OF CALIFORNIA
BERKELEY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS

Stanford University
Stanford, California

F. H. BROWNELL

University of Washington
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California
Los Angeles 7, California

L. J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
T. M. CHERRY

D. DERRY
M. OHTSUKA

H. L. ROYDEN
E. SPANIER

E. G. STRAUS
F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 11, No. 3 BadMonth, 1961

Errett Albert Bishop, <i>A generalization of the Stone-Weierstrass theorem</i>	777
Hugh D. Brunk, <i>Best fit to a random variable by a random variable measurable with respect to a σ-lattice</i>	785
D. S. Carter, <i>Existence of a class of steady plane gravity flows</i>	803
Frank Sydney Cater, <i>On the theory of spatial invariants</i>	821
S. Chowla, Marguerite Elizabeth Dunton and Donald John Lewis, <i>Linear recurrences of order two</i>	833
Paul Civin and Bertram Yood, <i>The second conjugate space of a Banach algebra as an algebra</i>	847
William J. Coles, <i>Wirtinger-type integral inequalities</i>	871
Shaul Foguel, <i>Strongly continuous Markov processes</i>	879
David James Foulis, <i>Conditions for the modularity of an orthomodular lattice</i>	889
Jerzy Górski, <i>The Sochocki-Plemelj formula for the functions of two complex variables</i>	897
John Walker Gray, <i>Extensions of sheaves of associative algebras by non-trivial kernels</i>	909
Maurice Hanan, <i>Oscillation criteria for third-order linear differential equations</i>	919
Haim Hanani and Marian Reichaw-Reichbach, <i>Some characterizations of a class of unavoidable compact sets in the game of Banach and Mazur</i>	945
John Grover Harvey, III, <i>Complete holomorphs</i>	961
Joseph Hersch, <i>Physical interpretation and strengthening of M. Protter's method for vibrating nonhomogeneous membranes; its analogue for Schrödinger's equation</i>	971
James Grady Horne, Jr., <i>Real commutative semigroups on the plane</i>	981
Nai-Chao Hsu, <i>The group of automorphisms of the holomorph of a group</i>	999
F. Burton Jones, <i>The cyclic connectivity of plane continua</i>	1013
John Arnold Kalman, <i>Continuity and convexity of projections and barycentric coordinates in convex polyhedra</i>	1017
Samuel Karlin, Frank Proschan and Richard Eugene Barlow, <i>Moment inequalities of Pólya frequency functions</i>	1023
Tilla Weinstein, <i>Imbedding compact Riemann surfaces in 3-space</i>	1035
Azriel Lévy and Robert Lawson Vaught, <i>Principles of partial reflection in the set theories of Zermelo and Ackermann</i>	1045
Donald John Lewis, <i>Two classes of Diophantine equations</i>	1063
Daniel C. Lewis, <i>Reversible transformations</i>	1077
Gerald Otis Losey and Hans Schneider, <i>Group membership in rings and semigroups</i>	1089
M. N. Mikhail and M. Nassif, <i>On the difference and sum of basic sets of polynomials</i>	1099
Alex I. Rosenberg and Daniel Zelinsky, <i>Automorphisms of separable algebras</i>	1109
Robert Steinberg, <i>Automorphisms of classical Lie algebras</i>	1119
Ju-Kwei Wang, <i>Multipliers of commutative Banach algebras</i>	1131
Neal Zierler, <i>Axioms for non-relativistic quantum mechanics</i>	1151