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STRONGLY CONTINUOUS MARKOV PROCESSES

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Introduction. This paper is a continuation of [3]. We deal here with Markov processes with continuous parameter, while in [3] the discrete parameter case was studied. The notion of a "Markov Process" (here and in [3]) is different from the standard one: A stationary probability measure is assumed to exist, but the Chapman-Kolmogoroff Equation is replaced by a weaker condition. The exact definitions are given in § 1.

All problems are discussed from a Hilbert space point of view and convergence will mean, always, either strong or weak convergence.

1. Notation and background. We shall repeat here, for completeness, the notation of [3] and some of the results.

Let (Ω, Σ, μ) be a given measure space where $\mu(\Omega) = 1$, and $\mu \geq 0$. The measure will be called the probability measure. The space of real square integrable functions is denoted by L_2 .

Let $X_t(\omega)$ be a family of measurable real functions where $0 \leq t < \infty$ and $\omega \in \Omega$. This will be called the Markov process and we assume:

If A is a Borel set on the real line and $t_1 < t_2 < t_3$ then the conditional probability that $X_{t_3} \in A$ given X_{t_1} and X_{t_2} is equal to the conditional probability that $X_{t_3} \in A$ given X_{t_2} .

Also we assume that the process is stationary. Namely:

$$\mu(X_{t_1+s} \in A_1 \cap X_{t_2+s} \in A_2) = \mu(X_{t_1} \in A_1 \cap X_{t_2} \in A_2)$$

for all t_1, t_2, s positive real numbers and A_1, A_2 Borel sets.

For any set $\sigma \subset \Omega$, χ_σ denotes the characteristic function of this set. Let B_t be the closed subspace of L_2 generated by the functions $\chi_{X_t \in A}$. The self adjoint projection on B_t is denoted by E_t . Finally, let T_t be the transformation from B_0 to B_t defined by

$$T_t \chi_{X_0 \in A} = \chi_{X_t \in A}$$

where we used additivity to extend it to whole of B_0 . In [3] the following equations are proved:

$$1.1 \quad E_{t_1} E_{t_2} E_{t_3} = E_{t_1} E_{t_3} \quad \text{if } t_1 < t_2 < t_3 .$$

$$1.2 \text{ a.} \quad \| T_t x \| = \| x \| , \quad \text{for } x \in B_0 .$$

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- b. $T_t B_0 = B_t$.
- c. $(T_{t_1+s}x, T_{t_2+s}y) = (T_{t_1}x, T_{t_2}y)$, for $x \in B_0$ $y \in B_0$.

See Theorem 2.1 and Lemma 2.4.

Let P_t be the operator on B_0 defined by $P_t = E_0 T_t$.

THEOREM 1.1. *The operators P_t form a semi group of contractions on B_0 . The adjoint semi group is given by $P_t^* = T_t^{-1} E_t$.*

Proof. It is clear that $\|P_t\| \leq 1$. Let x and y be vectors of B_0 and choose $z \in B_0$ so that $T_s z = E_s y$. Thus $z = T_s^{-1} E_s y$. Then

$$\begin{aligned} (P_s P_t x, y) &= (E_0 T_s E_0 T_t x, y) = (T_s E_0 T_t x, y) \\ &= (T_s E_0 T_t x, E_s y) = (E_0 T_t x, z) = (T_t x, z) . \end{aligned}$$

Where we used Equation 1.2c. On the other hand

$$\begin{aligned} (P_{s+t} x, y) &= (E_0 T_{s+t} x, y) = (E_0 E_s T_{s+t} x, y) = (E_s T_{s+t} x, y) \\ &= (T_{s+t} x, E_s y) = (T_{s+t} x, T_s z) = (T_t x, z) . \end{aligned}$$

Here we used Equations 1.1 and 1.2c. Now

$$(P_s x, y) = (T_s x, y) = (T_s x, E_s y) = (x, z) = (x, T_s^{-1} E_s y) .$$

The fact that P_t is a semi group is our version of the Chapman-Kolmogoroff Equation.

In most of this paper it will be assumed that the semi group P_t is strongly continuous. We shall say, in this case that the Markov process is strongly continuous.

THEOREM 2.1. *The Markov process is strongly continuous if and only if*

$$\lim_{t \rightarrow 0} \mu(X_0 \in A \cap X_t \in A) = \mu(X_0 \in A) .$$

Proof. Note that

$$\begin{aligned} \mu(X_0 \in A) &= \|\chi_{x_0 \in A}\|^2 \\ \mu(X_0 \in A \cap X_t \in A) &= (T_t \chi_{x_0 \in A}, \chi_{x_0 \in A}) = (P_t \chi_{x_0 \in A}, \chi_{x_0 \in A}) . \end{aligned}$$

Thus

$$\mu(X_0 \in A) - \mu(X_0 \in A \cap X_t \in A) = (\chi_{x_0 \in A} - P_t \chi_{x_0 \in A}, \chi_{x_0 \in A})$$

and this converges to zero if P_t converges to the identity operator strongly. On the other hand

$$\begin{aligned} \| P_t \chi_{x_0 \in A} - \chi_{x_0 \in A} \|^2 &= \| P_t \chi_{x_0 \in A} \|^2 + \| \chi_{x_0 \in A} \|^2 - 2(P_t \chi_{x_0 \in A}, \chi_{x_0 \in A}) \\ &\leq 2(\| \chi_{x_0 \in A} \|^2 - (P_t \chi_{x_0 \in A}, \chi_{x_0 \in A})) \\ &= 2(\mu(X_0 \in A) - \mu(X_0 \in A \cap X_t \in A)) . \end{aligned}$$

Thus the condition of the Theorem implies that $P_t x$ converges to x for a set of functions, x , that span B_0 and because $\| P_t \| \leq 1$ this must hold for every x in B_0 .

2. Limit of transition probabilities as $t \rightarrow \infty$. This section is an extension of § 3 of [3]. Throughout this section we assume:

CONDITION D. *There exist a finite a measure φ , on the real line, and an $\varepsilon > 0$ such that if A is a Borel set and $\varphi(A) < \varepsilon$ then*

$$E_0 \chi_{x_t \in A} \neq \chi_{x_t \in A} .$$

This condition was given in [3] and is similar to Doeblin's condition as given in [1] page 192. Another form of the condition is: if $\varphi(A) < \varepsilon$ then

$$\| T_t \chi_{x_0 \in A} \|^2 = \| \chi_{x_0 \in A} \|^2 > \| P_t \chi_{x_0 \in A} \|^2 .$$

In this form it is seen immediately that t can be replaced by any larger number. Thus one can choose t to be of the form $n\delta$ for any fixed $\delta > 0$. (n a positive integer). For a fixed $\delta > 0$ $X_{n\delta}$ form a discreet Markov process for which a Doeblin condition holds. Let H_δ be the space of all functions in B_0 such that

$$x \in \bigcap_{n=0}^{\infty} B_{n\delta}, T_{k\delta} x \in \bigcap_{n=0}^{\infty} B_{n\delta} \qquad k = 1, 2, \dots .$$

In [3] Theorem 3.7 it was proved that if x is orthogonal to H_δ then $T_{k\delta} x$ tends weakly to zero as k tends to infinity (k integer).

THEOREM 1.2. *$x \in H_\delta$ if and only if $T_t x = x$ for some $t > 0$. Thus H_δ is the same for all δ and will be denoted by H . The space H is generated by a finite number of disjoint characteristic functions and is invariant under T_t for all $t > 0$.*

Proof. It is enough to prove first statement for the rest follows from Theorem 3.8 and Corollary 2 of Theorem 3.11 of [3].

In Corollary 2 of Theorem 3.11 of [3] it was shown that if $x \in H_\delta$ then $T_{k\delta} x = x$ for some x . Thus it is enough to show that if $T_t x = x$ for some $t > 0$, then $x \in H_\delta$. Now if $T_t x = x$ then

$$(T_{t+a} x, T_a x) = (T_t x, x) = \| x \|^2 = \| T_a x \|^2$$

Thus

$$T_{t+a}x = T_a x .$$

In particular

$$x = T_t x = T_{2t} x = \dots .$$

Thus

$$x \in \bigcap_{k=0}^{\infty} B_{tk} .$$

But by Theorem 2.2 of [3]

$$\bigcap_{k=0}^{\infty} B_{tk} = \bigcap_{n=0}^{\infty} B_{\delta n} .$$

Now

$$T_{m\delta} x = T_{m\delta+t} x = T_{m\delta+2t} x = \dots$$

or

$$T_{m\delta} x \in \bigcap_{k=0}^{\infty} B_{m\delta+kt} = \bigcap_{n=m}^{\infty} B_{n\delta} .$$

Again by Theorem 2.2 of [3]. Thus it suffices to show that $T_{m\delta} x \in B_0$ for then $T_{m\delta} x \in \bigcap_{n=0}^{\infty} B_{n\delta}$ by the same Theorem. Now

$$\begin{aligned} \sup_{z \in B_0, \|z\|=1} (T_{m\delta} x, z) &= \sup_{z^1 \in B_{kt}, \|z^1\|=1} (T_{m\delta+kt} x, z^1) \\ &= \sup_{z^1 \in B_{kt}, \|z^1\|=1} (T_{m\delta} x, z^1) = \| T_{m\delta} x \| \end{aligned}$$

for

$$T_{m\delta} x \in \bigcap_{n=m}^{\infty} B_{n\delta} \subset B_{kt} \quad \text{if } kt > m\delta .$$

Thus

$$T_{m\delta} x \in B_0 \quad \text{and} \quad x \in H_{\delta} .$$

Notice that on $H P_t = T_t$, and P_t is a unitary operator.

In the rest of the paper we shall assume that the process $\{X_t\}$, is strongly continuous.

LEMMA 2.2. *On the space $H T_t$ is the identity operator for all t .*

Proof. Let χ be one of the atoms generating H . Thus χ is a characteristic function that is not the sum of two characteristic functions

in H . Let t be so small that $(T_t\chi, \chi) \neq 0$. Now $T_t\chi$ is also a characteristic function in H and $\|T_t\chi\| = \|\chi\|$. Thus $T_t\chi = \chi$ because χ is an atom. Also for every $nT_{ni}\chi = P_{ni}\chi = (P_i)^n\chi = \chi$, hence $T_i\chi = P_i\chi = \chi$ for all t .

THEOREM 3.2. *Let $x \in B_0$ and let y be the projection of x on H , then*

$$\text{weak limit}_{t \rightarrow \infty} P_t x = \text{weak limit}_{t \rightarrow \infty} T_t x = y .$$

Proof. By the previous lemma it suffices to show that if x is orthogonal to H then $T_t x$ tends weakly to zero. Let $z \in B_0, \|z\| = 1$ be a given vector and let $\varepsilon > 0$. Choose δ_0 so that $\|T_\delta x - x\| \leq \varepsilon/2$ if $\delta \leq \delta_0$. By Theorem 3.7 of [3] if n is large enough then

$$|(T_{n\delta_0} x, z)| \leq \varepsilon/2 .$$

Thus

$$\begin{aligned} |(T_t x, z)| &= |((T_t - T_{n\delta_0})x, z) + (T_{n\delta_0} x, z)| \\ &\leq \varepsilon/2 + \|(T_t - T_{n\delta_0})x\| . \end{aligned}$$

Now

$$\begin{aligned} \|(T_t - T_{n\delta_0})x\|^2 &= 2\|x\|^2 - 2(T_t x, T_{n\delta_0} x) \\ &= 2\|x\|^2 - 2(T_{t-n\delta_0} x, x) = \|T_{t-n\delta_0} x - x\|^2 \end{aligned}$$

by Equation 1.2.c. If n is so chosen that

$$t - n\delta_0 < \delta_0 \quad \text{then} \quad \|(T_t - T_{n\delta_0})x\| \leq \varepsilon/2 .$$

3. Differentiability. In this section we do not assume Condition D. The process $\{X_t\}$ is assumed to be strongly continuous. It is known that in this case the function $P_t x$ is differentiable at the origin for x in a dense subset of B_0 . The derivative, Q , of P_t is an unbounded closed operator. Let $D(Q)$ be the domain of Q . The simplest case is when Q is bounded. A necessary and sufficient condition for this is that the semi group P_t is continuous in the uniform topology. (See 2 Theorem VIII. 2)

THEOREM 1.3. *The operator Q is everywhere defined if and only if the expression*

$$1 - \frac{\mu(X_0 \in A \cap X_t \in A)}{\mu(X_0 \in A)}$$

tends to zero uniformly, for all Borel sets A .

Proof. If $\|I - P_t\| \rightarrow 0$ then

$$1 - \frac{\mu(X_0 \in A \cap X_t \in A)}{\mu(X_0 \in A)} = \frac{(\chi_{X_0 \in A} - P_t \chi_{X_0 \in A}, \chi_{X_0 \in A})}{\|\chi_{X_0 \in A}\|^2} \leq \|I - P_t\|.$$

Thus the condition is necessary. Conversely let

$$x = \sum a_i \chi_i \text{ where } \sum a_i^2 \|\chi_i\|^2 = 1 \text{ and } \chi_i = \chi_{X_0 \in A_i}, A_i \cap A_j = \phi.$$

Then

$$\begin{aligned} 1 - (P_t x, x) &= \sum_{i,j} a_i a_j ((\chi_i, \chi_j) - (P_t \chi_i, \chi_j)) \\ &\leq \left(\sum_{i,j} a_i^2 |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| \right)^{1/2} \left(\sum_{i,j} a_j^2 |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| \right)^{1/2}. \end{aligned}$$

By Schwarz's inequality. Let us consider each term separately.

$$\sum_{i,j} a_i^2 |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| = \sum_i a_i^2 \sum_j |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)|.$$

For a fixed i we have

$$\begin{aligned} \sum_j |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| &= \sum_{j \neq i} (P_t \chi_i, \chi_j) + \|\chi_i\|^2 - (P_t \chi_i, \chi_i) \\ &= \sum_j (P_t \chi_i, \chi_j) - (P_t \chi_i, \chi_i) + \|\chi_i\|^2 - (P_t \chi_i, \chi_i) \\ &= (P_t \chi_i, 1) - (P_t \chi_i, \chi_i) + \|\chi_i\|^2 - (P_t \chi_i, \chi_i) \end{aligned}$$

where 1 is the identity function. Now

$$(P_t \chi_i, 1) = (T_t \chi_i, 1) = (T_t \chi_i, T_t 1) = (\chi_i, 1) = \|\chi_i\|^2.$$

Thus the sum over j is equal to

$$2 \|\chi_i\|^2 \left(1 - \frac{(P_t \chi_i, \chi_i)}{\|\chi_i\|^2} \right)$$

and

$$\begin{aligned} \sum_{i,j} a_i^2 |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| &\leq 2 \sup_i \left(1 - \frac{(P_t \chi_i, \chi_i)}{\|\chi_i\|^2} \right). \\ \sum a_i^2 \|\chi_i\|^2 &= 2 \sup \left(1 - \frac{(P_t \chi_i, \chi_i)}{\|\chi_i\|^2} \right). \end{aligned}$$

For the second term we get

$$\sum a_j^2 |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| = \sum_j a_j^2 \sum_i |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)|$$

and

$$\begin{aligned} \sum_i |(\chi_i, \chi_j) - (P_t \chi_i, \chi_j)| &= \|\chi_j\|^2 - (P_t \chi_j, \chi_j) + \sum_{i \neq j} (P_t \chi_i, \chi_j) \\ &= \|\chi_j\|^2 - (P_t \chi_j, \chi_j) + \sum_i (P_t \chi_i, \chi_j) - (P_t \chi_j, \chi_j) \\ &= \|\chi_j\|^2 - (P_t \chi_j, \chi_j) + (P_t \mathbf{1}, \chi_j) - (P_t \chi_j, \chi_j) \\ &= 2(\|\chi_j\|^2 - (P_t \chi_j, \chi_j)). \end{aligned}$$

And the second term has the same bound. Thus

$$1 - (P_t x, x) \leq 2 \sup \left(1 - \frac{(P_t \chi_i, \chi_i)}{\|\chi_i\|^2} \right).$$

Now

$$\begin{aligned} \|P_t x - x\|^2 &= \|P_t x\|^2 + \|x\|^2 - 2(P_t x, x) \\ &\leq 2((I - P_t)x, x) \leq 4 \sup_i \left(1 - \frac{(P_t \chi_i, \chi_i)}{\|\chi_i\|^2} \right). \end{aligned}$$

By assumption this tends to zero uniformly. Hence $\|P_t x - x\|$ tends to zero uniformly, for x in a dense subset of B_0 , and hence everywhere because $\|P_t\| \leq 1$.

REMARKS. It is enough to assume the condition of the Theorem for a family of Borel sets, A , such that the functions χ_A generate B_0 . It follows, from the fact that Q is bounded, that

$$1 - \frac{\mu(X_0 \in A \cap X_t \in A)}{\mu(X_0 \in A)} \leq (\text{const})t.$$

Theorem 1.3 is well known for processes with countable state space. A brief discussion of this case is given in [1] page 265.

The function $P_t x$ is differentiable for many x 's even if Q is unbounded. In order to study this we will need:

LEMMA 2.3. *Let R_t be strongly continuous semi group of operators, defined on a reflexive space X . If $x \in X$ then $R_t x$ is differentiable if the expression $(1/t) \|R_t x - x\|$ is bounded for all t .*

This is included in Theorem 10.7.2 of [4]

Let $y \in L_2$ and Ω_1 be a subset of Ω such that $\chi_{\Omega_1} \in B_0$. Then

$$\|E_0 y\|^2 = \|\chi_{\Omega_1} \cdot E_0 y\|^2 + \|\chi_{\Omega_2} \cdot E_0 y\|^2$$

where $\Omega_2 = \Omega - \Omega_1$. Now $\chi_{\Omega_1} \cdot E_0 y$ is the projection of y on the subspace generated by characteristic function, in B_0 , of subsets of Ω_1 . Thus

$$\begin{aligned} \|\chi_{\Omega_1} \cdot E_0 y\| &= \sup \{ \sum (y, \chi_i) \alpha_i \mid \chi_i = \chi_{x_0 \in A_i} \in B_0 \text{ and } A_i \text{ are disjoint} \\ &\text{Borel sets, such that } X_0 \in A_i \subset \Omega_1, \text{ and } \sum \alpha_i^2 \|\chi_i\|^2 = 1 \}. \end{aligned}$$

But

$$| \sum (y, \chi_i) a_i | \leq \sum \frac{|(y, \chi_i)|}{\| \chi_i \|} | a_i | \| \chi_i \| \leq \left(\sum \frac{(y, \chi_i)^2}{\| \chi_i \|^2} \right)^{1/2} .$$

Hence

$$\| \chi_{a_1} \cdot E_0 y \|^2 = \sup \left\{ \sum \frac{(y, \chi_i)^2}{\| \chi_i \|^2} \mid \chi_i = \chi_{X_0 \in A_i} \in B_0, \right. \\ \left. A_i \text{ disjoint Borel sets and } X_0 \in A_i \subset \Omega_1 \right\}$$

A similar expression holds for $\| \chi_{a_2} \cdot E_0 y \|^2$.

THEOREM 3.3. *Let A be a Borel set. The function $P_t \chi_{X_0 \in A}$ is differentiable at zero if and only if the two expressions below, are bounded:*

1. $\frac{1}{t^2} \sup \left\{ \sum \frac{\mu(X_t \in A \cap X_0 \in A_i)^2}{\mu(X_0 \in A_i)} \mid A_i \text{ disjoint} \right. \\ \left. \text{Borel sets and } A_i \cap A = \phi \right\} .$
2. $\frac{1}{t^2} \sup \left\{ \sum \frac{(\mu(X_t \in A \cap X_0 \in A_i) - \mu(X_0 \in A_i))^2}{\mu(X_0 \in A_i)} \mid A_i \text{ disjoint} \right. \\ \left. \text{Borel sets and } A_i \subset A \right\} .$

Proof. By Lemma 2.3 and the above discussion it is enough to show that

$$\frac{1}{t^2} \sup \left\{ \sum \frac{(P_t \chi_{X_0 \in A} - \chi_{X_0 \in A}, \chi_{X_0 \in A_i})^2}{\| \chi_{X_0 \in A_i} \|^2} \mid A_i \text{ disjoint and } A_i \cap A = \phi \right\}$$

and

$$\frac{1}{t^2} \sup \left\{ \sum \frac{(P_t \chi_{X_0 \in A} - \chi_{X_0 \in A}, \chi_{X_0 \in A_i})^2}{\| \chi_{X_0 \in A_i} \|^2} \mid A_i \text{ disjoint and } A_i \subset A \right\}$$

are both bounded. But these expressions are equal to 1 and 2 respectively.

REMARK. If A is an atom for B_0 then the second expression is

$$\frac{1}{t^2} \left(\frac{\chi(X_t \in A \cap X_0 \in A) - \mu(X_0 \in A)}{\mu(X_0 \in A)} \right)^2 \mu(X_0 \in A) \\ = \left(\frac{1}{t} \left(1 - \frac{\mu(X_t \in A \cap X_0 \in A)}{\mu(X_0 \in A)} \right) \right)^2 \mu(X_0 \in A) .$$

A more precise information is available in the following special case.

THEOREM 4.3. *Let $x \in B_0$. Then $x \in D(Q)$ and $(Qx, x) = 0$ if and only if $(1/t^2)(\|x\|^2 - (P_t x, x))$ is bounded. In this case Q^*x exists and is equal to $-Qx$.*

Proof. If $y \in B_0$ then

$$\begin{aligned} \|y - P_t y\|^2 &= \|y\|^2 + \|P_t y\|^2 - 2(P_t y, y) \\ &\leq 2(\|y\|^2 - (T_t y, y)) = \|y - T_t y\|^2 \end{aligned}$$

thus

a.
$$\frac{\|T_t y - y\|}{\sqrt{t}} = \sqrt{2 \frac{(y - P_t y, y)}{t}} \geq \frac{\|P_t y - y\|}{\sqrt{t}}.$$

Also if y and z are any two vectors in B_0 then

b.
$$\begin{aligned} \left(\frac{1}{t}(P_t - 1)z, y\right) &= \frac{1}{t}(T_t z - z, y) = \frac{1}{t}(T_t z, y - T_t y) \\ &= \frac{1}{t}(T_t z - z, y - T_t y) + \frac{1}{t}(z, y - P_t y) \end{aligned}$$

where we used Equation 1.2.c for the third equality.

Let x be such that $(1/t^2)(\|x\|^2 - (P_t x, x))$ is bounded. Then from (a) we get

$$\left\| \frac{1}{t^2} (P_t x - x) \right\|^2 \leq 2 \frac{(x - P_t x, x)}{t^2}$$

and is bounded by assumption. Thus we know from Lemma 2.3 that $x \in D(Q)$. Moreover

$$(Qx, x) = - \lim t \frac{(x - P_t x, x)}{t^2} = 0.$$

Conversely let $x \in D(Q)$ and $(Qx, x) = 0$. If $y \in D(Q)$ then it follows from (b) that

$$\begin{aligned} (Qx, y) &= \lim_{t \rightarrow 0} \frac{1}{t} ((P_t - 1)x, y) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (T_t x - x, y - T_t y) + \frac{1}{t} (x, y - P_t y) \end{aligned}$$

the second term tends to $-(x, Qy)$ while the first is bounded by

$$\begin{aligned} \left| \frac{1}{t} (T_t x - x, y - T_t y) \right| &\leq \frac{\|T_t x - x\| \|y - T_t y\|}{\sqrt{t} \sqrt{t}} \\ &= \left(2 \frac{(x - P_t x, x)}{t} \cdot 2 \frac{(y - P_t y, y)}{t} \right)^{1/2} \end{aligned}$$

as $t \rightarrow 0$ this tends to

$$(4(Qx, x)(Qy, y))^{1/2} = 0.$$

Thus

$$(Qx, y) = - (x, Qy)$$

or

$$x \in D(Q^*) \quad \text{and} \quad Q^*x = -Qx.$$

Now

$$\begin{aligned} (x - P_t x, x) &= \int_0^t (QP_u x, x) du \leq t \max_{u \leq t} |(QP_u x, x)| \\ &= t \max_{u \leq t} |(P_u x, Qx)| = t \max_{u \leq t} |(P_u x - x, Qx)| \\ &\leq \text{const. } t^2 \end{aligned}$$

because $\|P_u x - x\| \leq \text{const. } u$.

REMARK. If x is a characteristic function then it is easy to see that $Qx = 0$ if $(Qx, x) = 0$.

The referee called my attention to the fact that this theorem generalizes to arbitrary semi groups of contraction operators, when T_t is replaced by the group of unitary operators which project down to P_t as in s_2 Nagy theorem (See Riesz Nagy appendix to the third edition). Some simple changes have to be done to take care of the complex case.

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