THE SOCHOCKI-PLEMEŁJ FORMULA FOR THE FUNCTIONS OF TWO COMPLEX VARIABLES

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Introduction. In the case of one complex variable the following theorems are well known [3]:

1. Let $C$ be a rectifiable oriented Jordan arc or curve and $f(\zeta)$ an integrable function defined on $C$, analytic at a point $z_0 \in C$ (in case $C$ is an arc we suppose $z_0$ is different from both endpoints of $C$). Then the function

$$ F(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta $$

possesses the left and right limit $F_l(z_0)$ and $F_r(z_0)$, respectively, when the point $\zeta$ approaches to the point $z_0$ remaining permanently on one side of $C$ and the relation

$$ F_l(z_0) - F_r(z_0) = f(z_0) $$

holds.

2. Under the same conditions concerning the curve $C$ suppose $f(\zeta)$ satisfies at every point $\zeta \in C$ the Hölder condition

$$ |f(\zeta) - f(\zeta_1)| \leq M|\zeta - \zeta_1|^\alpha, \quad M > 0, \quad 0 < \alpha \leq 1. $$

Then $F(z)$ possesses at almost every point $z_0 \in C$ the left and right limit when the point $\zeta$ approaches to $z_0$ along a non-tangent path to $C$ and

$$ F_l(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z_0} \, d\zeta + \frac{1}{2} f(z_0), $$

$$ F_r(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z_0} \, d\zeta - \frac{1}{2} f(z_0). $$

The improper integral on the right hand side is taken in the Cauchy sense.

The aim of the present note is to extend these theorems to the theory of functions of two complex variables. We start with Bergman’s integral formula [1], [2] which generalizes the Cauchy formula for the case of functions of several variables. It would be very interesting to obtain similar results starting with other integral formulas which are similar to Bergman’s formula e.g. A. Weil’s formula [6] or later forms.

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\footnote{Analogous results about the limits of exterior differential forms have been obtained by C. H. Look and T. D. Chung, see [4].}
of it, see [5].

The case of a bicylinder. Let $D$ be a bicylinder bounded by the hypersurfaces

$$z_1 - e^{i\lambda_1} = 0, \quad |z_2| \leq 1 \quad \lambda_j \in [0, 2\pi]$$

$$z_2 - e^{i\lambda_2} = 0, \quad |z_1| \leq 1$$

and let $f(\zeta_1, \zeta_2)$ be an integrable function defined on the distinguished boundary surface $d$ of $D$

$$(z_1 = e^{i\lambda_1}) \times (z_2 = e^{i\lambda_2}).$$

1. Suppose that $f(\zeta_1, \zeta_2)$ is analytic at a point $z_1^0, z_2^0 \in D$. We consider the function

$$(1) \quad F(z_1, z_2) = -\frac{1}{4\pi^2} \int_a \int_{x_{1,2}} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \, d\zeta_1 d\zeta_2.$$

Since $f(\zeta_1, \zeta_2)$ is analytic at the point $z_1^0, z_2^0 \in d$, there exists a small bicylinder $B$ which contains $z_1^0, z_2^0$ inside and such that $f(\zeta_1, \zeta_2)$ is analytic in $B$. B is bounded by the hypersurfaces

$$z_1 - z_1^0 - r_1 e^{i\lambda_1} = 0, \quad |z_1| \leq r_1$$

$$z_2 - z_2^0 - r_2 e^{i\lambda_2} = 0, \quad |z_2| \leq r_2$$

$$r_j > 0, \quad \lambda_{j+2} \in [0, 2\pi], \quad j = 1, 2.$$

Suppose that the point $z_1, z_2$ belongs to $DB$, the intersection of $D$ and $B$. Then using the integral formula for the function $f(\zeta_1, \zeta_2)$ and the domain $DB$ we obtain

$$(2) \quad f(z_1, z_2) = -\frac{1}{4\pi^2} \int_{a_{1,2}} \int_{a_{3,4}} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \, d\zeta_1 d\zeta_2$$

$$- \frac{1}{4\pi^2} \int_{a_{1,2}} \int_{a_{3,4}} - \frac{1}{4\pi^2} \int_{a_{3,4}} \int_{a_{1,2}} - \frac{1}{4\pi^2} \int_{a_{3,4}} \int_{a_{1,2}} ,$$

where $d_j, j = 1, 2, 3, 4$ denotes the positive oriented circle $z_j - e^{i\lambda_j} = 0$ and $z_j - z_j^0 - r_j e^{i\lambda_{j+2}} = 0, j = 1, 2$, respectively. (The integrands missing in the formula (2) are equal to that of the first integral.)

From (1) and (2) results

$$(3) \quad F(z_1, z_2) = -\frac{1}{4\pi^2} \int_{a_{1,2}} \int_{a_{3,4}} \frac{d\zeta_1}{\zeta_1 - z_1} \int_{a_{3,4}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} \, d\zeta_2 - \frac{1}{4\pi^2} \int_{a_{3,4}} \int_{a_{1,2}} \frac{d\zeta_2}{\zeta_2 - z_2}$$

$$+ f(z_1, z_2) + \frac{1}{4\pi^2} \int_{a_{3,4}} \int_{a_{1,2}} + \frac{1}{4\pi^2} \int_{a_{3,4}} \int_{a_{1,2}} + \frac{1}{4\pi^2} \int_{a_{3,4}} \int_{a_{1,2}} .$$

Let $z_1, z_2$ approach to the point $z_1^0, z_2^0$ remaining inside the bicylinder $D$, then
\[ f(z_1, z_2) \rightarrow f(z_1^0, z_2^0), \]

(4) \[ \frac{1}{4\pi^2} \int_{a_1, \bar{a}_1} \int_{a_2, \bar{a}_2} \frac{f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - \frac{1}{4\pi^2} \int_{a_1, \bar{a}_1} \int_{a_2, \bar{a}_2} \frac{f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)}. \]

Using the Cauchy formula for the domain which lies on the \( z_2 \)-plane and is bounded by the curves \( d_1 \bar{B} \) and \( d_2 \bar{B} \), we obtain

(4') \[ \int_{d_1 \bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 = 2\pi i f(\zeta_1, z_2) - \int_{d_2 \bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2. \]

When the point \( z_1, z_2 \) tends to \( z_1^0, z_2^0 \) it results from (4')

\[ \lim_{z_1, z_2 \to z_1^0, z_2^0} \int_{d_1 \bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 = 2\pi i f(\zeta_1, z_2^0) - \int_{d_2 \bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_2. \]

The convergence is uniform with respect to \( \zeta_1 \in d_1 \bar{B} \), therefore,

(5) \[ \lim_{z_1, z_2 \to z_1^0, z_2^0} \frac{1}{4\pi^2} \int_{d_1, \bar{d}_1} \int_{d_2, \bar{d}_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2 \]

\[ = \frac{1}{4\pi^2} \int_{d_1, \bar{d}_1} \frac{d\zeta_1}{\zeta_1 - z_1} \left\{ 2\pi i f(\zeta_1, z_2^0) - \int_{d_2, \bar{d}_2} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_2 \right\}. \]

In a similar way we obtain the formula

(6) \[ \lim_{z_1, z_2 \to z_1^0, z_2^0} \frac{1}{4\pi^2} \int_{d_1, \bar{d}_1} \int_{d_2, \bar{d}_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2 \]

\[ = \frac{1}{4\pi^2} \int_{d_1, \bar{d}_1} \frac{d\zeta_2}{\zeta_2 - z_2^0} \left\{ 2\pi i f(z_1^0, \zeta_2) - \int_{d_2, \bar{d}_2} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_1 \right\}. \]

For the first and second term on the right hand side of (3) we obtain the limits \( (z_1, z_2 \in DB) \):

(7') \[ \lim_{z_1, z_2 \to z_1^0, z_2^0} \left\{ - \frac{1}{4\pi^2} \int_{d_1, \bar{d}_1} \int_{d_2, \bar{d}_2} \frac{d\zeta_1}{\zeta_1 - z_1} \left\{ 2\pi i f(\zeta_1, \zeta_2) - \int_{d_2, \bar{d}_2} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_2 \right\} \right\} \]

\[ = \left\{ - \frac{1}{4\pi^2} \int_{d_1, \bar{d}_1} \int_{d_2, \bar{d}_2} \frac{d\zeta_1}{\zeta_1 - z_1} \left\{ 2\pi i f(\zeta_1, \zeta_2) - \int_{d_2, \bar{d}_2} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_2 \right\} \right\}, \]

and

(7'') \[ \lim_{z_1, z_2 \to z_1^0, z_2^0} \left\{ - \frac{1}{4\pi^2} \int_{d_1, \bar{d}_1} \int_{d_2, \bar{d}_2} \frac{d\zeta_2}{\zeta_2 - z_2^0} \left\{ 2\pi i f(z_1^0, \zeta_2) - \int_{d_2, \bar{d}_2} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_1 \right\} \right\}. \]

From (3), (4), (5), (6), (7') and (7'') results
(8) \[ F_{k}(z_{1}^{0}, z_{2}^{0}) = \lim_{z_{1}, z_{2} \to z_{1}^{0}, z_{2}^{0}} F(z_{1}, z_{2}) = f(z_{1}^{0}, z_{2}^{0}) + \frac{1}{2\pi i} \int_{a_{1}-a_{2} \to a_{1}-a_{2}} \frac{f(\xi_{1}, \xi_{2})}{\xi_{1} - z_{1}^{0}} d\xi_{1} \]
\[ + \frac{1}{2\pi i} \int_{a_{2}-a_{1} \to a_{2}-a_{1}} \frac{f(\xi_{1}, \xi_{2})}{\xi_{2} - z_{2}^{0}} d\xi_{2} - \frac{1}{4\pi^{2}} \int_{a_{1}-a_{2} \to a_{1}-a_{2}} \frac{d\xi_{1}}{\xi_{1} - z_{1}^{0}} \cdot \frac{d\xi_{2}}{\xi_{2} - z_{2}^{0}} \cdot \int_{a_{2}-a_{1} \to a_{2}-a_{1}} \frac{f(\xi_{1}, \xi_{2})}{\xi_{2} - z_{2}^{0}} d\xi_{2}. \]

When the point \( z_{1}, z_{2} \) does not belong to \( D \) and tends to \( z_{1}^{0}, z_{2}^{0} \), we obtain three values for the exterior limit \( F_{k}(z_{1}^{0}, z_{2}^{0}) \), \( k = 1, 2, 3 \), (in this case we need to put 0 instead of \( f(z_{1}, z_{2}) \) in (2) and similar changes ought to be made in (7') and (7''))

\[ F_{11}(z_{1}^{0}, z_{2}^{0}) = \lim_{z_{1}, z_{2} \to z_{1}^{0}, z_{2}^{0}} F(z_{1}, z_{2}) \]
\[ = - \frac{1}{4\pi^{2}} \int_{a_{1}-a_{2} \to a_{1}-a_{2}} \frac{d\xi_{1}}{\xi_{1} - z_{1}^{0}} \int_{a_{2}-a_{1} \to a_{2}-a_{1}} \frac{f(\xi_{1}, \xi_{2})}{\xi_{2} - z_{2}^{0}} d\xi_{2} ; \]
\[ F_{12}(z_{1}^{0}, z_{2}^{0}) = \lim_{z_{1}, z_{2} \to z_{1}^{0}, z_{2}^{0}} F(z_{1}, z_{2}) \]
\[ = - \frac{1}{4\pi^{2}} \int_{a_{1}-a_{2} \to a_{1}-a_{2}} \frac{d\xi_{1}}{\xi_{1} - z_{1}^{0}} \int_{a_{2}-a_{1} \to a_{2}-a_{1}} \frac{f(\xi_{1}, \xi_{2})}{\xi_{2} - z_{2}^{0}} d\xi_{2} + \frac{1}{2\pi i} \int_{a_{2}-a_{1} \to a_{2}-a_{1}} \frac{f(\xi_{1}, \xi_{2})}{\xi_{2} - z_{2}^{0}} d\xi_{2} ; \]
\[ F_{13}(z_{1}^{0}, z_{2}^{0}) = \lim_{z_{1}, z_{2} \to z_{1}^{0}, z_{2}^{0}} F(z_{1}, z_{2}) \]
\[ = - \frac{1}{4\pi^{2}} \int_{a_{1}-a_{2} \to a_{1}-a_{2}} \frac{d\xi_{1}}{\xi_{1} - z_{1}^{0}} \int_{a_{2}-a_{1} \to a_{2}-a_{1}} \frac{f(\xi_{1}, \xi_{2})}{\xi_{2} - z_{2}^{0}} d\xi_{2} + \frac{1}{2\pi i} \int_{a_{2}-a_{1} \to a_{2}-a_{1}} \frac{f(\xi_{1}, \xi_{2})}{\xi_{2} - z_{2}^{0}} d\xi_{2}. \]

Therefore

\[ F_{i}(z_{1}^{0}, z_{2}^{0}) - F_{1i}(z_{1}^{0}, z_{2}^{0}) = f(z_{1}^{0}, z_{2}^{0}) + \frac{1}{2\pi i} \int_{a_{1}-a_{2} \to a_{1}-a_{2}} \frac{f(\xi_{1}, \xi_{2})}{\xi_{1} - z_{1}^{0}} d\xi_{1} \]
\[ + \frac{1}{2\pi i} \int_{a_{2}-a_{1} \to a_{2}-a_{1}} \frac{f(\xi_{1}, \xi_{2})}{\xi_{2} - z_{2}^{0}} d\xi_{2} ; \]
\[ F_{i}(z_{1}^{0}, z_{2}^{0}) - F_{1i}(z_{1}^{0}, z_{2}^{0}) = f(z_{1}^{0}, z_{2}^{0}) + \frac{1}{2\pi i} \int_{a_{1}-a_{2} \to a_{1}-a_{2}} \frac{f(\xi_{1}, \xi_{2})}{\xi_{1} - z_{1}^{0}} d\xi_{1} ; \]
\[ F_{i}(z_{1}^{0}, z_{2}^{0}) - F_{1i}(z_{1}^{0}, z_{2}^{0}) = f(z_{1}^{0}, z_{2}^{0}) + \frac{1}{2\pi i} \int_{a_{2}-a_{1} \to a_{2}-a_{1}} \frac{f(\xi_{1}, \xi_{2})}{\xi_{2} - z_{2}^{0}} d\xi_{2} . \]

REMARK. The formulas (10) can be transformed as follows.
According to the well-known formula for the function \( f(\xi_{1}, \xi_{2}) \) of
one complex variable \( \zeta \), which is analytic at the point \( \zeta_1 = z_1 \), we have (see [3])

\[
G_{i1}^{(0)}(z_1^0) = \lim_{r_1 \to 0} \frac{1}{2\pi i} \int_{a_1 - a_1 \bar{a} - a_1 \bar{a}} \frac{f(\zeta_1, z_1^0)}{\zeta_1 - z_1} d\zeta_1 = \frac{1}{2\pi i} \int_{a_1 - a_1 \bar{a} - a_1 \bar{a}} \frac{f(\zeta_1, z_1^0)}{\zeta_1 - z_1} d\zeta_1 + f(z_1^0, z_1^0).
\]

Suppose the radius \( r_1 \) of the circle \( d_3 \) tends to 0, then

\[
\lim_{r_1 \to 0} \frac{1}{2\pi i} \int_{a_1 - a_1 \bar{a} - a_1 \bar{a}} \frac{f(\zeta_1, z_1^0)}{\zeta_1 - z_1^0} d\zeta_1 = G_{i1}^{(0)}(z_1^0) - f(z_1^0, z_1^0).
\]

Similarly, we have

\[
\lim_{r_2 \to 0} \frac{1}{2\pi i} \int_{a_2 - a_2 \bar{a} - a_2 \bar{a}} \frac{f(z_2^0, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_2 = G_{i2}^{(0)}(z_2^0) - f(z_2^0, z_2^0).
\]

On the other hand, we have

\[
\lim_{r_1 \to 0} \frac{1}{2\pi i} \int_{a_1 - a_1 \bar{a} - a_1 \bar{a}} \frac{f(\zeta_1, z_1^0)}{\zeta_1 - z_1} d\zeta_1 = \lim_{r_1 \to 0} \frac{1}{2\pi i} \int_{a_1 - a_1 \bar{a} - a_1 \bar{a}} \frac{f(\zeta_1, z_1^0)}{\zeta_1 - z_1} d\zeta_1
\]

\[
\lim_{r_2 \to 0} \frac{1}{2\pi i} \int_{a_2 - a_2 \bar{a} - a_2 \bar{a}} \frac{f(z_2^0, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 = \lim_{r_2 \to 0} \frac{1}{2\pi i} \int_{a_2 - a_2 \bar{a} - a_2 \bar{a}} \frac{f(z_2^0, \zeta_2)}{\zeta_2 - z_2} d\zeta_2.
\]

Therefore,

\[
F_1(z_1^0, z_2^0) - F_1(z_2^0, z_2^0) = G_{i1}^{(0)}(z_1^0) + G_{i1}^{(0)}(z_2^0),
\]

\[
(10*) \quad F_1(z_1^0, z_2^0) - F_1(z_1^0, z_2^0) = G_{i2}^{(0)}(z_2^0) + G_{i2}^{(0)}(z_2^0),
\]

\[
F_1(z_1^0, z_2^0) - F_1(z_1^0, z_2^0) = G_{i2}^{(0)}(z_1^0),
\]

\[
(10*) \quad F_1(z_1^0, z_2^0) - F_1(z_1^0, z_2^0) = G_{i2}^{(0)}(z_1^0),
\]

2. Suppose now that the function \( f(\zeta_1, \zeta_2) \) is not analytic at \( z_1^0, z_2^0 \) but satisfies the condition

\[
|f(\zeta_1, \zeta_2) - f(z_1^0, \zeta_2)| \leq M \cdot |\zeta_1 - z_1^0|^{\alpha_1} \cdot |\zeta_2 - z_2^0|^{\alpha_2},
\]

\[
M > 0, \quad \alpha_j > 0, \quad j = 1, 2.
\]

We have

\[
F(z_1, z_2) = -\frac{1}{4\pi^2} \int_{a} \frac{f(\zeta_1, \zeta_2) - f(z_1^0, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2
\]

\[
-\frac{1}{4\pi^2} \int_{a} \frac{f(z_1^0, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2.
\]

Since \( f(z_1^0, \zeta_2) \) is analytic, we can apply the formulas (8) and (9) to the second term of (12).

According to the assumption (11) the improper integral
exists. Let \( \rho_j(z_j, d_j) \) be the Euclidean distance of the point \( z_j \) from the circle \( d_j, j = 1, 2 \). We shall show that the limit

\[
\lim_{z_1, z_2 \rightarrow z_1^0, z_2^0} J(z_1, z_2) = \lim_{z_1, z_2 \rightarrow z_1^0, z_2^0} \frac{1}{4\pi^2} \int_{A, \sum_{d_1, \sum_{d_2}}} \frac{f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \, d\zeta_1 d\zeta_2
\]

exists when the point \( z_1, z_2 \) tends to \( z_1^0, z_2^0 \) in such a way that the rations \( |z_j - z_j^0| : \rho_j(z_j, d_j), j = 1, 2 \), are bounded, i.e.,

\[
(13) \quad \frac{|z_1 - z_1^0|}{\rho_j(z_j, d_j)} < A, \quad A > 0, \quad j = 1, 2.
\]

In fact, we have

\[
J(z_1, z_2) - J(z_1^0, z_2^0) = -\frac{1}{4\pi^2} \int_{A, \sum_{d_1, \sum_{d_2}}} \frac{f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \cdot \frac{|(\zeta_1 - z_1^0)(\zeta_2 - z_2^0) - (\zeta_1 - z_1)(\zeta_2 - z_2)|}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \, d\zeta_1 d\zeta_2
\]

\[
= -\frac{1}{4\pi^2} \left\{ \int_{a_1, a_2, a_1' a_2'} + \int_{a_1 - a_1' a_2 - a_2'} + \int_{a_1 - a_1' a_2' - a_2'} + \int_{a_2' a_2 a_1' a_1} \right\}
\]

(the integrands missing in the formula (14) are equal to that of the first term). The third term on the right hand side tends to 0 when \( z_1, z_2 \rightarrow z_1^0, z_2^0 \). The first term can be written in the form

\[
(15) \quad -\frac{1}{4\pi^2} \left\{ \int_{a_1, a_2, a_1' a_2'} + \int_{a_1 - a_1' a_2 - a_2'} + \int_{a_1 - a_1' a_2' - a_2'} + \int_{a_2' a_2 a_1' a_1} \right\}
\]

Suppose the radii \( r_j, j = 1, 2 \), of the bicylinder \( B \) are so small that for \( \zeta_j \in d_j B, j = 1, 2 \), we have \( |\zeta_j - z_j^0| \leq \delta \), where \( \delta > 0 \) is an arbitrary fixed number. Let \( z_1, z_2 \) satisfy the condition (13), then using (11) we obtain

\[
\left| -\frac{1}{4\pi^2} \int_{a_1, a_2, a_1' a_2'} + \int_{a_1 - a_1' a_2 - a_2'} + \int_{a_1 - a_1' a_2' - a_2'} + \int_{a_2' a_2 a_1' a_1} \right| \leq \frac{1}{4\pi^2} M(A + A(1 + A)) \left\{ \int_{a_1, a_2, a_1' a_2'} + \int_{a_1 - a_1' a_2 - a_2'} + \int_{a_1 - a_1' a_2' - a_2'} + \int_{a_2' a_2 a_1' a_1} \right\}
\]

\[
\leq \frac{1}{4\pi^2} M(A + A(1 + A)) \left\{ \int_{a_1, a_2, a_1' a_2'} + \int_{a_1 - a_1' a_2 - a_2'} + \int_{a_1 - a_1' a_2' - a_2'} + \int_{a_2' a_2 a_1' a_1} \right\} < \text{const.} \delta^{a_1 + a_2}.
\]

Therefore, for sufficiently small fixed \( \delta > 0 \) and \( z_1, z_2 \) sufficiently near to \( z_1^0, z_2^0 \) the first and third term on the right hand side of (14) are arbitrary
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small. Similarly, the remaining two terms of (14) tend to 0 when \( \delta \to 0 \).

For the difference between the interior and exterior limits of \( F(z_1, z_2) \) we obtain the same formulas (10), (10*) assuming that \( z_1, z_2 \) tends to \( z_1^0, z_2^0 \) in such a way that the conditions (13) are satisfied.

The interior limit \( F_i(z_1^0, z_2^0) \) is equal to \( J(z_1^0, z_2^0) \) plus the terms of the right hand side of (8). Similarly, we obtain three values of the exterior limits \( F_e(z_1^0, z_2^0), j = 1, 2, 3 \), adding \( J(z_1, z_2) \) to the terms of the right hand side of (9).

A general domain with the distinguished boundary surface. Suppose the given domain \( D \) is bounded by three \( ^2 \) analytic hypersurfaces (for definitions see [1], [2])

\[
\Phi_j(z_1, z_2, \lambda_j) = 0, \quad j = 1, 2, 3,
\]

and let \( z_1^0, z_2^0 \) be a fixed point which lies on the part of the intersection \( d_{12} \) of the hypersurfaces \( \Phi_j(z_1, z_2, \lambda_j) = 0, \Phi_j(z_1, z_2, \lambda_j) = 0 \) which belongs to the boundary of \( D \). We assume that \( z_1^0, z_2^0 \) does not belong to the hypersurface \( \Phi_j(z_1, z_2, \lambda_j) = 0 \).

1. Let \( f(z_1, z_2) \) be a continuous function defined on the distinguished boundary surface \( d \) of \( D \), analytic at the point \( z_1^0, z_2^0 \). We consider the function \( F(z_1, z_2) \) defined by Bergman’s integral formula \(^2 \) [2]

\[
F(z_1, z_2) = -\frac{1}{4\pi^2} \iint_{d_{12}} \frac{f(\xi_1, \xi_2)}{(\xi_1 - z_1)(\xi_2 - z_2)} \cdot \frac{\Phi_j(z_1, \xi_2, \lambda_j)\Phi_j(z_1, z_2, \lambda_j) - \Phi_j(z_1, z_2, \lambda_j)\Phi_j(z_1, \xi_2, \lambda_j)}{\Phi_j(z_1, z_2, \lambda_j)\Phi_j(z_1, z_2, \lambda_j)} d\xi_1, d\xi_2
\]

\[
-\frac{1}{4\pi^2} \iint_{d_{13}} -\frac{1}{4\pi^2} \iint_{d_{23}},
\]

where \( z_1, z_2 \) lies outside \( \Phi_j(z_1, z_2, \lambda_j) = 0, j = 1, 2, 3 \), and \( d_{j,k} \) denotes the part of intersection of the hypersurfaces \( \Phi_j(z_1, z_2, \lambda_j) = 0, \Phi_k(z_1, z_2, \lambda_k) = 0 \) which belongs to the boundary of \( D \).

Suppose the analytic hypersurface \( \Phi_j(z_1, z_2, \lambda_j) = 0 \) intersects the hypersurfaces \( \Phi_j(z_1, z_2, \lambda_j) = 0, \Phi_j(z_1, z_2, \lambda_j) = 0 \) and define a new domain \( B \subset D \) which is bounded by segments of \( \Phi_j(z_1, z_2, \lambda_j) = 0, \Phi_j(z_1, z_2, \lambda_j) = 0 \) and \( \Phi_j(z_1, z_2, \lambda_j) = 0 \). Further, suppose that the point \( z_1^0, z_2^0 \) does neither belong to the intersection of \( \Phi_j = 0, \Phi_j = 0 \) nor to that of \( \Phi_j = 0, \Phi_j = 0 \), and lies on the boundary of \( B \). Let \( B \) be sufficiently small so that \( f(\xi_1, \xi_2) \) is analytic in \( B \).

Let \( z_1, z_2 \) be an arbitrary point in \( B \). Using Bergman’s integral

\(^2\) For simplicity we assume that the number of the boundary surfaces is 3, but the considerations are valid for the general case.

\(^3\) The integrands of the second and third integrals equal to those of the first with \( \Phi_j \) and \( \Phi_j \) replaced by \( \Phi_1, \Phi_2 \) and \( \Phi_2, \Phi_3 \), respectively.
formula representing the function \( f(z_1, z_2) \) in the domain \( B \), we obtain
(comp. footnote 2)

\[
(17) \quad f(z_1, z_2) = - \frac{1}{4\pi^2} \iiint_{d_1 \cdots d_9} \frac{f(\xi_1, \xi_2)}{(\xi_1 - z_1)(\xi_2 - z_2)} \cdot \left\{ \frac{\Phi_1(z_1, \xi_2, \lambda_2)\Phi_2(z_1, z_2, \lambda_1) - \Phi_1(z_1, z_2, \lambda_1)\Phi_2(z_1, \xi_2, \lambda_2)}{\Phi_1(z_1, z_2, \lambda_1)\Phi_2(z_1, z_2, \lambda_2)} \right\} d\xi_1 d\xi_2
\]

Since \( \iiint_{d_1} = \iiint_{d_1 \cdots d_9} + \iiint_{d_1 - d_1 \cdots d_9} \) it follows from (16) and (17)

\[
(18) \quad F(z_1, z_2) = - \frac{1}{4\pi^2} \iiint_{d_1 \cdots d_9} + \iiint_{d_1 - d_1 \cdots d_9} + \frac{1}{4\pi^2} \iiint_{d_1 \cdots d_9} - \frac{1}{4\pi^2} \iiint_{d_1} - \frac{1}{4\pi^2} \iiint_{d_9} .
\]

If the point \( z_1, z_2 \) lies outside the domain \( B \) and the hypersurfaces \( \Phi_j = 0, j = 1, \cdots, 4 \), we ought to substitute 0 for \( f(z_1, z_2) \) in (18).

Consider the integrals on the right hand side of (18). As long as the point \( z_1, z_2 \) does not lie on any of the hypersurfaces \( \Phi_j(z_1, z_2, \lambda_j) = 0 \), \( j = 1, 2, 3, 4 \), we have \( \Phi_j(z_1, z_2, \lambda_j) \neq 0 \). According to the assumption under which the Bergman integral formula was proved (see [2]) the functions

\[
(19) \quad \psi_{jk}(z_1, z_2, \xi_1, \xi_2, \lambda_j, \lambda_k)
\]

\[
= \frac{\Phi_j(z_1, \xi_2, \lambda_2)\Phi_k(z_1, z_2, \lambda_1) - \Phi_j(z_1, z_2, \lambda_1)\Phi_k(z_1, \xi_2, \lambda_2)}{(\xi_1 - z_1)(\xi_2 - z_2)},
\]

\( j, k = 1, \cdots, 4 \)

are continuous provided that \( \xi_1, \xi_2 \in d \) and \( z_1, z_2 \) does not lie on the distinguished boundary surface \( d \) of \( D \). (It can happen that \( \xi_1 = z_1 \) or \( \xi_2 = z_2 \), but the case \( \xi_1, \xi_2 = z_1, z_2 \) is excluded.)

We denote by \( \lambda_1^0 \) and \( \lambda_2^0 \) the values of the parameters \( \lambda_1 \) and \( \lambda_2 \) which correspond to the point \( z_1^0, z_2^0 \), i.e., \( \Phi_1(z_1^0, z_2^0, \lambda_1^0) = 0, \Phi_2(z_1^0, z_2^0, \lambda_2^0) = 0 \).

Let \( z_1 = z_1^0, z_2 = z_2^0 \), then the integrals in (18) are improper since the factors \( \Phi_j^{-1}(z_1^0, z_2^0, \lambda_j) \) and \( \Phi_j^{-1}(z_1^0, z_2^0, \lambda_j) \) are indefinite for \( \lambda_1 = \lambda_1^0 \) and \( \lambda_2 = \lambda_2^0 \), respectively. The functions \( \psi_{jk}(z_1^0, z_2^0, \xi_1, \xi_2, \lambda_j, \lambda_k) \) are continuous for \( (\xi_1, \xi_2) \in d_{12} - d_{11}B + d_{11}B + d_{21}B + d_{12} + d_{13} \) (according to Bergman's assumption) because the point \( \xi_1, \xi_2 \) does not coincide with \( z_1^0, z_2^0 \).

In general, the integrals on the right hand side of (18) are divergent for \( (z_1, z_2) = (z_1^0, z_2^0) \).

Suppose the functions \( \Phi_j(z_1^0, z_2^0, \lambda_j), j = 1, 2 \), satisfy the conditions

\[
(19^*) \quad |\Phi_j(z_1^0, z_2^0, \lambda_j)| \geq A |\lambda_j - \lambda_j^0|^\alpha, \quad A > 0, \ 0 < \alpha < 1 ,
\]
then $F(z_1, z_2)$ exists. We denote by $\rho(z_1, z_2; z_1^0, z_2^0)$ the Euclidean distance between the points $z_1, z_2$ and $z_1^0, z_2^0$ and by $\rho_j(z_1, z_2; \Phi_j)$ the distance of the point $z_1, z_2$ to the hypersurface $\Phi_j = 0$. If the functions $\Phi_j(z_1, z_2, \lambda_j)$, $j = 1, 2$, satisfy the conditions

\begin{equation}
|\Phi_j(z_1, z_2, \lambda_j)| \geq A |\lambda_j - \lambda_j^0|^\alpha, \quad 0 < \alpha < \frac{1}{2},
\end{equation}

for $z_1, z_2$ belonging to $\Delta$, where $\Delta$ is defined by the inequalities

\begin{equation}
\Delta: 0 < \rho(z_1, z_2; z_1^0, z_2^0) < M, \quad M > 0, j = 1, 2,
\end{equation}

then

\begin{equation}
\lim_{z_1, z_2 \to z_1^0, z_2^0} F(z_1, z_2) = F(z_1^0, z_2^0).
\end{equation}

The proof of (21) is similar to that given in § 1.

Similarly, if the point $z_1, z_2$ lies outside the domain $D$ and tends to $z_1^0, z_2^0$ there exists the exterior limit

\begin{equation}
\lim_{z_1, z_2 \to z_1^0, z_2^0} F(z_1, z_2) = F_i(z_1^0, z_2^0)
\end{equation}

provided that (20) and (20*) hold. The difference of both limits is equal to $f(z_1^0, z_2^0)$:

\begin{equation}
F_i(z_1^0, z_2^0) - F_i(z_1^0, z_2^0) = f(z_1^0, z_2^0).
\end{equation}

**Remark.** Under the conditions (20), (20*) there exists one interior and only one exterior limit of the function $F(z_1, z_2)$ for $z_1, z_2 \rightarrow z_1^0, z_2^0$.

2. Suppose now the function $f(\zeta_1, \zeta_2)$ is not analytic at the point $z_1^0, z_2^0$ but satisfies the condition

\begin{equation}
|f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)| \leq A |\zeta_1 - z_1^0| |\zeta_2 - z_2^0|, \quad A > 0.
\end{equation}

The function $F(z_1, z_2)$ can be represented as follows

\begin{equation}
F(z_1, z_2) = \sum_{j \leq k \leq 3} - \frac{1}{4\pi^2} \int_{\Delta_{jk}} \left[ f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0) \right] + f(z_1^0, z_2^0)
\end{equation}

\begin{equation}
\cdot \frac{\nu_{jk}(\zeta_1, \zeta_2, \zeta_1^0, \zeta_2^0, \lambda_j, \lambda_k)}{\Phi_j(z_1, z_2, \lambda_j) \Phi_k(z_1, z_2, \lambda_k)} d\zeta_1 d\zeta_2.
\end{equation}

Since $f(z_1^0, z_2^0) = \text{const.}$ is an analytic function, we can apply to the latter terms in (24) the results obtained in § 1. Under the conditions (20), (20*) there exists the exterior and interior limit of those terms.

Consider the first term in (24). If the function
\[
\psi_{12}(\zeta_1^0, \zeta_2^0, \zeta_1, \zeta_2, \lambda_1, \lambda_2) = \frac{\Phi_1(\zeta_1^0, \zeta_2^0, \lambda_1)\Phi_2(\zeta_1^0, \zeta_2^0, \lambda_2) - \Phi_3(\zeta_1^0, \zeta_2^0, \lambda_1)\Phi_4(\zeta_1^0, \zeta_2^0, \lambda_2)}{(\zeta_1 - \zeta_1^0)(\zeta_2 - \zeta_2^0)}
\]

is continuous for \(\zeta_1, \zeta_2 \in d_{12}\), the integral

\[
(25) - \frac{1}{4\pi^2} \int d_{12} \left[ f(\zeta_1, \zeta_2) - f(\zeta_1^0, \zeta_2^0) \right] \cdot \psi_{12}(\zeta_1^0, \zeta_2^0, \zeta_1, \zeta_2, \lambda_1, \lambda_2) \cdot \frac{1}{\Phi_1(\zeta_1^0, \zeta_2^0, \lambda_1)\Phi_4(\zeta_1^0, \zeta_2^0, \lambda_2)} \, d\zeta_1 \, d\zeta_2
\]

exists provided that \(\Phi_1(\zeta_1^0, \zeta_2^0, \lambda_1)\) and \(\Phi_4(\zeta_1^0, \zeta_2^0, \lambda_2)\) satisfy the condition (19*). If in addition \(\psi_{12}(\zeta_1^0, \zeta_2^0, \zeta_1, \zeta_2, \lambda_1, \lambda_2)\) is continuous for \(\zeta_1, \zeta_2 \in d_{12}\) and \(\zeta_1, \zeta_2 \to \zeta_1^0, \zeta_2^0\) and if \(\Phi_1(\zeta_1, \zeta_2, \lambda_1), \Phi_3(\zeta_1, \zeta_2, \lambda_2)\) satisfy (20), (20*), there exists the limit of (25) for \(\zeta_1, \zeta_2 \to \zeta_1^0, \zeta_2^0\). In the case where \(\psi_{12}(\zeta_1, \zeta_2, \zeta_1, \zeta_2, \lambda_1, \lambda_2)\) is not continuous for \(\zeta_1, \zeta_2 \in d_{12}\) and \(\zeta_1, \zeta_2 \to \zeta_1^0, \zeta_2^0\) we use the condition (23). Then the limit of (25) exists provided that \(\zeta_1, \zeta_2 \to \zeta_1^0, \zeta_2^0\) under the conditions (20), (20*).

Observe that for the difference between the interior and exterior limit of \(F(\zeta_1, \zeta_2)\) we obtain the formula (22).

3. If one of the hypersurfaces \(\Phi_j(\zeta_1, \zeta_2, \lambda_j) = 0, \ j = 1, 2,\) depends on one of the variables \(\zeta_1, \zeta_2,\) e.g., if \(\Phi_1(\zeta_1, \zeta_2, \lambda_1)\) is independent from \(\zeta_2\)

\[
(26) \quad \Phi_1(\zeta_1, \zeta_2, \lambda_1) = \zeta_1 - \varphi(\lambda_1),
\]

then the integrand in the first term on the right hand side of (18) can be represented in the form

\[
(27) \quad \omega_{12} = f(\zeta_1, \zeta_2) \left[ \frac{\Phi_2(\zeta_1, \zeta_2, \lambda_2) - \Phi_3(\zeta_1, \zeta_2, \lambda_2)}{(\zeta_1 - \zeta_1^0)(\zeta_2 - \zeta_2^0)\Phi_4(\zeta_1, \zeta_2, \lambda_2)} \right].
\]

According to Bergman's assumption (26) is continuous for \(\zeta_1, \zeta_2 \in d, \zeta_2 = \zeta_2^0, \zeta_1 \neq \zeta_1\). For \(\zeta_1 = \zeta_1^0, \zeta_2 = \zeta_2^0\) the integral \(\int_{d_{12} - d_{12} \bar{B}}\) in (18) and the remaining integrals are improper. If \(\Phi_2(\zeta_1^0, \zeta_2^0, \lambda_2)\) satisfies the condition (19*) it is sufficient to take into account the singularity due to the factor \((\zeta_1 - \zeta_1^0)^{-1}\).

According to (26) the first coordinate of every point \(\zeta_1, \zeta_2\) of \(d\) belongs to the curve \(C_1: \zeta_1 = \varphi(\lambda_1)\). Suppose, the double integral over \(d_{12} - d_{12} \bar{B}\) can be represented as follows

\[
(28) \quad \int_{d_{12} - d_{12} \bar{B}} \omega_{12} \, d\zeta_1 \, d\zeta_2 = \int_{\zeta_1^0}^{\zeta_1} \int_{d_{12} - d_{12} \bar{B}} \omega_{12} \cdot \frac{f(\zeta_1, \zeta_2)\left[\Phi_2(\zeta_1, \zeta_2, \lambda_2) - \Phi_3(\zeta_1, \zeta_2, \lambda_2)\right]}{(\zeta_2 - \zeta_2^0)\Phi_4(\zeta_1, \zeta_2, \lambda_2)} \, d\zeta_2,
\]

where \([d_{12} - d_{12} \bar{B}]_z\) is the projection of the set \(d_{12} - d_{12} \bar{B}\) on the \(z_2\) plane.

Under the conditions (19*) assuming that
THE SOCHOCKI-PLEMELJ FORMULA FOR THE FUNCTIONS 

\[
\frac{f(z_1, z_2)[\Phi_2(z_1, z_2, \lambda_2) - \Phi_2(z_1, \xi_2, \lambda_2)]}{\Phi_2(z_1, z_2, \lambda_2)}
\]

is analytic at the point \(z_1', z_2'\), the integral (28) possesses one interior and two exterior limits when \(z_1, z_2 \to z_1'z_2'\). Similarly, the integrals \(\int_{d_4} d_1\) and \(\int_{d_4} d_4\) in (18) possesses one interior and two exterior limits.

In the case where \(\Phi_1(z_1, z_2, \lambda_1) = z_1 - \varphi(\lambda_1), \ \Phi_2(z_1, z_2, \lambda_2) = z_2 - \psi(\lambda_2)\), we obtain the same result as for a bicylinder.

**REMARK.** The Sochocki-Plemelj formula (22) was proved for a special class of domains-domains with the distinguished boundary surface. The basic tool was the Bergman’s integral formula (16). It arises the problem to generalize the Bergman formula for more general domains with maximal manifold (Bergman-Silov boundary) and to extend the Sochocki-Plemelj formula for such domains.

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