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**EXTENSIONS OF SHEAVES OF ASSOCIATIVE ALGEBRAS BY
NON-TRIVIAL KERNELS**

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Introduction. Let X be a topological space, \mathcal{A} a sheaf of associative algebras over X and \mathcal{A} a sheaf of two-sided \mathcal{A} -modules considered as a sheaf of algebras with trivial multiplication. It was shown in [1] that the group $F(\mathcal{A}, \mathcal{A})$ of equivalence classes of algebra extensions of \mathcal{A} with \mathcal{A} as kernel occurs naturally in an exact sequence

$$\dots \rightarrow H^1(X, \mathcal{A}) \rightarrow F(\mathcal{A}, \mathcal{A}) \rightarrow \text{Ext}^2(\mathcal{A}, \mathcal{A}) \rightarrow H^2(X, \mathcal{A}) \rightarrow \dots$$

where $H^*(X, \mathcal{A})$ denotes the Čech cohomology of X with coefficients in \mathcal{A} . In this paper the same question will be discussed for the case in which \mathcal{A} has a non-trivial multiplication. It will be shown that under appropriate hypotheses $F(\mathcal{A}, \mathcal{A})$ occurs in a similar exact sequence, except that in the other terms of the sequence, \mathcal{A} must be replaced by the "bicenter" $K_{\mathcal{A}}$ of \mathcal{A} . A precise statement of the main result of this paper is given in Theorem 2. The methods used here are an adaptation of those used by S. MacLane in [2].

1. The extension problem. Let R be a sheaf of rings on a topological space X . If C and D are sheaves of R -modules, then $\text{Hom}_R(C, D)$ will denote the sheaf of germs of R -homomorphisms of C into D and $\text{Ext}_R^n(C, D)$ will denote the n th derived functor of $\text{Hom}_R(C, D)$. If A is a sheaf of associative R -algebras, then, as usual, A^* will denote the opposite of R -algebras and $A^e = A \otimes_R A^*$ will denote the enveloping sheaf of A . If M is a sheaf of A^e -modules, the operation of A^e on M being given by the formula $(\lambda \otimes \mu^*)(\gamma) = \lambda\gamma\mu$.

$$\text{Now, let } M'_A = \text{Hom}_{A^*}(A, A) \oplus \text{Hom}_A(A, A)$$

where \oplus denotes the direct sum. Then M'_A , being the direct sum of sheaves of rings, is itself a sheaf of rings and A can be considered as a sheaf of left and right M'_A -modules as follows: Let $\sigma = (\sigma_1, \sigma_2) \in M'_A$. Then the left action is given by $\sigma(a) = \sigma_1(a)$ and the right action by $(a)\sigma = \sigma_2(a)$. Let

$$M_A = \{ \sigma \in M'_A \mid a(\sigma b) = (a\sigma)b \text{ for all } a, b \in A \}.$$

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Then M_A is a subsheaf of subrings of M'_A . M_A will be called the sheaf of germs of bimultiplications of A . Note that we cannot assert that A is a sheaf of M'_A -modules since we do not know that $(\sigma a)\tau = \sigma(a\tau)$. If σ and τ satisfy this relation then they are called permutable bimultiplications. The natural ring homomorphisms $A \rightarrow \text{Hom}_{M_A^*}(A, A)$ and $A \rightarrow \text{Hom}_A(A, A)$ given respectively by left and right multiplication induce a ring homomorphism $\mu: A \rightarrow M_A$ whose image is a sheaf of two-sided ideals. The kernel K_A of μ will be called the bicenter of A and the cokernel P_A of μ will be called the sheaf of germs of outer bimultiplications of A . P_A is a sheaf of rings and K_A is a sheaf of left and right P_A -modules. As above, K_A is not a sheaf of P_A -modules. Elements $\bar{\sigma}$ and $\bar{\tau}$ of P_A such that $(\bar{\sigma}a)\bar{\tau} = \bar{\sigma}(a\bar{\tau})$ for all $a \in K_A$ will be called permutable. Note that $\bar{\sigma}$ and $\bar{\tau}$ are permutable if and only if representative elements σ and τ in M_A are also permutable.

An extension of a sheaf \mathcal{A} of R -algebras by a sheaf A of R -algebras is an exact sequence.

$$(1) \quad 0 \rightarrow A \xrightarrow{i} \Gamma \xrightarrow{p} \mathcal{A} \rightarrow 0$$

of sheaves of R -algebras and R -algebra homomorphisms. As in [1], we shall say that such a sequence is locally trivial if there exists a covering $\mathcal{U} = \{U_\alpha\}$ of X such that the restriction of the sequence to each U_α splits as an exact sequence of sheaves of R -modules. Hence if (1) is locally trivial then there exist R -module homomorphisms $j_\alpha: \mathcal{A}|U_\alpha \rightarrow \Gamma|U_\alpha$ with $p \circ j_\alpha = \text{identity}$. Furthermore, since A is a sheaf of two-sided ideals in Γ , the map $\mu: A \rightarrow M_A$ extends to a map $\mu_\Gamma: \Gamma \rightarrow M_A$. Thus, we may define the composition

$$\theta_\alpha = (\text{coker } \mu) \circ \mu_\Gamma \circ j_\alpha: \mathcal{A}|U_\alpha \rightarrow P_A|U_\alpha.$$

Since $(j_\beta - j_\alpha): \mathcal{A}|U_{\alpha\beta} \rightarrow \mathcal{A}|U_{\alpha\beta}$, we see that $\theta_\beta = \theta_\alpha$ on $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Hence $\{\theta_\alpha\}$ determines an element $\theta \in \text{Hom}_R(\mathcal{A}, P_A)$. We shall say that this θ is induced by the extension (1). Clearly θ is an algebra homomorphism whose image consists of permutable elements. Note that this implies that K_A is a sheaf of \mathcal{A}^e -modules via the operation of P_A on K_A .

If $\theta \in \text{Hom}_R(\mathcal{A}, P_A)$ is an algebra homomorphism whose image consists of permutable elements, then, with respect to the usual equivalence relation, we wish to classify the extensions which induce θ in the manner described above.

2. The complexes. From [1], we recall that a sheaf B of R -modules is said to be weakly R -projective if each stalk B_x is an R_x -projective module and it is said to be R -coherent if there exists a covering $\mathcal{U} = \{U_\alpha\}$ such that for each U_α there are integers p and q and R -homomorphisms so that the sequence

$$R^p | U_\alpha \longrightarrow R^q | U_\alpha \longrightarrow B | U_\alpha \longrightarrow 0$$

is exact. Also, as in [1], $C^*(X, B)$ will denote the direct limit over coverings \mathcal{U} indexed by X of the Čech cohomology complexes $C^*(\mathcal{U}, B)$. If $S_0(A) = R$ and $S_n(A)$, $n > 0$ denotes the n -fold tensor product of A with itself, then we define

$$L^{i,j}(B) = C^i(X, \text{Hom}_R(S_j(A), B)) .$$

PROPOSITION 1. If X is paracompact Hausdorff and if A is weakly R -projective and R -coherent, then, for each $n \geq 0$,

$$0 \longrightarrow L^{*,n}(K_A) \longrightarrow L^{*,n}(A) \xrightarrow{\mu^*} L^{*,n}(M_A) \xrightarrow{\pi^*} L^{*,n}(P_A) \longrightarrow 0$$

is an exact sequence of complexes, the mappings being those induced by the exact sequence of sheaves

$$0 \longrightarrow K_A \longrightarrow A \xrightarrow{\mu} M_A \xrightarrow{\pi} P_A \longrightarrow 0$$

Proof. In [1] it was shown that if A is weakly R -projective and R -coherent then so is $S_n(A)$ and hence the sheaves $\text{Ext}_R^i(S_n(A), B) = 0$ for $i > 0$, $n \geq 0$ and for all B . Hence, for each $n \geq 0$, there is an exact sequence of sheaves

$$\begin{aligned} 0 \longrightarrow \text{Hom}_R(S_n(A), K_A) \longrightarrow \text{Hom}_R(S_n(A), A) \longrightarrow \text{Hom}_R(S_n(A), M_A) \\ \longrightarrow \text{Hom}_R(S_n(A), P_A) \longrightarrow 0 . \end{aligned}$$

If X is paracompact Hausdorff then $C^*(X, -)$ is an exact functor and hence we get the indicated sequence of complexes.

We would like to consider each of the complexes $L^{i,j}(-)$ in the preceding proposition as a bicomplex in some manner which reflects a given structure of K_A as a sheaf of A -modules and which coincides with the usual structure of $\text{Hom}_R(S_n(A), -)$ as a complex. This is too much to ask, but such a structure on $L^{i,j}(A)$ can be approximated as follows: Let $\theta \in \text{Hom}(A, P_A)$ be an algebra homomorphism whose image consists of permutable elements. If θ is regarded as an element of $L^{0,1}(P_A)$, then by exactness there is an element $\sigma \in L^{0,1}(M_A)$ such that $\pi_*(\sigma) = \theta$. Let σ be represented by cocycle $\{\sigma_\alpha\}$ on some sufficiently fine covering \mathcal{U} . Given this date, we can define a ‘‘coboundary’’ operator δ_σ on $L^{m,n}(A)$ by the following formula. Let $k \in L^{m,n}(A)$ be represented by a cochain $\{k_{\alpha_0, \dots, \alpha_m}\}$ on \mathcal{U} . Then

$$\begin{aligned} \delta_\sigma k_{\alpha_0, \dots, \alpha_m}(\lambda_1, \dots, \lambda_{n+1}) &= \sigma_{\alpha_0}(\lambda_1) k_{\alpha_0, \dots, \alpha_m}(\lambda_2, \dots, \lambda_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i k_{\alpha_0, \dots, \alpha_m}(\lambda_1, \dots, \lambda_i \lambda_{i+1}, \dots, \lambda_{n+1}) \\ &+ (-1)^{n+1} k_{\alpha_0, \dots, \alpha_m}(\lambda_1, \dots, \lambda_n) \sigma_{\alpha_0}(\lambda_{n+1}) . \end{aligned}$$

We shall see that the restriction of δ_σ to $L^{i,j}(K_A)$ is in fact a good coboundary operator.

In order to investigate the properties of δ_σ and the relations between δ_σ and the Čech coboundary operator $\widehat{\delta}$, we must introduce some more notation.

(2.1) To avoid constantly writing variables we make the following convention: If r is a function of p variables and s is a function of q variables, both with values in an algebra, then $r \cdot s$ is the function of $p + q$ variables defined by

$$r \cdot s(\lambda_1, \dots, \lambda_{p+q}) = r(\lambda_1, \dots, \lambda_p) \cdot s(\lambda_{p+1}, \dots, \lambda_{p+q}) .$$

(2.2) m will denote ambiguously the multiplication in all of the algebras which appear here.

(2.3) Since θ is an algebra homomorphism,

$\pi_*(\sigma_\alpha \cdot \sigma_\alpha - \sigma_\alpha \circ m) = 0$ Hence there exists an $f \in L^{0,2}(A)$ which is represented by a cochain $\{f_\alpha\}$ on \mathcal{U} such that

$$\mu_* f_\alpha = \sigma_\alpha \cdot \sigma_\alpha - \sigma_\alpha \circ m .$$

(2.4) Since $\pi_*(\widehat{\delta}\sigma) = \widehat{\delta}\pi_*(\sigma) = 0$, there exists an $h \in L^{1,1}(A)$ which is represented by a cochain $\{h_{\alpha\beta}\}$ on \mathcal{U} such that

$$\mu_* h_{\alpha\beta} = (\widehat{\delta}\sigma)_{\alpha\beta} .$$

(2.5) If $\sigma' \in L^{0,1}(M_A)$ also satisfies $\pi_*(\sigma') = \theta$, then $\pi_*(\sigma' - \sigma) = 0$ and hence there exists a $\bar{\sigma} \in L^{0,1}(A)$ which is represented by a cochain $\{\bar{\sigma}_\alpha\}$ on \mathcal{U} such that

$$\mu_* \bar{\sigma}_\alpha = \bar{\sigma}'_\alpha - \bar{\sigma}_\alpha .$$

Using these notations the following result is easily checked:

PROPOSITION 2. If $k \in L^{m,n}(A)$ is represented by $\{k_{\alpha_0, \dots, \alpha_m}\}$ on \mathcal{U} , then

$$(2.6) \quad \delta_\sigma \delta_\sigma k_{\alpha_0, \dots, \alpha_m} = f_{\alpha_0} \cdot k_{\alpha_0, \dots, \alpha_m} - k_{\alpha_0, \dots, \alpha_m} \cdot f_{\alpha_0}$$

$$(2.7) \quad \delta_\sigma (\widehat{\delta} k)_{\alpha_0, \dots, \alpha_{m+1}} = (\widehat{\delta} \delta_\sigma k)_{\alpha_0, \dots, \alpha_{m+1}} - h_{\alpha_0, \alpha_1} k_{\alpha_1, \dots, \alpha_{m+1}} \\ - (-1)^{n+1} k_{\alpha_1, \dots, \alpha_{m+1}} h_{\alpha_0, \alpha_1}$$

$$(2.8) \quad \delta_{\sigma'} k_{\alpha_0, \dots, \alpha_m} = \delta_\sigma k_{\alpha_0, \dots, \alpha_m} + \bar{\sigma}_{\alpha_0} k_{\alpha_0, \dots, \alpha_m} + (-1)^{n+1} k_{\alpha_0, \dots, \alpha_m} \bar{\sigma}_{\alpha_0} .$$

COROLLARY. $L^{i,j}(K_A)$ is a bicomplex with respect to the pair of differential operators $\widehat{\delta}, \delta_\sigma$. The total differential operator is given by

$$\delta = (-1)^{j+1} \widehat{\delta} + \delta_\sigma .$$

This differential operator depends only on θ .

Finally, we shall need to know something about the behavior of $\widehat{\delta}$ on products of low dimensional cochains, where Čech cochains are multiplied by multiplying the values (suitably restricted when necessary) on corresponding elements of the nerve of a covering according to the convention of 2.1. It is easy to verify the following statements by explicit calculation.

PROPOSITION 3. If $r \in L^{0,p}(A)$ and $s \in L^{0,q}(A)$ are represented on \mathcal{U} by $\{r_\alpha\}$ and $\{s_\alpha\}$ respectively, then $\widehat{\delta}(r \cdot s) \in L^{1,p+q}(A)$ and

$$(2.9) \quad \widehat{\delta}(r \cdot s)_{\alpha\beta} = (\widehat{\delta}r)_{\alpha\beta} \cdot s_\alpha + r_\alpha \cdot (\widehat{\delta}s)_{\alpha\beta} + (\widehat{\delta}r)_{\alpha\beta} \cdot (\delta s)_{\alpha\beta} .$$

If $t \in L^{1,p}(A)$ and $u \in L^{1,q}(A)$ are represented on \mathcal{U} by $\{t_{\alpha\beta}\}$ and $\{u_{\alpha\beta}\}$ respectively then $\widehat{\delta}(t \cdot u) \in L^{2,p+q}(A)$ and

$$(2.10) \quad \begin{aligned} \widehat{\delta}(t \cdot u)_{\alpha\beta\gamma} = & (\widehat{\delta}t)_{\alpha\beta\gamma} \cdot u_{\alpha\gamma} + t_{\alpha\gamma} \cdot (\widehat{\delta}u)_{\alpha\beta\gamma} + (\widehat{\delta}t)_{\alpha\beta\gamma} \cdot (\widehat{\delta}u)_{\alpha\beta\gamma} - t_{\alpha\beta} \cdot u_{\beta\gamma} \\ & - t_{\beta\gamma} \cdot u_{\alpha\beta} . \end{aligned}$$

Finally, if $r \in L^{m,p}(A)$ and $s \in L^{m,q}(A)$ then δ_σ satisfies the good co-boundary formula.

$$(2.11) \quad \delta_\sigma(r \cdot s) = (\delta_\sigma r) \cdot s + (-1)^p r \cdot \delta_\sigma s .$$

3. The obstruction. We shall regard the complex $L^{i,j}(K_A)$ as being filtered by the second degree and we define $F^p(L) = \sum_{j \geq p} L^{i,j}(K_A)$. In analogy with the proceedings of [1], the classical results for extensions of algebras suggest that each algebra homomorphism $\theta \in \text{Hom}(A, P_A)$ whose range consists of permutable elements determines an "obstruction" in $H^3(F^1(L))$; this obstruction being zero if and only if there exists an extension which induces θ in the manner described in § 1. A representative cocycle for such a cohomology class would be an element of $L^{2,1}(K_A) \oplus L^{1,2}(K_A) \oplus L^{0,3}(K_A)$.

Let $\sigma \in L^{0,1}(A)$ satisfy $\pi_* \sigma = \theta$ and let

$f \in L^{0,2}(A)$ and $h \in L^{1,1}(A)$ be defined as in 2.3 and 2.4. Then the components of a representative cocycle of the "obstruction" to θ are defined as follows:

(i) Since $\mu_* (\widehat{\delta}h) = \widehat{\delta} \mu_* h = 0$, there exists an element $a \in L^{2,1}(K_A)$ which is represented by a cochain $\{a_{\alpha\beta\gamma}\}$ on \mathcal{U} such that

$$a_{\alpha\beta\gamma} = (\widehat{\delta}h)_{\alpha\beta\gamma}$$

(ii) A standard elementary calculation shows that $\mu_* (\delta_\sigma f) = 0$.

Hence there exists an element $c \in L^{0,3}(K_A)$ which is represented by a cochain $\{c_\alpha\}$ on \mathcal{U} such that $c_\alpha = \delta_\sigma f_\alpha$.

(iii) An equally elementary calculation shows that $\mu_*[\widehat{\delta}f - \delta_\sigma h - h \cdot h] = 0$. Hence there exists an element $b \in L^{1,2}(K_A)$ which is represented by a cochain $\{b_{\alpha\beta}\}$ on \mathcal{U} such that

$$b_{\alpha\beta} = -(\widehat{\delta}f)_{\alpha\beta} + \delta_\sigma h_{\alpha\beta} + h_{\alpha\beta} \cdot h_{\alpha\beta}$$

THEOREM 1. *Let $s = a \oplus b \oplus c$. Then s is a cocycle of $F^1(L)$ whose cohomology class depends only on θ .*

DEFINITION. The cohomology class of s will be denoted by $Ob(\theta)$ and will be called the obstruction to θ .

THEOREM 2. *Let X be paracompact Hausdorff and let Λ be weakly R -projective and R -coherent. Then $Ob(\theta) = 0$ if and only if there is an extension of Λ by A which induces θ . If $Ob(\theta) = 0$, then the set $F_\theta(\Lambda, A)$ of equivalence classes of extensions which induce θ is in one-to-one correspondence with the set of elements of the group $H^2(F^1K)$, and hence the following two sequences are exact.*

$$\begin{aligned} (1) \quad & 0 \longrightarrow H^1[\text{Hom}_R(S_*(\Lambda), K_A)] \longrightarrow \text{Ext}_{\mathcal{A}}^1(\Lambda, K_A) \longrightarrow H^1(X, K_A) \\ & \longrightarrow F_\theta(\Lambda, A) \longrightarrow \text{Ext}_{\mathcal{A}}^2(\Lambda, K_A) \longrightarrow H^2(X, K_A) \longrightarrow \\ (2) \quad & 0 \longrightarrow H^2[\text{Hom}_R(S_*(\Lambda), K_A)] \longrightarrow F_\theta(\Lambda, A) \longrightarrow \\ & H^1(X, \text{Hom}_R(\Lambda, K_A)) \longrightarrow \end{aligned}$$

Proof of Theorem 1. It is clear that $\widehat{\delta}a = \widehat{\delta}\widehat{\delta}h = 0$, and, by 2.6, that $\delta_\sigma c = \delta_\sigma \delta_\sigma f = 0$. Thus, to prove that s is a cocycle we must show that $\widehat{\delta}b = \delta_\sigma a$ and that $\widehat{\delta}c = -\delta_\sigma b$. To derive the first expression, we have by definition that

$$(\widehat{\delta}b)_{\alpha\beta\gamma} = -(\widehat{\delta}\widehat{\delta}f)_{\alpha\beta\gamma} + (\widehat{\delta}\delta_\sigma h)_{\alpha\beta\gamma} + \widehat{\delta}(h \cdot h)_{\alpha\beta\gamma}$$

The first term is zero and the second and third terms can be expanded by 2.7 and 2.10 respectively. After obvious cancellations, this yields

$$(\widehat{\delta}b)_{\alpha\beta\gamma} = \delta_\sigma(\widehat{\delta}h)_{\alpha\beta\gamma} + (\widehat{\delta}h)_{\alpha\beta\gamma} \cdot h_{\alpha\gamma} + h_{\alpha\gamma} \cdot (\widehat{\delta}h)_{\alpha\beta\gamma} + (\widehat{\delta}h)_{\alpha\beta\gamma} \cdot (\widehat{\delta}h)_{\alpha\beta\gamma}$$

Since $\widehat{\delta}h = a \in L^{2,1}(K_A)$, on a sufficiently fine covering multiplication by $(\widehat{\delta}h)_{\alpha\beta\gamma}$ is zero and hence $\widehat{\delta}b = \delta_\sigma a$. Similarly, since $c = \delta_\sigma f$, $\widehat{\delta}c$ can be expanded by the equation, 2.7, for commuting $\widehat{\delta}$ and δ_σ . The resulting expression can be simplified by using equations 2.6 and 2.11 and the definition of b in (ii). This yields easily that

$$(\widehat{\delta}c)_{\alpha\beta} = \delta_\sigma[(\widehat{\delta}f)_{\alpha\beta} - \delta_\sigma h_{\alpha\beta} - h_{\alpha\beta} \cdot h_{\alpha\beta}] = -\delta_\sigma b_{\alpha\beta}.$$

Thus s is a cocycle.

The definition of s depends on the choices of b , h , and f . We shall show that changing any of these changes s by a coboundary and that any cocycle cohomologous to s can be obtained by such a choice.

Suppose that h' satisfies $\mu_* h' = \widehat{\delta}\sigma$ and f' satisfies $\mu_* f' = \sigma \cdot \sigma - \sigma \circ m$. Then $h' - h = \bar{h} \in L^{1,1}(K_A)$ and $f' - f = \bar{f} \in L^{0,2}(K_A)$. If s' denotes the cocycle corresponding to σ , h' and f' , then it is easy to see that

$$s' - s = (\widehat{\delta} + \delta_\sigma)\bar{h} + (-\widehat{\delta} + \delta_\sigma)\bar{f} = \delta(\bar{h} + \bar{f}).$$

Conversely, if $\bar{h} \oplus \bar{f}$ is any 2-cochain of $F^1(L)$, then $h + \bar{h}$ and $f + \bar{f}$ are admissible liftings of $\widehat{\delta}_\sigma$ and $\sigma \cdot \sigma - \sigma \circ m$ respectively and this change alters s by $\delta(\bar{h} \oplus \bar{f})$. Hence, in this manner we obtain all cocycles cohomologous to s .

It remains to show that if $\pi_* \sigma' = \theta$, then h' and f' can be chosen so that the corresponding cocycle $s' = a' + b' + c' = s$. Since $\pi_*(\sigma' \cdot \sigma) = 0$, there is a $\bar{\sigma} \in L^{0,1}(A)$ such that $\mu_* \bar{\sigma} = \sigma' - \sigma$. Let $h' = h + \widehat{\delta}\bar{\sigma}$ and $f' = f + \delta_\sigma \bar{\sigma} + \bar{\sigma} \cdot \bar{\sigma}$. Then it is immediate that h' and f' are liftings of $\widehat{\delta}\sigma'$ and $\sigma' \cdot \sigma' - \sigma' \cdot m$ respectively and that $a' = \widehat{\delta}h' = a$. The difference $\delta_\sigma f' - \delta_\sigma f$ can be expressed by 2.8. Using 2.6 and 2.11, it is easily seen that this difference is zero and hence $c' = c$. The only difficult point is to show that $b' = b$. By definition

$$b' = -\widehat{\delta}f' + \delta_\sigma h' + h' \cdot h'$$

Using the definitions of f' and h' and rearranging terms, we arrive at the equality

$$\begin{aligned} b_{\alpha\beta}' - b_{\alpha\beta} &= [\delta_\sigma \widehat{\delta}\bar{\sigma}_{\alpha\beta} - \widehat{\delta}\delta_\sigma \bar{\sigma}_{\alpha\beta}] + [\bar{\sigma}_\alpha \cdot h_{\alpha\beta} + h_{\alpha\beta} \cdot \bar{\sigma}_\alpha + \widehat{\delta}\bar{\sigma}_{\sigma\beta} \cdot h_{\alpha\beta} + h_{\alpha\beta} \cdot \widehat{\delta}\bar{\sigma}_{\alpha\beta}] \\ &\quad + [\bar{\sigma}_\alpha \cdot \widehat{\delta}\bar{\sigma}_{\alpha\beta} + \widehat{\delta}\bar{\sigma}_{\alpha\beta} \cdot \bar{\sigma}_\alpha + \widehat{\delta}\bar{\sigma}_{\alpha\beta} \cdot \widehat{\delta}\bar{\sigma}_{\alpha\beta} - \widehat{\delta}(\bar{\sigma} \cdot \bar{\sigma})_{\alpha\beta}]. \end{aligned}$$

The third bracket is zero by the formula 2.9 for the Čech coboundary of a product and the first bracket equals $-h_{\alpha\beta} \cdot \bar{\sigma}_\beta - \bar{\sigma}_\beta \cdot h_{\alpha\beta}$ by the rule 2.7 for interchanging $\widehat{\delta}_\sigma$ and δ . Hence the sum of the first two brackets is zero and therefore $b' = b$.

Proof of Theorem 2. Suppose $0 \longrightarrow A \xrightarrow{i} \Gamma \xrightarrow{p} A \longrightarrow 0$ is an extension. By Proposition 3.1 of [1], the hypotheses imply that any such extension is locally trivial considered as an extension of sheaves of R -modules. Hence there exists a covering $\mathcal{U} = \{U_\alpha\}$ which carries R -module homomorphisms $j_\alpha \cdot A|_{U_\alpha} \longrightarrow \Gamma|_{U_\alpha}$ with $p \cdot j_\alpha = \text{identity}$. If $\sigma_\alpha: A|_{U_\alpha} \longrightarrow M_A|_{U_\alpha}$ is defined by $[\sigma_\alpha(\lambda)](a) = j_\alpha(\lambda) \cdot a$ and $(a)[\sigma_\alpha(\lambda)] = a \cdot j_\alpha(\lambda)$ then $\{\sigma_\alpha\}$ determines an element $\sigma \in L^{0,1}(M_A)$ which is a lifting of the homomorphism θ induced as in §1 by the given extension. If we define $h_{\alpha\beta} = j_\beta - j_\alpha$ and $f_\alpha = j_\alpha j_\alpha - j_\alpha \circ m$, then the corresponding elements $h \in L^{1,1}(A)$ and $f \in L^{0,2}(A)$ satisfy $\mu_* h = \delta\sigma$ and $\mu_* f = \sigma \cdot \sigma -$

$\sigma \circ m$. Elementary calculations show that for this choice of h and f we get that $s = a \oplus b \oplus c = 0$ and hence $Ob(\theta) = 0$.

Conversely, if $Ob(\theta) = 0$, then on some sufficiently fine covering \mathcal{U} , we may choose $\{f_\alpha\} \in \hat{C}^0(\mathcal{U}, Hom_R(S_2(\mathcal{A}), A))$ and $\{h_{\alpha\beta}\} \in \hat{C}^1(U, Hom_R(\mathcal{A}, A))$ so that $\delta_\sigma f_\alpha = 0$, $(\hat{\delta} h)_{\alpha\beta\gamma} = 0$ and $(\hat{\delta} f)_{\alpha\beta} = \delta_\sigma h_{\alpha\beta} + h_{\alpha\beta} \cdot h_{\alpha\beta}$. As in [1], we define Γ to be the sheaf which is the quotient of $\bigcup_\alpha (A \oplus \mathcal{A})|_{U_\alpha}$ by the relation

$$(a + h_{\alpha\beta}(\lambda), \lambda)_\alpha \sim (a, \lambda)_\beta \text{ for } (a, \lambda) \in A \oplus \mathcal{A}|_{U_{\alpha\beta}}.$$

Multiplication in Γ is given by the formula

$$(a, \lambda)_\alpha \cdot (a', \lambda')_\alpha = (aa' + \sigma_\alpha(\lambda)a' + a\sigma_\alpha(\lambda) + f_\alpha(\lambda, \lambda'), \lambda\lambda')_\alpha.$$

It is easy to show that this multiplication is associative since $\delta_\sigma f = 0$ and that it agrees with the equivalence relation since $\hat{\delta} f = \delta_\sigma h + h \cdot h$.

It follows then, exactly as in MacLane [2] that the set of equivalence classes of extensions which realize a given θ with $Ob(\theta) = 0$ is in one-to-one correspondence with the set of elements of the group $H^2(F^1(L))$. The exact sequences are derived exactly as in [1] from the exact sequences of complexes

$$0 \longrightarrow F^1 L \longrightarrow F^0 L \longrightarrow E_0^{*,0} \longrightarrow 0$$

and

$$0 \longrightarrow F^2 L \longrightarrow F^1 L \longrightarrow E_0^{*,1} \longrightarrow 0.$$

4. Examples. (1) If $K_A = 0$ then all obstructions are zero and all terms involving K_A in the exact sequence containing $F_\theta(\mathcal{A}, A)$ are zero. Hence there is a unique extension of \mathcal{A} by A which induces a given $\theta \in Hom_R(\mathcal{A}, P_A)$. As in MacLane [2], this extension can be described as the ‘‘graph’’ of θ ; i.e., the pull-back of the pair of maps $\theta: \mathcal{A} \longrightarrow P_A, \pi: M_A \longrightarrow P_A$.

(2) If $K_A = A$, then the map $\mu: \mathcal{A} \longrightarrow M_A$ is the zero map and hence $M_A = P_A$. Consequently, if $\theta \in Hom_R(\mathcal{A}, P_A)$ is given, then σ may be chosen equal to θ and so $\delta\sigma$ and $\sigma \cdot \sigma - \sigma \circ m$ are both zero. Therefore, any cocycle $f \oplus h \in L^{0,2}(\mathcal{A}) \oplus L^{1,1}(\mathcal{A})$ is a lifting of these two terms. It follows that $Ob(\theta) = 0$ and that $F_\theta(\mathcal{A}, A) = H^2(F^1 L)$. Thus the results of [1] are a special case of the results of this paper.

(3) We wish to discuss more thoroughly a remark in § 3.3 of [1]. Let X be paracompact Hausdorff and let \mathcal{A} be a weakly R -projective and R -coherent sheaf of R -algebras. Suppose that A is a sheaf of R -algebras and that

$$0 \longrightarrow A \longrightarrow \Gamma \longrightarrow \mathcal{A} \longrightarrow 0$$

is an exact sequence of R -modules. Let $\mathcal{U} = \{U_\alpha\}$ be a sufficiently fine covering of X and let $\{j_\alpha\} \in \hat{C}^0(\mathcal{U}, Hom_R(\mathcal{A}, \Gamma))$ determine the locally

trivial structure of Γ and let $h_{\alpha\beta} = (\widehat{\delta}j)_{\alpha\beta}$. An algebra homomorphism $\theta \in \text{Hom}_R(A, P_A)$ whose image consists of permutable elements will be called compatible with the locally trivial structure of Γ if there exists a lifting $\sigma \in L^{0,1}(M_A)$ of θ which is represented by a cochain $\{\sigma_\alpha\}$ on \mathcal{U} such that $\mu_*h = \widehat{\delta}\sigma$. Furthermore, an element $f \in L^{0,2}(A)$ will be called a multiplication compatible with θ and h if $\mu_*f = \sigma \cdot \sigma - \sigma \circ m$, $\widehat{\delta}f = \delta_\sigma h + h \cdot h$ and $\delta_\sigma f = 0$. The set of equivalence classes with respect to the usual equivalence relation of such multiplications will be denoted by $F_{\theta, h}(A, A)$. We wish to calculate $F_{\theta, h}(A, A)$.

Proceeding as in § 2, let $f \in L^{0,2}(A)$ be a cochain such that $\mu_*f = \sigma \cdot \sigma - \sigma \circ m$. Corresponding to $f \oplus h$ there is an obstruction cocycle $s(h) = c \oplus b \oplus 0$. The only relevant changes of $s(h)$ are given by varying f by an element $\bar{f} \in L^{0,2}(K_A)$. Such a change alters s by a coboundary in F^2L . Hence we obtain the result:

THEOREM. *Corresponding to θ and h , there is an obstruction cohomology class $Ob(\theta, h) \in H^3(F^2L)$ which is zero if and only if there exists a multiplication compatible with θ and h . If $Ob(\theta, h) = 0$ then $F_{\theta, h}(A, A)$ is in one-to-one correspondence with the elements of the group $H^2[\text{Hom}_R(S_*(A), K_A)]$.*

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