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**PHYSICAL INTERPRETATION AND STRENGTHING OF M.
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The origin of this work lies partly in *M. H. Protter's* method [7], [8], partly in two papers [3], [5], developing the idea, found in *Payne-Weinberger* [6], of auxiliary one-dimensional problems; the physical interpretation in § 3 rejoins that of [2] and [4].

1. We consider the first eigenvalue λ_1 of a nonhomogeneous membrane with specific mass $\rho(x, y) \geq 0$ covering a plane domain D and elastically supported (elastic coefficient $k(s)$) along its boundary Γ :

$$\Delta u + \lambda_1 \rho(x, y)u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} + k(s)u = 0 \text{ along } \Gamma,$$

where \vec{n} is the outward normal.

Every continuous and piecewise smooth function $v(x, y)$ furnishes an upper bound for λ_1 : By Rayleigh's principle

$$\lambda_1 = \text{Min}_v \frac{D(v) + \oint_{\Gamma} k(s)v^2 ds}{\iint_D \rho v^2 dA},$$

where ds is the length element, dA the element of area, and $D(v)$ the Dirichlet integral $\iint_D \text{grad}^2 v dA$. *The Minimum is realized if $v = u_1(x, y)$ (first eigenfunction, satisfying $\Delta u_1 + \lambda_1 \rho u_1 = 0$).*

In the opposite direction, we are here in search of a Maximum principle for λ_1 , from which we could calculate lower bounds.

2. Let us consider in D a sufficiently regular vector field \vec{p} (we shall discuss presently what discontinuities are allowed), satisfying the *condition*

$$(1) \quad \vec{p} \cdot \vec{n} \leq k(s) \quad \text{along } \Gamma.$$

$$\text{grad}^2 u_1 + (\vec{p}^2 - \text{div } \vec{p}) u_1^2 = -\text{div}(u_1^2 \vec{p}) + \text{grad}^2 u_1 + u_1^2 \vec{p}^2 + 2u_1 \text{grad } u_1 \cdot \vec{p}$$

$$= -\text{div}(u_1^2 \vec{p}) + (\text{grad } u_1 + u_1 \vec{p})^2 \geq -\text{div}(u_1^2 \vec{p}).$$

Let us integrate this inequality:

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$$\begin{aligned} 0 &\leq D(u_1) + \oint u_1^2 \vec{p} \cdot \vec{n} ds + \iint (\vec{p}^2 - \operatorname{div} \vec{p}) u_1^2 dA \\ &\leq D(u_1) + \oint k(s) u_1^2 ds + \iint (\vec{p}^2 - \operatorname{div} \vec{p}) u_1^2 dA = \iint (\lambda_1 \rho + \vec{p}^2 - \operatorname{div} \vec{p}) u_1^2 dA, \end{aligned}$$

whence the lower bound

$$(2) \quad \lambda_1 \geq \inf_D \left(\frac{\operatorname{div} \vec{p} - \vec{p}^2}{\rho} \right).$$

We have equality if (and only if) $\vec{p} = -\operatorname{grad} u_1 / u_1$, whence the Maximum principle

$$(3) \quad \lambda_1 = \operatorname{Max}_{\vec{p} \cdot \vec{n} \leq k(s) \text{ along } \Gamma} \inf_D \left(\frac{\operatorname{div} \vec{p} - \vec{p}^2}{\rho} \right).$$

Allowed discontinuities (see also [5]): the same as in Thomson’s principle for boundary value problems. — If D is cut into subdomains D_1, D_2, \dots, D_n by analytic arcs, it is sufficient that the vector field \vec{p} be continuous and differentiable in each D_i and that its normal component be continuous across all those analytic arcs; the tangential component need not be continuous. — Other sufficient condition: $\vec{p} = \{p_1, p_2\}$, p_1 continuous in x and differentiable with respect to x , p_2 continuous in y and differentiable with respect to y .

Two properties of a “good” concurrent vector field: One should try to construct \vec{p} such that $\vec{p} \cdot \vec{n} = k(s)$ along Γ and $(\operatorname{div} \vec{p} - \vec{p}^2) / \rho = \text{const}$ in D (such is the case for the extremal field $-\operatorname{grad} u_1 / u_1$); the examples calculated in [5] show that such a “good” field may be easy to construct.

REMARK. For a fixed boundary ($u = 0$ along Γ), $k \equiv \infty$ and condition (1) falls off. — A “good” field will then be singular along Γ .

3. A physical interpretation.

3.1. One verifies immediately that *the nonhomogeneous membrane upon D , with specific mass $= \lambda_1 \rho(x, y)$ and elastic coefficient $k(s)$, vibrates with ground eigenfrequency 1:* $\Delta u_1 + 1 \cdot (\lambda_1 \rho) u_1 = 0$.

We shall presently establish the following *theorem:* Given an admissible vector field \vec{p} in D , *the nonhomogeneous membrane with specific mass $\tilde{\rho}(x, y) = \operatorname{div} \vec{p} - \vec{p}^2$ in D and elastic coefficient $\tilde{k}(s) = \vec{p} \cdot \vec{n}$ along Γ , vibrates with ground frequency ≥ 1 .*

The inequality (2) follows as a corollary: according to two general principles regarding vibrating systems (cf. [1], pp. 354 and 357), a homo-

geneous membrane with specific mass $\leq \bar{\rho}$ and elastic coefficient $k(s) \geq \bar{k}(s)$ vibrates *a fortiori* with ground frequency ≥ 1 ; whence (2).

3.2. The above theorem will be established by proving the following statement to be true: If we *cut* the membrane (specific mass $\tilde{\rho}(x, y) = \text{div } \vec{p} - \vec{p}^2$, elastic coefficient $\tilde{k}(s) = \vec{p} \cdot \vec{n}$) into slices D_j of infinitesimal breadth along all trajectories of \vec{p} , it *then* vibrates with ground frequency 1.

Indeed: Each slice D_j has the first eigenfrequency 1: Call s the arc length along the trajectory (measured from an arbitrary origin on D_j); we define in D_j a function $f(x, y) = f(s) = c_j \exp \left\{ - \int_{s=0}^s \vec{p} \cdot \vec{ds} \right\}$, $c_j > 0$ arbitrary. Then $\text{grad } f = -f\vec{p}$;

$$\Delta f = -f \text{div } \vec{p} - \vec{p} \cdot \text{grad } f = (\vec{p}^2 - \text{div } \vec{p})f = -\tilde{\rho}f,$$

$$\frac{\partial f}{\partial n} = -(\vec{p} \cdot \vec{n})f = \begin{cases} -\tilde{k}f & \text{on } \Gamma_j \text{ (infinitesimal part of } \Gamma \text{ bounding } D_j); \\ 0 & \text{along the cuts;} \end{cases}$$

$f > 0$ in D . Thus, our function f is the first eigenfunction of the vibrating slice D_j with specific mass $\tilde{\rho}$, free along the cuts and with elastic coefficient \tilde{k} on Γ_j ; its first eigenfrequency is 1, because $\Delta f + 1 \cdot \tilde{\rho}f = 0$: this proves the theorem and justifies our physical interpretation of (2).—The light in which the Maximum principle is viewed here, is in agreement with [2] and [4].

4. An inequality of M. H. Protter.

Let $\vec{p} = \frac{\vec{t}}{a} - \frac{\text{grad } a}{2a}$, where $\vec{t}(x, y)$ is a vector field and $a(x, y) > 0$ a scalar field. Then

$$\text{div } \vec{p} - \vec{p}^2 = \frac{\text{div } \vec{t}}{a} - \frac{\vec{t}^2}{a^2} - \frac{\Delta a}{2a} + \frac{\text{grad}^2 a}{4a^2} \geq \frac{\text{div } \vec{t}}{a} - \frac{\vec{t}^2}{a^2} - \frac{\Delta a}{2a}.$$

For a membrane with fixed boundary, Condition (1) falls off, so we have by (2)

$$(4) \quad \lambda_1 \geq \inf_D \left[\frac{\text{div } \vec{t} - \frac{\vec{t}^2}{a} - \frac{\Delta a}{2}}{a\rho} \right].$$

This is M. H. Protter's inequality [7], [8] (if we write $\vec{t} = \{P, Q\}$) —although he requires $P(x, y)$ and $Q(x, y)$ to be C^1 in D , which is unnecessarily restrictive (cf. also [5] and [3]): P might be discontinuous in y and Q in x .

M. H. Protter indicates in [8] very interesting applications of (4) to comparison theorems between ground eigenfrequencies of two non-homogeneous membranes spanning the same domain D .

Critical remark.—In the proof of (4) we neglected the positive term $\text{grad}^2 a/4a^2$: equality is impossible in (4) unless $a(x, y) = \text{const}$, in which case (4) reduces back to (2) with $\vec{p} = \vec{t}/a$.

5. Strengthening of Protter's inequality. Let first (a little more generally) $\vec{p} = \frac{\vec{t}}{a} + \vec{v}$ with $\vec{t}(x, y), \vec{v}(x, y), a(x, y) > 0$; $\text{div } \vec{p} - \vec{p}^2 = \frac{\text{div } \vec{t}}{a} - \frac{\vec{t}^2}{a^2} + \text{div } \vec{v} - \vec{v}^2 - \frac{\text{grad } a}{a^2} \cdot \vec{t} - 2\frac{\vec{v}}{a} \cdot \vec{t}$; in order that the two last terms may cancel everywhere, let (with Protter) $\vec{v} = -\frac{\text{grad } a}{2a} = -\frac{\text{grad } \sqrt{a}}{\sqrt{a}}$; then $\text{div } \vec{v} - \vec{v}^2 = -\frac{\Delta \sqrt{a}}{\sqrt{a}}$; let $\sqrt{a(x, y)} = b(x, y) > 0$ in D , i.e.

$$\vec{p} = \frac{\vec{t}}{b^2} - \frac{\text{grad } b}{b}; \text{div } \vec{p} - \vec{p}^2 = \frac{\text{div } \vec{t}}{b^2} - \frac{\vec{t}^2}{b^4} - \frac{\Delta b}{b}.$$

— Under the condition

$$(5) \quad \frac{\vec{t} \cdot \vec{n}}{b^2} - \frac{1}{b} \frac{\partial b}{\partial n} \leq k(s) \quad (\text{identically satisfied if } k \equiv \infty),$$

we have the lower bound

$$(6) \quad \lambda_1 \geq \inf_D \left[\frac{1}{\rho} \left(\frac{\text{div } \vec{t}}{b^2} - \frac{\vec{t}^2}{b^4} - \frac{\Delta b}{b} \right) \right],$$

with equality whenever $\frac{\vec{t}}{b^2} - \frac{\text{grad } b}{b} = -\frac{\text{grad } u}{u}$, as no term has been neglected.—If, for example, we take $\vec{t} \equiv 0$, we get an inequality of Barta-Pólya $\lambda_1 \geq \inf_D \left(-\frac{\Delta b}{\rho b} \right)$.—[In fact, if $\frac{\partial b}{\partial n} + k(s)b = 0$ on Γ , λ_1 is comprised between the two Barta-Pólya bounds

$$\inf_D \left(-\frac{\Delta b}{\rho b} \right) \leq \lambda_1 \leq \sup_D \left(-\frac{\Delta b}{\rho b} \right).]$$

The expression in square brackets in (6) is larger than that in (4), because

$$-\frac{\Delta a}{2a} = -\frac{\Delta(b^2)}{2b^2} = -\frac{\text{div}(b \text{ grad } b)}{b^2} = -\frac{\Delta b}{b} - \frac{\text{grad}^2 b}{b^2};$$

this does not diminish M. H. Protter's merit, as his inequality (4)

contains (2) as a special case, whence (6) follows.

6. Applications.

6.1. The inequalities obtained by M. H. Protter in [8] may be sharpened by using (6) instead of (4).

6.2. *Small variation of the elastic coefficient along the boundary.*

First case: $\rho(x, y), k(s); \lambda_1, u_1(x, y).$

Second case: $\tilde{\rho}(x, y) = \rho(x, y), \tilde{k}(s) = k(s) + \varepsilon K(s); \tilde{\lambda}_1, \tilde{u}_1(x, y).$

By Rayleigh's principle,

$$(7) \quad \tilde{\lambda}_1 \leq \frac{D(u_1) + \oint \tilde{k}u_1^2 ds}{\iint \rho u_1^2 dA} = \lambda_1 + \varepsilon Q, \quad \text{where } Q = \frac{\oint K u_1^2 ds}{\iint \rho u_1^2 dA}.$$

We now introduce $b = u_1(x, y)$ into (6):

$$\tilde{\lambda}_1 \geq \lambda_1 + \inf_D \left\{ \frac{1}{\rho} \left(\operatorname{div} \vec{t} - \frac{\vec{t}^2}{u_1^4} \right) \right\} \text{ under the condition } \frac{\vec{t} \cdot \vec{n}}{u_1^2} \leq \varepsilon K(s),$$

whence $\iint \operatorname{div} \vec{t} dA = \oint \vec{t} \cdot \vec{n} ds \leq \varepsilon \oint K u_1^2 ds = \varepsilon Q \iint \rho u_1^2 dA$.—There exists a vector field \vec{t} such that

$\operatorname{div} \vec{t} = \varepsilon Q \rho(x, y) u_1^2$ in D and $\vec{t} \cdot \vec{n} = \varepsilon K(s) u_1^2$ along Γ : indeed, we can even impose the supplementary condition $\operatorname{rot} \vec{t} = 0, \vec{t} = \operatorname{grad} v$; v (determined up to an additive constant) is the solution of the Poisson-Neumann problem

$$\Delta v = \varepsilon Q \rho(x, y) u_1^2 \text{ in } D \text{ and } \frac{\partial v}{\partial n} = \varepsilon K(s) u_1^2 \text{ along } \Gamma.$$

Clearly, v and \vec{t} are proportional to ε . Thus,

$$(7') \quad \tilde{\lambda}_1 \geq \lambda_1 + \varepsilon Q - \sup_D \left(\frac{\vec{t}^2}{\rho u_1^4} \right) = \lambda_1 + \varepsilon Q - O(\varepsilon^2).$$

(7) and (7') give

$$(7'') \quad \tilde{\lambda}_1 = \lambda_1 + \varepsilon Q - O(\varepsilon^2).$$

The *first perturbation calculus* gives $\tilde{\lambda}_1 = \lambda_1 + \varepsilon Q$; we thus verify that this is the *tangent* to the exact curve $\tilde{\lambda}_1 = \tilde{\lambda}_1(\varepsilon)$.

6.3. *Small variation of the specific mass $\rho(x, y)$.*

First case: $\rho(x, y), k(s); \lambda_1, u_1(x, y).$

Second case: $\tilde{\rho}(x, y) = \rho(x, y) + \varepsilon \sigma(x, y), \tilde{k}(s) = k(s); \tilde{\lambda}_1, \tilde{u}_1(x, y).$

By Rayleigh's principle,

$$(8) \quad \tilde{\lambda}_1 \leq \frac{D(u_1) + \oint k(s)u_1^2 ds}{\iint \tilde{\rho}u_1^2 dA} = \frac{\lambda_1}{1 + \varepsilon R}, \text{ where } R = \frac{\iint \sigma u_1^2 dA}{\iint \rho u_1^2 dA}.$$

We now introduce again $b = u_1(x, y)$ into (6):

$$\tilde{\lambda}_1 \geq \inf_D \left\{ \frac{1}{\tilde{\rho}} \left(\frac{\operatorname{div} \vec{t}}{u_1^2} - \frac{\vec{t}^2}{u_1^4} + \lambda_1 \rho \right) \right\} \text{ under the condition } \vec{t} \cdot \vec{n} \leq 0 \text{ along } \Gamma;$$

we want to use a vector field \vec{t} such that $\vec{t} \cdot \vec{n} = 0$ along Γ and $\frac{1}{\tilde{\rho}} \left(\frac{\operatorname{div} \vec{t}}{u_1^2} + \lambda_1 \rho \right) = c = \text{const}$ in D , so $\operatorname{div} \vec{t} = u_1^2(c\tilde{\rho} - \lambda_1\rho)$; the constant c is determined by the condition

$$0 = \oint \vec{t} \cdot \vec{n} ds = \iint \operatorname{div} \vec{t} dA = c \iint \tilde{\rho}u_1^2 dA - \lambda_1 \iint \rho u_1^2 dA,$$

whence

$$c = \frac{\lambda_1}{1 + \varepsilon R}; \quad \operatorname{div} \vec{t} = \lambda_1 u_1^2 \left(\frac{\rho + \varepsilon \sigma}{1 + \varepsilon R} - \rho \right) = \varepsilon \lambda_1 u_1^2 \frac{\sigma - \rho R}{1 + \varepsilon R};$$

such a vector field \vec{t} exists: we can even request that it be a gradient field; $\vec{t} = O(\varepsilon)$.

$$(8') \quad \tilde{\lambda}_1 \geq \frac{\lambda_1}{1 + \varepsilon R} - \sup_D \left(\frac{\vec{t}^2}{\tilde{\rho}u_1^4} \right) = \frac{\lambda_1}{1 + \varepsilon R} - O(\varepsilon^2).$$

(8) and (8') give

$$(8'') \quad \tilde{\lambda}_1 = \frac{\lambda_1}{1 + \varepsilon R} - O(\varepsilon^2).$$

7. Schrödinger's equation.

7.1. We consider an equation of Schrödinger's type in 3-space:

$$(9) \quad \Delta u + [\lambda - W(x, y, z)]u = 0$$

with some boundary conditions not specified here, but which must permit partial integrations analogous to those of § 2; $W = \frac{2m}{\hbar^2} V(x, y, z)$,

$\lambda_1 = \frac{2m}{\hbar^2} E_1$, where V is the potential, and E_1 the lowest energy level.

Rayleigh's principle:

$$(10) \quad \lambda_1 = \text{Min}_v \frac{D(v) + \iiint W(x, y, z)v^2 d\tau}{\iiint v^2 d\tau},$$

with, possibly, a supplementary term at the numerator, owing to the boundary conditions; $d\tau$ is the volume element.—The Minimum is realized for the first eigenfunction $u_1(x, y, z)$ of the differential equation.

7.2. An argument almost identical to that of § 2 (cf. also [5]) gives the Maximum principle:

$$(11) \quad \boxed{\lambda_1 = \text{Max}_{\vec{p}} \inf_D \{W(x, y, z) + \text{div } \vec{p} - \vec{p}^2\}},$$

where the concurrent vector fields \vec{p} must satisfy corresponding boundary conditions.—The Maximum is realized for $\vec{p} = -\text{grad } u_1/u_1$.—Allowed discontinuities: cf. § 2 (continuity of the normal derivative, etc.).—To get a good lower bound, one should try to construct a vector field \vec{p} such that $W(x, y, z) + \text{div } \vec{p} - \vec{p}^2 = \text{const}$.

7.3. *A physical interpretation.*—For expository purposes, we shall consider here equation (9) for 2 dimensions only.—This is exactly the equation of a vibrating homogeneous membrane covering a plane domain D , on which each area element $dxdy$ (at the point (x, y)) is pulled towards its equilibrium position $u = 0$ by a weak spring of infinitesimal elastic coefficient $W(x, y)dxdy$.—We suppose that the membrane’s boundary Γ is also elastically supported with elastic coefficient $k(s)$: $\partial u/\partial n + k(s)u = 0$ along Γ .

Analogously to § 3.1, we verify immediately: *The homogeneous membrane covering D , with specific mass $\equiv \lambda_1$ and “interior springs” $W(x, y)$, vibrates with the ground eigenfrequency 1.*

Let us now consider another vibrating system: Given in D an admissible vector field \vec{p} with $\vec{p} \cdot \vec{n} \leq k(s)$, we study the system formed by:

(a) A nonhomogeneous membrane covering a copy D_a of D , with specific mass $= (\text{div } \vec{p} - \vec{p}^2)$ and elastic coefficient $= \vec{p} \cdot \vec{n}$ along Γ ;

(b) Another copy D_b of D , without any “transversal elasticity”, where every area element $dxdy$ contains a mass $W(x, y)dxdy$ vibrating independently under the action of a spring with elastic coefficient $W(x, y)dxdy$.

According to § 3, the nonhomogeneous membrane (a) has ground eigenfrequency ≥ 1 ; each infinitesimal mass of the system (b) vibrates

with the exact frequency $\omega = 1$, as this mass is equal to the spring coefficient.—Therefore 1 is the ground eigenfrequency of the system (a) + (b).

By superposing D_a and D_b and *welding*, in each point (x, y) , the two masses there placed, we synthesize a nonhomogeneous membrane with specific mass $W(x, y) + \operatorname{div} \vec{p} - \vec{p}^2$, elastic coefficient $= \vec{p} \cdot \vec{n}$ along Γ , and “interior springs” $W(x, y)$.—As the addition of supplementary constraints (welding!) can only make the ground eigenfrequency higher ([1], p. 354), our “synthetic” membrane vibrates with a ground frequency ≥ 1 .

Consider now the homogeneous membrane with specific mass $\equiv \inf_D [W(x, y) + \operatorname{div} \vec{p} - \vec{p}^2]$, elastic coefficient $k(s)$ along Γ , and the same “interior springs” $W(x, y)$; this membrane has smaller masses and greater constraints: therefore ([1], pp. 354 and 357), its ground frequency is *a fortiori* ≥ 1 .

As our initial membrane [specific mass $\equiv \lambda_1$; elastic coefficient $= k(s)$; interior springs $W(x, y)$] has ground eigenfrequency 1, its specific mass λ_1 must be $\geq \inf_D [W(x, y) + \operatorname{div} \vec{p} - \vec{p}^2]$, which explains (11).

7.4. (Analogous to § 5): Let $\vec{p} = \frac{\vec{t}}{b^2} - \frac{\operatorname{grad} b}{b}$; we get

$$(12) \quad \lambda_1 \geq \inf_D \left\{ W(x, y, z) + \frac{\operatorname{div} \vec{t}}{b^2} - \frac{\vec{t}^2}{b^4} - \frac{\Delta b}{b} \right\},$$

where adequate boundary restrictions must be imposed on the concurrent vector fields $\vec{t}(x, y, z)$ and scalar fields $b(x, y, z)$.

7.5. *An application.*—*Small variation of the potential*; boundary conditions on the surface Γ of D : $\partial u / \partial n + k(X)u = 0$ ($X \in \Gamma$).

Boundary conditions to be satisfied by \vec{t} and b :

$$(5) \quad \frac{\vec{t} \cdot \vec{n}}{b^2} - \frac{1}{b} \frac{\partial b}{\partial n} \leq k(X) \text{ on } \Gamma.$$

First case:

$$W(x, y, z), \quad k(X); \quad \lambda_1, \quad u_1(x, y, z).$$

Second case:

$$\tilde{W}(x, y, z) = W(x, y, z) + \varepsilon w(x, y, z), \quad \tilde{k}(X) = k(X); \quad \tilde{\lambda}_1, \quad \tilde{u}_1(x, y, z).$$

By Rayleigh’s principle (10),

$$(13) \quad \tilde{\lambda}_1 \leq \frac{D(u_1) + \iiint \tilde{W} u_1^2 d\tau}{\iiint u_1^2 d\tau} = \lambda_1 + \varepsilon U, \quad \text{where } U = \frac{\iiint w u_1^2 d\tau}{\iiint u_1^2 d\tau}.$$

Now let $b = u_1(x, y, z)$ into (12):

$$\tilde{\lambda}_1 \geq \lambda_1 + \inf_D \left[\varepsilon w + \frac{\operatorname{div} \vec{t}}{u_1^2} - \frac{\vec{t}^2}{u_1^4} \right] \text{ under the condition } \vec{t} \cdot \vec{n} \leq 0 \text{ on } \Gamma. \text{ We}$$

want to use a vector field \vec{t} such that $\vec{t} \cdot \vec{n} = 0$ and $\frac{\operatorname{div} \vec{t}}{u_1^2} + \varepsilon w = c = \text{const}$, $\operatorname{div} \vec{t} = u_1^2(c - \varepsilon w)$; the constant c is determined by the condition $0 = \oint \vec{t} \cdot \vec{n} dS = \iiint \operatorname{div} \vec{t} d\tau = c \iiint u_1^2 d\tau - \varepsilon \iiint w u_1^2 d\tau$, where dS is the surface element; hence, $c = \varepsilon U$; $\operatorname{div} \vec{t} = \varepsilon u_1^2(U - w)$; there exists such a vector field \vec{t} : we can even impose that it be a gradient field; \vec{t} is proportional to ε .

$$(13') \quad \tilde{\lambda}_1 \geq \lambda_1 + \varepsilon U - \sup_D (\vec{t}^2/u_1^4) = \lambda_1 + \varepsilon U - O(\varepsilon^2);$$

(13) and (13') give

$$(13'') \quad \tilde{\lambda}_1 = \lambda_1 + \varepsilon U - O(\varepsilon^2).$$

The *first approximation* $\tilde{\lambda}_1 = \lambda_1 + \varepsilon U$ of the *perturbation calculus* is, as we see, the *tangent* to the exact curve $\tilde{\lambda}_1 = \tilde{\lambda}_1(\varepsilon)$.

Post-scriptum. For the case $k \equiv \infty$ and $\rho \equiv 1$, the inequality (2), written for the components $\vec{p} = \{\varphi(x, y), \psi(x, y)\}$ instead of vectorially, was known (except for the allowed discontinuities) to E. Picard as early as 1893: *Traité d'Analyse*, t. II, p. 25–26, and to T. Boggio: *Sull'equazione del moto vibratorio delle membrane elastiche*, Atti Accad. Lincei, ser. 5, vol. 16 (2° sem., 1907), 386–393, especially p. 390.—They also chose φ and ψ to be continuous in the domain, which is criticized here and in [5] as an unnecessary restriction.—In contrast with M. H. Protter, both Picard and Boggio seem to have under-estimated the importance of inequality (2): it just incidentally appears (in the quoted places) in the course of demonstrations for very simple monotony properties.

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Vol. 11, No. 3

BadMonth, 1961

Errett Albert Bishop, <i>A generalization of the Stone-Weierstrass theorem</i>	777
Hugh D. Brunk, <i>Best fit to a random variable by a random variable measurable with respect to a σ-lattice</i>	785
D. S. Carter, <i>Existence of a class of steady plane gravity flows</i>	803
Frank Sydney Cater, <i>On the theory of spatial invariants</i>	821
S. Chowla, Marguerite Elizabeth Dunton and Donald John Lewis, <i>Linear recurrences of order two</i>	833
Paul Civin and Bertram Yood, <i>The second conjugate space of a Banach algebra as an algebra</i>	847
William J. Coles, <i>Wirtinger-type integral inequalities</i>	871
Shaul Foguel, <i>Strongly continuous Markov processes</i>	879
David James Foulis, <i>Conditions for the modularity of an orthomodular lattice</i>	889
Jerzy Górski, <i>The Sochocki-Plemelj formula for the functions of two complex variables</i>	897
John Walker Gray, <i>Extensions of sheaves of associative algebras by non-trivial kernels</i>	909
Maurice Hanan, <i>Oscillation criteria for third-order linear differential equations</i>	919
Haim Hanani and Marian Reichaw-Reichbach, <i>Some characterizations of a class of unavoidable compact sets in the game of Banach and Mazur</i>	945
John Grover Harvey, III, <i>Complete holomorphs</i>	961
Joseph Hersch, <i>Physical interpretation and strengthening of M. Protter's method for vibrating nonhomogeneous membranes; its analogue for Schrödinger's equation</i>	971
James Grady Horne, Jr., <i>Real commutative semigroups on the plane</i>	981
Nai-Chao Hsu, <i>The group of automorphisms of the holomorph of a group</i>	999
F. Burton Jones, <i>The cyclic connectivity of plane continua</i>	1013
John Arnold Kalman, <i>Continuity and convexity of projections and barycentric coordinates in convex polyhedra</i>	1017
Samuel Karlin, Frank Proschan and Richard Eugene Barlow, <i>Moment inequalities of Pólya frequency functions</i>	1023
Tilla Weinstein, <i>Imbedding compact Riemann surfaces in 3-space</i>	1035
Azriel Lévy and Robert Lawson Vaught, <i>Principles of partial reflection in the set theories of Zermelo and Ackermann</i>	1045
Donald John Lewis, <i>Two classes of Diophantine equations</i>	1063
Daniel C. Lewis, <i>Reversible transformations</i>	1077
Gerald Otis Losey and Hans Schneider, <i>Group membership in rings and semigroups</i>	1089
M. N. Mikhail and M. Nassif, <i>On the difference and sum of basic sets of polynomials</i>	1099
Alex I. Rosenberg and Daniel Zelinsky, <i>Automorphisms of separable algebras</i>	1109
Robert Steinberg, <i>Automorphisms of classical Lie algebras</i>	1119
Ju-Kwei Wang, <i>Multipliers of commutative Banach algebras</i>	1131
Neal Zierler, <i>Axioms for non-relativistic quantum mechanics</i>	1151