

# Pacific Journal of Mathematics

**THE CYCLIC CONNECTIVITY OF PLANE CONTINUA**

F. BURTON JONES

# THE CYCLIC CONNECTIVITY OF PLANE CONTINUA

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Suppose that  $p$  and  $q$  are distinct points of the compact plane continuum  $M$ . If no point separates  $p$  from  $q$  in  $M$  and  $M$  is *locally connected*, then it is known [5] that  $M$  contains a simple closed curve which contains both  $p$  and  $q$ . But in the absence of local connectivity such a simple closed curve may fail to exist. Even if no point *cuts*<sup>1</sup>  $p$  from  $q$  in  $M$ , there does not necessarily exist in  $M$  a simple closed curve which contains both  $p$  and  $q$ . For example, no point of the continuum  $C$  indicated in Figure 1 cuts  $p$  from  $q$  in  $C$ , but  $C$  contains no simple closed curve whatsoever. However, if  $M$  is the continuum obtained by adding to  $C$  either of its complementary domains, there does exist in  $M$  a simple closed curve which contains both  $p$  and  $q$ . Here  $M$  fails to separate the plane and this is indicative of the general situation.

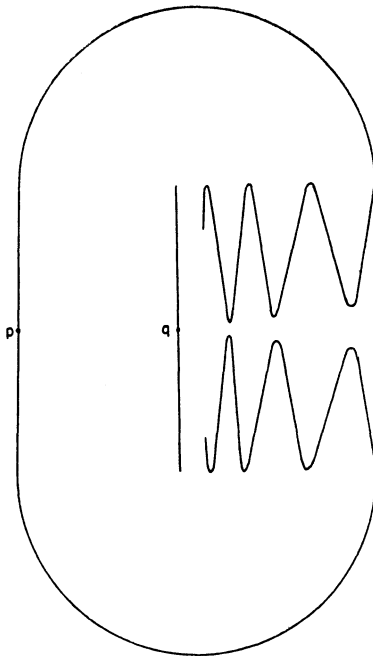


Fig. 1

LEMMA. *If  $p$  is a point of the compact subcontinuum  $M'$  of the plane  $S$  and  $L'$  is a nondegenerate compact continuum containing  $p$*

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<sup>1</sup> A point  $x$  ( $p \neq x \neq q$ ) cuts  $p$  from  $q$  in  $M$  if every subcontinuum of  $M$  containing both  $p$  and  $q$  also contains  $x$ . Obviously a *separating* point is a cut point but for continua in general a cut point is not necessarily a separating point.

and lying in  $(S - M') + p$  such that  $L' - p$  is connected, then there exists a connected open subset  $D'$  of  $S - M'$  such that

- (1)  $D' + p$  contains  $L'$ ,
- (2)  $D' + p$  is a connected, locally connected, complete, metric space, and
- (3)  $D' + p$  is strongly regular (i.e., the author's Axiom  $5_1^*$  [1, p. 54] holds true in  $D' + p$ ).

*Proof.* Let  $q$  denote a point of  $L' - p$ , let  $n$  denote a natural number such that  $d(p, q) > 1/n$ , and let  $R_0, R_1, R_2, \dots$  denote a sequence of circular regions centered on  $p$  of radii  $1/n, 1/n + 1, 1/n + 2, \dots$  respectively. Now for each integer  $i$  ( $i > -1$ ), add to  $M'$  every open interval  $I$  of the boundary  $C_i$  of  $R_i$  such that  $I$  contains no point of  $L' + M'$  but has both of its end points in  $M'$ , and call the resulting pointset  $N$ . Let  $D_1$  denote the sum of the components of  $(S - N) \cdot (S - \bar{R}_1)$  which contain points of  $L'$  and for each integer  $i > 1$ , let  $D_i$  denote the sum of the components of  $(S - N) \cdot (R_{i-2} - \bar{R}_i)$  which contain points of  $L'$ . Furthermore let  $D' = \sum D_i$ . Certainly  $D'$  is open and since  $L' - p$  is connected,  $D'$  is connected. Also it is easy to see that  $D' + p$  contains  $L'$  and is a connected, complete, metric space. It remains only to show that  $D' + p$  is strongly regular for it follows that such a space is locally connected [2, p. 623]. Obviously  $D' + p$  is strongly regular at each point of  $D'$ . To see that  $D' + p$  is strongly regular at  $p$  (relative to  $D' + p$ , of course) one has merely to observe that if  $k$  is a positive integer, the boundary of  $p + \sum D_i$  ( $i > k$ ) relative to  $D' + p$  is a subset of the sum of those components of  $(S - M') \cdot C_{k-1}$  which intersect  $L'$  and since  $L'$  contains no point of  $M'$  except  $p$ , this set of components is finite.

**THEOREM.** *Let  $M$  be a compact subcontinuum of the plane  $S$  which does not separate  $S$ . Then if  $p$  and  $q$  are distinct points of  $M$  and no point cuts  $p$  from  $q$  in  $M$ , there exists a simple closed curve  $J$  lying in  $M$  which contains both  $p$  and  $q$ .*

*Proof.* Three cases arise depending upon the location of  $p$  and  $q$ . If both  $p$  and  $q$  are inner points (non-boundary points) of  $M$ , then it follows from [3] that both  $p$  and  $q$  belong to the same component of the set of inner points of  $M$ . For this case the theorem is known to hold true (see for example [4], p. 124).

If both  $p$  and  $q$  are boundary points of  $M$ , then the argument outlined in [3] shows that  $M$  contains a compact continuum  $L$  which contains both  $p$  and  $q$  such that every point of  $L - (p + q)$  is an inner point of  $M$ . Since  $L$  must contain a subcontinuum irreducible from  $p$  to  $q$  it is no loss of generality to assume that  $L$  itself has this property.

In this case  $L - (p + q)$  is a connected subset of a component  $D$  of the set of inner points of  $M$  and the theorem follows with the help of the lemma in somewhat the same manner as the next case.

Finally, if  $q$  is an inner point of  $M$  and  $p$  is a boundary point of  $M$ , it follows from [3] that some component  $D$  of the set of inner points of  $M$  contains  $q$  and has  $p$  in its boundary. To show that  $D + p$  contains a continuum  $L$  containing both  $p$  and  $q$  requires a modification of the argument given in [3].

Suppose that  $\varepsilon$  is a positive number such that  $\varepsilon < d(p, q)$ . Let  $C_p(\varepsilon)$  denote a circle of radius  $\varepsilon$  centered on  $p$  and let  $C_q$  denote a straight line through  $q$  which is perpendicular to the line  $pq$ . There exists a simple domain  $I(\varepsilon)$  which contains  $M$  such that if  $J(\varepsilon)$  denotes the boundary of  $I(\varepsilon)$ ,  $y$  is a boundary point of  $M$ , and  $z$  is a point of  $I(\varepsilon) + J(\varepsilon)$ , then  $d[y, J(\varepsilon)] < \varepsilon$  and  $d(z, M) < \varepsilon$ . There exist arcs  $T_p(\varepsilon)$  and  $T_q(\varepsilon)$  in  $C_p(\varepsilon)$  and  $C_q$  respectively such that each is minimal with respect to separating  $I(\varepsilon) + J(\varepsilon)$ ,  $q$  belongs to  $T_q(\varepsilon)$ , and  $T_p(\varepsilon)$  separates  $p$  from  $T_q(\varepsilon)$  in  $I(\varepsilon) + J(\varepsilon)$ .

Since  $T_p(\varepsilon)$  and  $T_q(\varepsilon)$  have only their endpoints in  $J(\varepsilon)$ , and except for these points lie entirely in  $I(\varepsilon)$ , there exist in  $J(\varepsilon)$  two nonintersecting unique arcs  $A(\varepsilon)$  and  $B(\varepsilon)$  such that  $T_p(\varepsilon) + A(\varepsilon) + T_q(\varepsilon) + B(\varepsilon)$  is a simple closed curve  $H(\varepsilon)$ . Let  $D(\varepsilon)$  denote the bounded complementary domain of  $H(\varepsilon)$ . If  $z$  is a point of  $D(\varepsilon) + H(\varepsilon)$ , then  $d(z, M) < \varepsilon$ . Any subcontinuum of  $M$  which contains  $p + q$  contains a subcontinuum irreducible from  $T_p(\varepsilon)$  to  $T_q(\varepsilon)$  which lies in  $T_p(\varepsilon) + D(\varepsilon) + T_q(\varepsilon)$ .

Now let  $L(\varepsilon)$  denote a continuum lying in  $T_p(\varepsilon) + D(\varepsilon) + T_q(\varepsilon)$  which intersects both  $T_p(\varepsilon)$  and  $T_q(\varepsilon)$  such that if  $z$  belongs to  $L(\varepsilon)$ , then  $d[z, A(\varepsilon)] = d[z, B(\varepsilon)]$ . The continuum  $L(\varepsilon)$  must exist; for if it did not, the set  $W$  of all points of  $D(\varepsilon) + H(\varepsilon)$  equidistant from  $A(\varepsilon)$  and  $B(\varepsilon)$  would be the sum of two mutually separated sets one containing  $W \cdot T_p(\varepsilon)$  and the other containing  $W \cdot T_q(\varepsilon)$  and consequently some simple closed curve would separate  $T_p(\varepsilon)$  from  $T_q(\varepsilon)$  but at the same time would fail to contain a point of  $W$  which involves a contradiction. So there exists a simple infinite sequence  $\alpha$  of values of  $\varepsilon$  such that  $D(\varepsilon) + H(\varepsilon)$  converges to a subset of  $M$ ,  $T_q(\varepsilon) \rightarrow T_q$  and  $L(\varepsilon) \rightarrow L$  as  $\varepsilon \rightarrow 0$  in  $\alpha$ . The set  $L$  has the following properties:

- (a)  $L$  is a continuum containing both  $p$  and point of  $T_q$ ,
- (b)  $L$  is a subset of  $M$ , and
- (c) every point of  $L - (p + L \cdot T_q)$  is an inner point of  $M$ .

Properties (a) and (b) are evident. So it remains only to prove property (c).

Let  $x$  be a point of  $L - (p + L \cdot T_q)$ . Since  $x$  does not cut  $p$  from  $q$  in  $M$ , there exists a subcontinuum  $K$  of  $M$  which contains  $p + q$  but not  $x$ . Let  $\delta$  be a positive number such that  $4\delta = d(x, K + T_q)$  and let

$U_\delta(x)$  and  $U_{3\delta}(x)$  be the circular regions centered on  $x$  of radius  $\delta$  and  $3\delta$  respectively. When  $\varepsilon$  (in  $\alpha$ ) is sufficiently small  $[T_p(\varepsilon) + T_q(\varepsilon)] \cdot [U_{3\delta}(x)] = 0$  but  $L(\varepsilon) \cdot U_\delta(x) \neq 0$ . Let  $y$  be some point of  $L(\varepsilon) \cdot U_\delta(x)$ , let  $r = \delta + d(x, y)$  and let  $U_r(y)$  be a circular region of radius  $r$  and center  $y$ . Obviously  $U_{3\delta}(x) \supset U_r(y) \supset U_\delta(x)$ . So  $[T_p(\varepsilon) + T_q(\varepsilon)] \cdot U_r(y) = 0$ . If  $A(\varepsilon) \cdot U_r(y) \neq 0$ , let  $f$  be a point of  $A(\varepsilon) \cdot U_r(y)$  such that  $d(f, y) = d[y, A(\varepsilon)]$ . But  $y$  belongs to  $L(\varepsilon)$ . Hence there exists in  $U_r(y)$  a point  $g$  of  $B(\varepsilon)$  such that  $d(g, y) = d[g, B(\varepsilon)] = d(f, y)$ . The sum of the straight line intervals  $yf$  and  $yg$  from  $y$  to  $f$  and from  $y$  to  $g$  respectively is an arc  $T_y$  lying in  $U_r(y)$ , having only its endpoints  $f$  and  $g$  in  $H(\varepsilon)$ , and containing the point  $y$  of  $D(\varepsilon)$ . Hence  $T_y - (f + g) \subset D(\varepsilon)$  for clearly  $yf$  cannot intersect  $B(\varepsilon)$  and  $yg$  cannot intersect  $A(\varepsilon)$ . But  $T_y \cdot K = 0$  and  $K$  contains a continuum lying in  $T_p(\varepsilon) + D(\varepsilon) + T_q(\varepsilon)$  irreducible from  $T_p(\varepsilon)$  to  $T_q(\varepsilon)$ . Since the points  $f$  and  $g$  separate  $T_p(\varepsilon)$  from  $T_q(\varepsilon)$  in  $H(\varepsilon)$  this involves a contradiction [4, Th. 17, p. 167]. Hence  $U_r(y) \cdot H(\varepsilon) = 0$  and since  $y$  belongs to  $D(\varepsilon)$ ,  $U_r(y) \subset D(\varepsilon)$ ; so for sufficiently small values of  $\varepsilon$  (in  $\alpha$ ),  $U_\delta(x) \subset D(\varepsilon)$ . Consequently  $U_\delta(x)$  is a subset of  $M$  and  $x$  is an inner point of  $M$ .

Now let  $C$  denote a circle which separates  $p$  from  $T_q$ . Obviously  $L$  intersects  $C$ . Hence  $L$  contains a subcontinuum  $L'$  irreducible from  $C$  to  $p$ . Let  $q'$  denote a point of  $L' \cdot C$ . Clearly  $L' - p$  is a connected subset of  $D$ . Let  $M'$  denote the boundary of  $D$ . Since  $M'$  is a continuum and contains only the point  $p$  of  $L'$ , by the lemma there exists a connected open subset  $D'$  of  $S - M'$  which contains  $L' - p$  and has the other properties of the set designated as  $D'$  in the lemma. It now follows from Theorem A of [1] that there exists a simple closed curve  $J'$  lying in  $D' + p$  and containing  $p + q'$ . Since  $D'$  is a connected subset of  $S - M'$  and contains a point of  $L$ , it follows that  $D'$  is a subset of  $D$  and that  $J'$  is a subset of  $M$ . Of course using  $J'$  it is now easy to construct a simple closed curve  $J$  which lies in  $D + p$  and contains  $p + q$ .

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