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**CONTINUITY AND CONVEXITY OF PROJECTIONS AND  
BARYCENTRIC COORDINATES IN CONVEX POLYHEDRA**

JOHN ARNOLD KALMAN

# CONTINUITY AND CONVEXITY OF PROJECTIONS AND BARYCENTRIC COORDINATES IN CONVEX POLYHEDRA

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If  $s_0, \dots, s_n$  are linearly independent points of real  $n$ -dimensional Euclidean space  $R^n$  then each point  $x$  of their convex hull  $S$  has a (unique) representation  $x = \sum_{i=0}^n \lambda_i(x) s_i$  with  $\lambda_i(x) \geq 0$  ( $i = 0, \dots, n$ ) and  $\sum_{i=0}^n \lambda_i(x) = 1$ , and the barycentric coordinates  $\lambda_0, \dots, \lambda_n$  are continuous convex functions on  $S$  (cf. [3, p. 288]). We shall show in this paper that given any finite set  $s_0, \dots, s_m$  of points of  $R^n$  we can assign barycentric coordinates  $\lambda_0, \dots, \lambda_m$  to their convex hull  $S$  in such a way that each coordinate is continuous on  $S$  and that one prescribed coordinate ( $\lambda_0$  say) is convex on  $S$  (Theorem 2); the author does not know whether it is always possible to make all the coordinates convex simultaneously (cf. Example 3). In proving Theorem 2 we shall use certain "projections" which we now define; these projections are in general distinct from those of [1, p. 614] and [2, p. 12]. Given two distinct points  $s_0$  and  $s$  of  $R^n$ , let  $s_0s$  be the open half-line consisting of all points  $s_0 + \lambda(s - s_0)$  with  $\lambda > 0$ ; given a point  $s_0$  of  $R^n$  and a closed subset  $S$  of  $R^n$  such that  $s_0 \notin S$ , let  $C(s_0, S)$  be the "cone" formed by the union of all open half-lines  $s_0s$  with  $s$  in  $S$ ; and given a point  $x$  in such a cone  $C(s_0, S)$ , let  $\pi(x)$  be the (unique) point of  $s_0S \cap S$  which is closest to  $s_0$ . Then we shall call the function  $\pi$  the "projection of  $C(s_0, S)$  on  $S$ ." Our proof of Theorem 2 depends on the fact that if  $S$  is a convex polyhedron then  $\pi$  is continuous (Theorem 1). This result may appear to be obvious, but it is not immediately obvious how a formal proof should be given; moreover, as we shall show (Examples 1 and 2), the conclusion need not remain true for polyhedra  $S$  which are not convex or for convex sets  $S$  which are not polyhedra. The author is indebted to the referee for improvements to Lemma 3, Example 1, and Example 2, and for the remark at the end of § 1.

**1. Projections.** For any subset  $A$  of  $R^n$  we shall denote by  $H(A)$  the convex hull of  $A$  and by  $L(A)$  the affine subspace of  $R^n$  spanned by  $A$  (cf. [2, pp. 21, 15]). If  $A = \{s_1, \dots, s_p\}$  we shall write  $H(A) = H(s_1, \dots, s_p)$  and  $L(A) = L(s_1, \dots, s_p)$ . Given two points  $x$  and  $y$  of  $R^n$  we shall denote by  $(x, y)$  the inner product of  $x$  and  $y$  and by  $|x - y|$  the Euclidean distance  $\sqrt{(x - y, x - y)}$  between  $x$  and  $y$ .

**LEMMA 1.** *Let  $s_0$  be a point of  $R^n$ , let  $S$  be a closed convex subset of  $R^n$  such that  $s_0 \notin S$ , and let  $\pi$  be the projection of  $C(s_0, S)$  on  $S$ . Suppose that points  $x, s_1, \dots, s_p$  of  $S$  and real numbers  $\lambda_1, \dots, \lambda_p$  are*

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such that  $x = \sum_{i=1}^p \lambda_i s_i$ ,  $\lambda_i > 0$  ( $i = 1, \dots, p$ ),  $\sum_{i=1}^p \lambda_i = 1$ , and  $\pi(x) = x$ . Then

- (i)  $\pi(y) = y$  for all  $y$  in  $H(s_1, \dots, s_p)$ ; and
- (ii)  $s_0 \notin L(s_1, \dots, s_p)$ .

*Proof.* (i) Given  $y$  in  $H(s_1, \dots, s_p)$  we can find nonnegative real numbers  $\mu_1, \dots, \mu_p$  such that  $y = \sum_{i=1}^p \mu_i s_i$  and  $\sum_{i=1}^p \mu_i = 1$ . Since each  $\lambda_i > 0$ , there exists  $\alpha$  with  $0 < \alpha < 1$  such that  $\lambda_i - \alpha \mu_i > 0$  for each  $i = 1, \dots, p$ . Let

$$z = \frac{x}{1 - \alpha} - \frac{\alpha y}{1 - \alpha} = \sum_{i=1}^p \left( \frac{\lambda_i - \alpha \mu_i}{1 - \alpha} \right) s_i;$$

then  $z \in H(s_1, \dots, s_p) \subseteq S$  and  $x = \alpha y + (1 - \alpha)z$ . We now use an indirect argument. Suppose that  $\pi(y) \neq y$ ; then for some  $\beta$  with  $0 < \beta < 1$  we have  $\pi(y) = (1 - \beta)s_0 + \beta y$  and

$$(1) \quad \frac{\alpha(1 - \beta)s_0 + \beta x}{\alpha(1 - \beta) + \beta} = \frac{\alpha\pi(y) + \beta(1 - \alpha)z}{\alpha + \beta(1 - \alpha)} = x'$$

say. It follows from (1) that  $x' \in s_0 x \cap S$  and that  $|s_0 - x'| < |s_0 - x|$ , contradicting the hypothesis that  $\pi(x) = x$ . This completes the proof of (i).

(ii) Suppose that  $s_0 \in L(s_1, \dots, s_p)$ . Then we can find real numbers  $\nu_1, \dots, \nu_p$  such that  $s_0 = \sum_{i=1}^p \nu_i s_i$  and  $\sum_{i=1}^p \nu_i = 1$ . Since each  $\lambda_i > 0$ , there exists  $\gamma$  with  $0 < \gamma < 1$  such that  $\lambda_i - \gamma(\lambda_i - \nu_i) > 0$  for each  $i = 1, \dots, p$ . But then if

$$w = \gamma s_0 + (1 - \gamma)x = \sum_{i=1}^p [\lambda_i - \gamma(\lambda_i - \nu_i)] s_i$$

we have  $w \in s_0 x \cap S$  and  $|s_0 - w| < |s_0 - x|$ , contradicting the hypothesis that  $\pi(x) = x$ . This completes the proof of (ii).

Let  $s_0, S$ , and  $\pi$  be as in Lemma 1. Then we shall call a subset  $A$  of  $S$  " $\pi$ -admissible" if  $\pi(x) = x$  for all  $x$  in  $H(A)$ .

**LEMMA 2.** *Let  $s_0, S$ , and  $\pi$  be as in Lemma 1, let  $A$  be a finite  $\pi$ -admissible subset of  $S$ , and let  $\pi'$  be the projection of  $C(s_0, H(A))$  on  $H(A)$ . Then*

- (i)  $\pi(x) = \pi'(x)$  for all  $x$  in  $C(s_0, H(A))$ ; and
- (ii)  $\pi'$  is a continuous mapping of  $C(s_0, H(A))$  into  $H(A)$ .

*Proof.* (i) Let  $x$  be any point of  $C(s_0, H(A))$ . Then  $\pi(\pi'(x)) = \pi'(x)$  since  $A$  is  $\pi$ -admissible, hence  $\pi'(x)$  is the point of  $s_0 \pi'(x) \cap S = s_0 x \cap S$  which is closest to  $s_0$ , and hence  $\pi(x) = \pi'(x)$ .

- (ii) Let  $A = \{s_1, \dots, s_p\}$  and let  $x_0 = \sum_{i=1}^p (1/p)s_i$ ; then  $\pi(x_0) = x_0$

since  $A$  is  $\pi$ -admissible, and hence  $s_0 \notin L(A)$  by Lemma 1. It follows that if  $s_*$  is the point of  $L(A)$  which is closest to  $s_0$ , and  $x$  is any point of  $C(s_0, H(A))$ , then

$$\pi'(x) = \frac{x - \lambda(x)s_0}{1 - \lambda(x)}, \quad \text{where} \quad \lambda(x) = \frac{(x - s_*, s_0 - s_*)}{|s_0 - s_*|^2}.$$

Hence  $\pi'$  is continuous.

**LEMMA 3.** *Let  $s_0$  be a point of  $R^n$  and let  $T$  be a closed bounded subset of  $R^n$  such that  $s_0 \notin T$ . Then  $\{s_0\} \cup C(s_0, T)$  is a closed subset of  $R^n$ .*

With the help of the Bolzano-Weierstrass theorem it is not difficult to prove Lemma 3.

**THEOREM 1.** *Let  $s_0, s_1, \dots, s_m$  be points of  $R^n$  such that  $s_0 \notin H(s_1, \dots, s_m) = S$  say, and let  $\pi$  be the projection of  $C(s_0, S)$  on  $S$ . Then  $\pi$  is a continuous mapping of  $C(s_0, S)$  into  $S$ .*

*Proof.* Let  $A_1, \dots, A_q$  be the subsets of  $\{s_1, \dots, s_m\}$  which are  $\pi$ -admissible subsets of  $S$ . Then each  $x$  in  $C(s_0, S)$  belongs to at least one  $C(s_0, H(A_j))$  ( $1 \leq j \leq q$ ); indeed, given  $x$  in  $C(s_0, S)$ , there exist positive integers  $x(1), \dots, x(p)$  and positive real numbers  $\lambda_1, \dots, \lambda_p$  such that  $\pi(x) = \sum_{i=1}^p \lambda_i s_{x(i)}$  and  $\sum_{i=1}^p \lambda_i = 1$ , and then  $A = \{s_{x(1)}, \dots, s_{x(p)}\}$  is  $\pi$ -admissible by Lemma 1 (i), and  $x \in C(s_0, H(A))$ . For each  $j = 1, \dots, q$  let  $\pi_j$  be the projection of  $C(s_0, H(A_j))$  on  $H(A_j)$ .

To prove the theorem it will be enough to show that, if  $x, x_1, x_2, \dots$  in  $C(s_0, S)$  are such that  $x = \lim_k x_k$ , then it follows that  $\pi(x) = \lim_k \pi(x_k)$ . Let  $J$  be the set of all  $j$  ( $1 \leq j \leq q$ ) such that  $x_k \in C(s_0, H(A_j))$  for infinitely many values of  $k$ , and for each  $j$  in  $J$  let  $j(1) < j(2) < \dots$  be the values of  $k$  such that  $x_k \in C(s_0, H(A_j))$ . Now, for each  $j$  in  $J$ ,  $x \in C(s_0, H(A_j))$  by Lemma 3, and hence, by Lemma 2,  $\pi(x) = \pi_j(x) = \lim_l \pi_j(x_{j(l)}) = \lim_l \pi(x_{j(l)})$ . Since all but a finite number of the positive integers are of the form  $j(l)$  for some  $j$  in  $J$  and some  $l = 1, 2, \dots$ , it follows that  $\pi(x) = \lim_k \pi(x_k)$ , as we wished to prove.

The following example shows that if  $S$  is a non-convex polyhedron in  $R^2$ , and  $s_0 \notin S$ , then the projection of  $C(s_0, S)$  on  $S$  need not be continuous.

**EXAMPLE 1.** Let  $s_0 = (0, 2)$ ,  $s_1 = (0, 1)$ ,  $s_2 = (0, 0)$ , and  $s_3 = (1, 0)$ ; and let  $S = H(s_1, s_2) \cup H(s_2, s_3)$ . Then the projection of  $C(s_0, S)$  on  $S$  is not continuous at  $s_1$ .

The following example shows that if  $S$  is a closed convex set in  $R^3$ , and  $s_0 \notin S$ , then the projection of  $C(s_0, S)$  on  $S$  need not be continuous.

**EXAMPLE 2.** Let  $s_0 = (0, 0, 2)$ , let  $s_1 = (0, 0, 1)$ , let  $K$  be the circle consisting of all points  $(\xi, \eta, \zeta)$  in  $R^3$  such that  $(\xi - 1)^2 + \eta^2 = 1$  and  $\zeta = 0$ , let  $S = H(\{s_1\} \cup K)$ , and let  $\pi$  be the projection of  $C(s_0, S)$  on  $S$ . Then if we set  $x_k = (1 - \cos k^{-1}, \sin k^{-1}, 0)$  ( $k = 1, 2, \dots$ ) we have  $x_k \in C(s_0, S)$  and  $\pi(x_k) = x_k$  ( $k = 1, 2, \dots$ ). When  $k \rightarrow \infty$ ,  $x_k \rightarrow (0, 0, 0) = s_2$ , say, and  $\pi(x_k) \rightarrow s_2$ ; since  $\pi(s_2) = s_1$ , this shows that  $\pi$  is not continuous at  $s_2$ .

**REMARK.** Theorem 1 is valid for each closed convex set  $S \subseteq R^2$ , and for each strictly convex closed set  $S \subseteq R^n$ .

**2. Barycentric coordinates.** Let  $s_0$  be a point of  $R^n$ , let  $S$  be a closed convex subset of  $R^n$  such that  $s_0 \notin S$ , and let  $D(s_0, S)$  be the union of all segments  $H(s_0, s)$  joining  $s_0$  to points  $s$  of  $S$ ; then  $D(s_0, S)$  is a convex set. Define a real-valued function  $\lambda_0$  on  $D(s_0, S)$  as follows: let  $\lambda_0(s_0) = 1$ , let  $\lambda_0(x) = 0$  if  $x \in S$ , and if  $x \neq s_0$  and  $x \notin S$  let  $\lambda_0(x)$  be defined by the equation  $x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]\pi(x)$ , where  $\pi$  is the projection of  $C(s_0, S)$  on  $S$ ; then each  $x$  in  $D(s_0, S)$  has a representation of the form

$$(2) \quad x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]s,$$

with  $s$  in  $S$ . We shall call  $\lambda_0$  the "barycentric function of  $D(s_0, S)$ ."

**LEMMA 4.** Let  $s_0$  be a point of  $R^n$ , let  $S$  be a closed convex subset of  $R^n$  such that  $s_0 \notin S$ , and let  $\lambda_0$  be the barycentric function of  $D(s_0, S)$ . Then  $0 \leq \lambda_0(x) \leq 1$  for all  $x$  in  $D(s_0, S)$  and  $\lambda_0$  is a convex function on  $D(s_0, S)$ . If  $S$  is a polyhedron then  $\lambda_0$  is continuous on  $D(s_0, S)$ .

*Proof.* It is clear that  $\lambda_0(x) \leq 1$  for all  $x$  in  $D(s_0, S)$ ; the proof that  $\lambda_0(x) \geq 0$  for all  $x$  in  $D(s_0, S)$  depends on the convexity of  $S$ , and will be left to the reader. To prove that  $\lambda_0$  is convex on  $D(s_0, S)$  we show that if  $x, x' \in D(s_0, S)$  and  $0 < \alpha < 1$  then

$$(3) \quad \lambda_0(\alpha x + (1 - \alpha)x') \leq \alpha\lambda_0(x) + (1 - \alpha)\lambda_0(x').$$

Let  $x^* = \alpha x + (1 - \alpha)x'$  and let  $\beta = \alpha\lambda_0(x) + (1 - \alpha)\lambda_0(x')$ ; we may assume that  $\beta < 1$  since otherwise (3) is trivial. Then if  $\gamma = \alpha[1 - \lambda_0(x)](1 - \beta)^{-1}$ , and  $s, s'$  in  $S$  are such that

$$x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]s, \quad x' = \lambda_0(x')s_0 + [1 - \lambda_0(x')]s'$$

(cf. (2)), we have

$$(4) \quad \gamma s + (1 - \gamma)s' = -\beta(1 - \beta)^{-1}s_0 + (1 - \beta)^{-1}x^*,$$

and  $\gamma s + (1 - \gamma)s' \in S$  since  $S$  is convex. It follows from (4) that  $x^* \neq s_0$ .

If  $x^* \notin S$  and  $\pi$  is the projection of  $C(s_0, S)$  on  $S$  then

$$\pi(x^*) = -\lambda_0(x^*)[1 - \lambda_0(x^*)]^{-1}s_0 + [1 - \lambda_0(x^*)]^{-1}x^* ,$$

and hence by (4) and the definition of  $\pi$ ,  $\lambda_0(x^*) \leq \beta$ , as asserted by (3). If  $x^* \in S$  then (3) is trivial. This completes the proof that  $\lambda_0$  is convex on  $D(s_0, S)$ .

We next show that  $\lambda_0$  is continuous at  $s_0$ . Given  $\varepsilon$  with  $0 < \varepsilon < 1$ , let  $\delta = M\varepsilon$ , where  $M > 0$  is the shortest distance from  $s_0$  to  $S$ . Then if  $x \in D(s_0, S)$  and  $0 < |x - s_0| < \delta$  we have  $x \neq s_0$ ,  $x \notin S$ , and

$$M \leq |\pi(x) - s_0| = [1 - \lambda_0(x)]^{-1} |x - s_0| < [1 - \lambda_0(x)]^{-1}M\varepsilon ,$$

and hence  $0 < 1 - \lambda_0(x) < \varepsilon$ . This proves that  $\lambda_0$  is continuous at  $s_0$ . It remains to prove that  $\lambda_0$  is continuous on  $D(s_0, S) - \{s_0\}$  if  $S$  is a polyhedron. For each  $x$  in  $C(s_0, S)$  define  $\mu_0(x)$  by the equation  $x = \mu_0(x)s_0 + [1 - \mu_0(x)]\pi(x)$ ; then

$$(5) \quad \mu_0(x) = 1 - |x - s_0| / |\pi(x) - s_0| .$$

It follows that  $\mu_0(x) \leq 0$  if  $x \in S$ , and that  $\mu_0(x) = \lambda_0(x) > 0$  if  $x \in D(s_0, S)$ ,  $x \neq s_0$ , and  $x \notin S$ ; thus

$$(6) \quad \lambda_0(x) = \max[\mu_0(x), 0] \quad (x \in D(s_0, S), x \neq s_0) .$$

If  $S$  is a polyhedron then  $\mu_0$  is continuous on  $C(s_0, S)$  by Theorem 1 and (5), and hence  $\lambda_0$  is continuous on  $D(s_0, S) - \{s_0\}$  by (6). This completes the proof of the lemma.

**THEOREM 2.** *Let  $s_0, \dots, s_m$  be points of  $R^n$ , and let  $S = H(s_0, \dots, s_m)$ . Then there exist nonnegative real-valued continuous functions  $\lambda_0, \dots, \lambda_m$  on  $S$ , with  $\lambda_0$  a convex function, such that, for each  $x$  in  $S$ ,*

$$x = \sum_{i=0}^m \lambda_i(x)s_i , \quad \text{and} \quad \sum_{i=0}^m \lambda_i(x) = 1 .$$

*Proof.* We use induction on  $m$ . The case  $m = 0$  is trivial. We assume the theorem to have been proved for  $m = M - 1$  and deduce it for  $m = M$ . Let  $T = H(s_1, \dots, s_M)$ . If  $s_0 \in T$  we may set  $\lambda_0(x) = 0$  for all  $x$  in  $S$ , and deduce the existence of  $\lambda_1, \dots, \lambda_M$  directly from the induction hypothesis; we therefore assume that  $s_0 \notin T$ . By the induction hypothesis there exist nonnegative real-valued continuous functions  $\mu_1, \dots, \mu_M$  on  $T$  such that, for each  $y$  in  $T$ ,  $y = \sum_{i=1}^M \mu_i(y)s_i$ , and  $\sum_{i=1}^M \mu_i(y) = 1$ . Let  $\lambda_0$  be the barycentric function of  $D(s_0, T)$ . Then each  $x$  in  $S = D(s_0, T)$  has a representation of the form  $x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]s_x$  with  $s_x$  in  $T$  (cf. (2)), and if we now set  $\lambda_i(x) = \mu_i(s_x)[1 - \lambda_0(x)]$  ( $x \in S$ ;  $i = 1, \dots, M$ ) then it follows that the  $\lambda_i$  ( $i = 1, \dots, M$ ) are well-defined

functions on  $S$ , and, by Lemma 4, that the functions  $\lambda_0, \dots, \lambda_M$  satisfy all the conditions in the statement of the theorem.

To show that the functions  $\lambda_i$  defined in the proof of Theorem 2 need not all be convex we can let  $s_0, s_1, s_2$ , and  $s_3$  be the points  $(0, 2)$ ,  $(1, 0)$ ,  $(-1, 0)$ , and  $(0, 1)$  respectively of  $R^2$  and let  $S = H(s_0, s_1, s_2, s_3)$ ; however in this example we obtain convex barycentric coordinates if we interchange the roles of  $s_0$  and  $s_3$ . In the following example some of the barycentric coordinates determined as in the proof of Theorem 2 fail to be convex no matter how  $s_0$  is chosen.

EXAMPLE 3. Define  $t_0, \dots, t_4$  in  $R^3$  as follows:  $t_0 = (0, 0, 1)$ ,  $t_1 = (0, 1, 0)$ ,  $t_2 = (0, -1, 0)$ ,  $t_3 = (1, 0, -1)$ , and  $t_4 = (-1, 0, -1)$ ; let  $S = H(t_0, \dots, t_4)$ ; and let barycentric coordinates be defined for  $S$  as in the proof of Theorem 2, with

- (i)  $t_0$ ,
- (ii)  $t_1$  or  $t_2$ , and
- (iii)  $t_3$  or  $t_4$  playing the role of  $s_0$ . Then if we write  $\theta_{\pm}$  for  $\max[\pm\theta, 0]$  ( $\theta$  real) we obtain

$$(i) \quad (\xi, 0, 0) = |\xi| t_0 + (\frac{1}{2} - |\xi|)(t_1 + t_2) + \xi_+ t_3 + \xi_- t_4 \quad (|\xi| \leq \frac{1}{2}),$$

$$(ii) \quad (0, \eta, 0) = \frac{1}{2}(1 - |\eta|)t_0 + \eta_+ t_1 + \eta_- t_2 + \frac{1}{4}(1 - |\eta|)(t_3 + t_4) \quad (|\eta| \leq 1), \text{ and}$$

$$(iii) \quad (0, 0, \zeta) = \zeta_+ t_0 + \frac{1}{2}(1 - |\zeta|)(t_1 + t_2) + \frac{1}{2}\zeta_-(t_3 + t_4) \quad (|\zeta| \leq 1),$$

respectively, and hence in no case are the barycentric coordinates all convex.

The argument in the proof of Theorem 2 amounts to determining barycentric coordinates  $\lambda_0, \dots, \lambda_m$  for  $H(s_0, \dots, s_m)$  by first choosing  $\lambda_0$  as small as possible, then choosing  $\lambda_1$  as small as possible with this choice of  $\lambda_0$ , etc. We remark in conclusion that if we first choose  $\lambda_0$  as large as possible, then choose  $\lambda_1$  as large as possible with this choice of  $\lambda_0$ , etc., we do not in general obtain convex barycentric coordinates; this may be seen by considering the case of a square in  $R^2$ .

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