Pacific Journal of Mathematics

CONTINUITY AND CONVEXITY OF PROJECTIONS AND BARYCENTRIC COORDINATES IN CONVEX POLYHEDRA

JOHN ARNOLD KALMAN

Vol. 11, No. 3 BadMonth 1961

CONTINUITY AND CONVEXITY OF PROJECTIONS AND BARYCENTRIC COORDINATES IN CONVEX POLYHEDRA

J. A. KALMAN

If $s_0, \cdots, s_n$ are linearly independent points of real n-dimensional Euclidean space R^n then each point x of their convex hull S has a (unique) representation $x = \sum_{i=0}^{n} \lambda_i(x) s_i$ with $\lambda_i(x) \geqq 0$ $(i = 0, \cdots, n)$ and $\sum_{i=0}^{n} \lambda_i(x) = 1$, and the barycentric coordinates $\lambda_0, \cdots, \lambda_n$ are continuous convex functions on S (cf. [3, p. 288]). We shall show in this paper that given any finite set $s_0, \cdots, s_m$ of points of R^n we can assign barycentric coordinates $\lambda_0, \cdots, \lambda_m$ to their convex hull S in such a way that each coordinate is continuous on S and that one prescribed coordinate (λ_0 say) is convex on S (Theorem 2); the author does not know whether it is always possible to make all the coordinates convex simultaneously (cf. Example 3). In proving Theorem 2 we shall use certain "projections" which we now define; these projections are in general distinct from those of [1, p. 614] and [2, p. 12]. Given two distinct points s_0 and s of R^n, let s_0s be the open half-line consisting of all points $s_0 + \lambda(s - s_0)$ with $\lambda > 0$; given a point s_0 of R^n and a closed subset S of R^n such that $s_0 \notin S$, let $C(s_0, S)$ be the "cone" formed by the union of all open half-lines s_0s with s in S; and given a point x in such a cone $C(s_0, S)$, let $\pi(x)$ be the (unique) point of $s_0x \cap S$ which is closest to s_0. Then we shall call the function π the "projection of $C(s_0, S)$ on S." Our proof of Theorem 2 depends on the fact that if S is a convex polyhedron then π is continuous (Theorem 1). This result may appear to be obvious, but it is not immediately obvious how a formal proof should be given; moreover, as we shall show (Examples 1 and 2), the conclusion need not remain true for polyhedra S which are not convex or for convex sets S which are not polyhedra. The author is indebted to the referee for improvements to Lemma 3, Example 1, and Example 2, and for the remark at the end of § 1.

1. Projections. For any subset A of R^n we shall denote by $H(A)$ the convex hull of A and by $L(A)$ the affine subspace of R^n spanned by A (cf. [2, pp. 21, 15]). If $A = \{s_1, \cdots, s_p\}$ we shall write $H(A) = H(s_1, \cdots, s_p)$ and $L(A) = L(s_1, \cdots, s_p)$. Given two points x and y of R^n we shall denote by (x, y) the inner product of x and y and by $|x - y|$ the Euclidean distance $\sqrt{(x - y, x - y)}$ between x and y.

LEMMA 1. *Let s_0 be a point of R^n, let S be a closed convex subset of R^n such that $s_0 \notin S$, and let π be the projection of $C(s_0, S)$ on S. Suppose that points $x, s_1, \cdots, s_p$ of S and real numbers $\lambda_1, \cdots, \lambda_p$ are*

Received July 7, 1960.

such that $x = \sum_{i=1}^{p} \lambda_i s_i$, $\lambda_i > 0$ $(i = 1, \cdots, p)$, $\sum_{i=1}^{p} \lambda_i = 1$, *and* $\pi(x) = x$. *Then*

(i) $\pi(y) = y$ *for all* y *in* $H(s_1, \cdots, s_p)$; *and*

(ii) $s_0 \notin L(s_1, \cdots, s_p)$.

Proof. (i) Given y in $H(s_1, \cdots, s_p)$ we can find nonnegative real numbers $\mu_1, \cdots, \mu_p$ such that $y = \sum_{i=1}^{p} \mu_i s_i$ and $\sum_{i=1}^{p} \mu_i = 1$. Since each $\lambda_i > 0$, there exists α with $0 < \alpha < 1$ such that $\lambda_i - \alpha\mu_i > 0$ for each $i = 1, \cdots, p$. Let

$$z = \frac{x}{1-\alpha} - \frac{\alpha y}{1-\alpha} = \sum_{i=1}^{p} \left(\frac{\lambda_i - \alpha\mu_i}{1-\alpha}\right) s_i\ ;$$

then $z \in H(s_1, \cdots, s_p) \subseteqq S$ and $x = \alpha y + (1-\alpha)z$. We now use an indirect argument. Suppose that $\pi(y) \neq y$; then for some β with $0 < \beta < 1$ we have $\pi(y) = (1-\beta)s_0 + \beta y$ and

$$(1) \qquad \frac{\alpha(1-\beta)s_0 + \beta x}{\alpha(1-\beta) + \beta} = \frac{\alpha\pi(y) + \beta(1-\alpha)z}{\alpha + \beta(1-\alpha)} = x'$$

say. It follows from (1) that $x' \in s_0 x \cap S$ and that $|s_0 - x'| < |s_0 - x|$, contradicting the hypothesis that $\pi(x) = x$. This completes the proof of (i).

(ii) Suppose that $s_0 \in L(s_1, \cdots, s_p)$. Then we can find real numbers $\nu_1, \cdots, \nu_p$ such that $s_0 = \sum_{i=1}^{p} \nu_i s_i$ and $\sum_{i=1}^{p} \nu_i = 1$. Since each $\lambda_i > 0$, there exists γ with $0 < \gamma < 1$ such that $\lambda_i - \gamma(\lambda_i - \nu_i) > 0$ for each $i = 1, \cdots, p$. But then if

$$w = \gamma s_0 + (1-\gamma)x = \sum_{i=1}^{p} [\lambda_i - \gamma(\lambda_i - \nu_i)]s_i$$

we have $w \in s_0 x \cap S$ and $|s_0 - w| < |s_0 - x|$, contradicting the hypothesis that $\pi(x) = x$. This completes the proof of (ii).

Let s_0, S, and π be as in Lemma 1. Then we shall call a subset A of S "π-admissible" if $\pi(x) = x$ for all x in $H(A)$.

LEMMA 2. *Let* s_0, S, *and* π *be as in Lemma* 1, *let* A *be a finite* π*-admissible subset of* S, *and let* π' *be the projection of* $C(s_0, H(A))$ *on* $H(A)$. *Then*

(i) $\pi(x) = \pi'(x)$ *for all* x *in* $C(s_0, H(A))$; *and*

(ii) π' *is a continuous mapping of* $C(s_0, H(A))$ *into* $H(A)$.

Proof. (i) Let x be any point of $C(s_0, H(A))$. Then $\pi(\pi'(x)) = \pi'(x)$ since A is π-admissible, hence $\pi'(x)$ is the point of $s_0\pi'(x) \cap S = s_0 x \cap S$ which is closest to s_0, and hence $\pi(x) = \pi'(x)$.

(ii) Let $A = \{s_1, \cdots, s_p\}$ and let $x_0 = \sum_{i=1}^{p} (1/p)s_i$; then $\pi(x_0) = x_0$

since A is π-admissible, and hence $s_0 \notin L(A)$ by Lemma 1. It follows that if s_* is the point of $L(A)$ which is closest to s_0, and x is any point of $C(s_0, H(A))$, then

$$\pi'(x) = \frac{x - \lambda(x)s_0}{1 - \lambda(x)}, \quad \text{where} \quad \lambda(x) = \frac{(x - s_*, s_0 - s_*)}{|s_0 - s_*|^2}.$$

Hence π' is continuous.

LEMMA 3. *Let s_0 be a point of R^n and let T be a closed bounded subset of R^n such that $s_0 \notin T$. Then $\{s_0\} \cup C(s_0, T)$ is a closed subset of R^n.*

With the help of the Bolzano-Weierstrass theorem it is not difficult to prove Lemma 3.

THEOREM 1. *Let $s_0, s_1, \cdots, s_m$ be points of R^n such that $s_0 \notin H(s_1, \cdots, s_m) = S$ say, and let π be the projection of $C(s_0, S)$ on S. Then π is a continuous mapping of $C(s_0, S)$ into S.*

Proof. Let $A_1, \cdots, A_q$ be the subsets of $\{s_1, \cdots, s_m\}$ which are π-admissible subsets of S. Then each x in $C(s_0, S)$ belongs to at least one $C(s_0, H(A_j))$ $(1 \leqq j \leqq q)$; indeed, given x in $C(s_0, S)$, there exist positive integers $x(1), \cdots, x(p)$ and positive real numbers $\lambda_1, \cdots, \lambda_p$ such that $\pi(x) = \sum_{i=1}^{p} \lambda_i s_{x(i)}$ and $\sum_{i=1}^{p} \lambda_i = 1$, and then $A = \{s_{x(1)}, \cdots, s_{x(p)}\}$ is π-admissible by Lemma 1 (i), and $x \in C(s_0, H(A))$. For each $j = 1, \cdots, q$ let π_j be the projection of $C(s_0, H(A_j))$ on $H(A_j)$.

To prove the theorem it will be enough to show that, if $x, x_1, x_2, \cdots$ in $C(s_0, S)$ are such that $x = \lim_k x_k$, then it follows that $\pi(x) = \lim_k \pi(x_k)$. Let J be the set of all j $(1 \leqq j \leqq q)$ such that $x_k \in C(s_0, H(A_j))$ for infinitely many values of k, and for each j in J let $j(1) < j(2) < \cdots$ be the values of k such that $x_k \in C(s_0, H(A_j))$. Now, for each j in J, $x \in C(s_0, H(A_j))$ by Lemma 3, and hence, by Lemma 2, $\pi(x) = \pi_j(x) = \lim_l \pi_j(x_{j(l)}) = \lim_l \pi(x_{j(l)})$. Since all but a finite number of the positive integers are of the form $j(l)$ for some j in J and some $l = 1, 2, \cdots$, it follows that $\pi(x) = \lim_k \pi(x_k)$, as we wished to prove.

The following example shows that if S is a non-convex polyhedron in R^2, and $s_0 \notin S$, then the projection of $C(s_0, S)$ on S need not be continuous.

EXAMPLE 1. Let $s_0 = (0, 2)$, $s_1 = (0, 1)$, $s_2 = (0, 0)$, and $s_3 = (1, 0)$; and let $S = H(s_1, s_2) \cup H(s_2, s_3)$. Then the projection of $C(s_0, S)$ on S is not continuous at s_1.

The following example shows that if S is a closed convex set in R^3, and $s_0 \notin S$, then the projection of $C(s_0, S)$ on S need not be continuous.

EXAMPLE 2. Let $s_0 = (0, 0, 2)$, let $s_1 = (0, 0, 1)$, let K be the circle consisting of all points (ξ, η, ζ) in R^3 such that $(\xi - 1)^2 + \eta^2 = 1$ and $\zeta = 0$, let $S = H(\{s_1\} \cup K)$, and let π be the projection of $C(s_0, S)$ on S. Then if we set $x_k = (1 - \cos k^{-1}, \sin k^{-1}, 0)$ $(k = 1, 2, \cdots)$ we have $x_k \in C(s_0, S)$ and $\pi(x_k) = x_k$ $(k = 1, 2, \cdots)$. When $k \to \infty$, $x_k \to (0, 0, 0) = s_2$ say, and $\pi(x_k) \to s_2$; since $\pi(s_2) = s_1$, this shows that π is not continuous at s_2.

REMARK. Theorem 1 is valid for each closed convex set $S \subseteqq R^2$, and for each strictly convex closed set $S \subseteqq R^n$.

2. Barycentric coordinates. Let s_0 be a point of R^n, let S be a closed convex subset of R^n such that $s_0 \notin S$, and let $D(s_0, S)$ be the union of all segments $H(s_0, s)$ joining s_0 to points s of S; then $D(s_0, S)$ is a convex set. Define a real-valued function λ_0 on $D(s_0, S)$ as follows: let $\lambda_0(s_0) = 1$, let $\lambda_0(x) = 0$ if $x \in S$, and if $x \neq s_0$ and $x \notin S$ let $\lambda_0(x)$ be defined by the equation $x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]\pi(x)$, where π is the projection of $C(s_0, S)$ on S; then each x in $D(s_0, S)$ has a representation of the form

$$x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]s \,, \tag{2}$$

with s in S. We shall call λ_0 the "barycentric function of $D(s_0, S)$."

LEMMA 4. *Let s_0 be a point of R^n, let S be a closed convex subset of R^n such that $s_0 \notin S$, and let λ_0 be the barycentric function of $D(s_0, S)$. Then $0 \leqq \lambda_0(x) \leqq 1$ for all x in $D(s_0, S)$ and λ_0 is a convex function on $D(s_0, S)$. If S is a polyhedron then λ_0 is continuous on $D(s_0, S)$.*

Proof. It is clear that $\lambda_0(x) \leqq 1$ for all x in $D(s_0, S)$; the proof that $\lambda_0(x) \geqq 0$ for all x in $D(s_0, S)$ depends on the convexity of S, and will be left to the reader. To prove that λ_0 is convex on $D(s_0, S)$ we show that if $x, x' \in D(s_0, S)$ and $0 < \alpha < 1$ then

$$\lambda_0(\alpha x + (1 - \alpha)x') \leqq \alpha\lambda_0(x) + (1 - \alpha)\lambda_0(x') \,. \tag{3}$$

Let $x^* = \alpha x + (1 - \alpha)x'$ and let $\beta = \alpha\lambda_0(x) + (1 - \alpha)\lambda_0(x')$; we may assume that $\beta < 1$ since otherwise (3) is trivial. Then if $\gamma = \alpha[1 - \lambda_0(x)](1 - \beta)^{-1}$, and s, s' in S are such that

$$x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]s \,, \quad x' = \lambda_0(x')s_0 + [1 - \lambda_0(x')]s'$$

(cf. (2)), we have

$$\gamma s + (1 - \gamma)s' = -\beta(1 - \beta)^{-1}s_0 + (1 - \beta)^{-1}x^* \,, \tag{4}$$

and $\gamma s + (1 - \gamma)s' \in S$ since S is convex. It follows from (4) that $x^* \neq s_0$.

If $x^* \notin S$ and π is the projection of $C(s_0, S)$ on S then

$$\pi(x^*) = -\lambda_0(x^*)[1 - \lambda_0(x^*)]^{-1}s_0 + [1 - \lambda_0(x^*)]^{-1}x^* ,$$

and hence by (4) and the definition of π, $\lambda_0(x^*) \leqq \beta$, as asserted by (3). If $x^* \in S$ then (3) is trivial. This completes the proof that λ_0 is convex on $D(s_0, S)$.

We next show that λ_0 is continuous at s_0. Given ε with $0 < \varepsilon < 1$, let $\delta = M\varepsilon$, where $M > 0$ is the shortest distance from s_0 to S. Then if $x \in D(s_0, S)$ and $0 < |x - s_0| < \delta$ we have $x \neq s_0$, $x \notin S$, and

$$M \leqq |\pi(x) - s_0| = [1 - \lambda_0(x)]^{-1} |x - s_0| < [1 - \lambda_0(x)]^{-1}M\varepsilon ,$$

and hence $0 < 1 - \lambda_0(x) < \varepsilon$. This proves that λ_0 is continuous at s_0. It remains to prove that λ_0 is continuous on $D(s_0, S) - \{s_0\}$ if S is a polyhedron. For each x in $C(s_0, S)$ define $\mu_0(x)$ by the equation $x = \mu_0(x)s_0 + [1 - \mu_0(x)]\pi(x)$; then

$$\mu_0(x) = 1 - |x - s_0| / |\pi(x) - s_0| . \tag{5}$$

It follows that $\mu_0(x) \leqq 0$ if $x \in S$, and that $\mu_0(x) = \lambda_0(x) > 0$ if $x \in D(s_0, S)$, $x \neq s_0$, and $x \notin S$; thus

$$\lambda_0(x) = \max[\mu_0(x), 0] \quad (x \in D(s_0, S),\ x \neq s_0) . \tag{6}$$

If S is a polyhedron then μ_0 is continuous on $C(s_0, S)$ by Theorem 1 and (5), and hence λ_0 is continuous on $D(s_0, S) - \{s_0\}$ by (6). This completes the proof of the lemma.

THEOREM 2. *Let $s_0, \cdots, s_m$ be points of R^n, and let $S = H(s_0, \cdots, s_m)$. Then there exist nonnegative real-valued continuous functions $\lambda_0, \cdots, \lambda_m$ on S, with λ_0 a convex function, such that, for each x in S,*

$$x = \sum_{i=0}^{m} \lambda_i(x)s_i , \quad \text{and} \quad \sum_{i=0}^{m} \lambda_i(x) = 1 .$$

Proof. We use induction on m. The case $m = 0$ is trivial. We assume the theorem to have been proved for $m = M - 1$ and deduce it for $m = M$. Let $T = H(s_1, \cdots, s_M)$. If $s_0 \in T$ we may set $\lambda_0(x) = 0$ for all x in S, and deduce the existence of $\lambda_1, \cdots, \lambda_M$ directly from the induction hypothesis; we therefore assume that $s_0 \notin T$. By the induction hypothesis there exist nonnegative real-valued continuous functions $\mu_1, \cdots, \mu_M$ on T such that, for each y in T, $y = \sum_{i=1}^{M} \mu_i(y)s_i$, and $\sum_{i=1}^{M} \mu_i(y) = 1$. Let λ_0 be the barycentric function of $D(s_0, T)$. Then each x in $S = D(s_0, T)$ has a representation of the form $x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]s_x$ with s_x in T (cf. (2)), and if we now set $\lambda_i(x) = \mu_i(s_x)[1 - \lambda_0(x)]$ ($x \in S$; $i = 1, \cdots, M$) then it follows that the λ_i ($i = 1, \cdots, M$) are well-defined functions on S, and, by Lemma 4, that the functions $\lambda_0, \cdots, \lambda_M$ satisfy all the conditions in the statement of the theorem.

To show that the functions λ_i defined in the proof of Theorem 2 need not all be convex we can let s_0, s_1, s_2, and s_3 be the points $(0, 2)$, $(1, 0)$, $(-1, 0)$, and $(0, 1)$ respectively of R^2 and let $S = H(s_0, s_1, s_2, s_3)$; however in this example we obtain convex barycentric coordinates if we interchange the roles of s_0 and s_3. In the following example some of the barycentric coordinates determined as in the proof of Theorem 2 fail to be convex no matter how s_0 is chosen.

EXAMPLE 3. Define $t_0, \cdots, t_4$ in R^3 as follows: $t_0 = (0, 0, 1)$, $t_1 = (0, 1, 0)$, $t_2 = (0, -1, 0)$, $t_3 = (1, 0, -1)$, and $t_4 = (-1, 0, -1)$; let $S = H(t_0, \cdots, t_4)$; and let barycentric coordinates be defined for S as in the proof of Theorem 2, with

(i) t_0,
(ii) t_1 or t_2, and
(iii) t_3 or t_4 playing the role of s_0. Then if we write $\theta_\pm$ for $\max[\pm\theta, 0]$ (θ real) we obtain

(i) $(\xi, 0, 0) = |\xi|\, t_0 + (\tfrac{1}{2} - |\xi|)(t_1 + t_2) + \xi_+ t_3 + \xi_- t_4 \qquad (|\xi| \leqq \tfrac{1}{2})$,

(ii) $(0, \eta, 0) = \tfrac{1}{2}(1 - |\eta|)t_0 + \eta_+ t_1 + \eta_- t_2 + \tfrac{1}{4}(1 - |\eta|)(t_3 + t_4) \qquad (|\eta| \leqq 1)$, and

(iii) $(0, 0, \zeta) = \zeta_+ t_0 + \tfrac{1}{2}(1 - |\zeta|)(t_1 + t_2) + \tfrac{1}{2}\zeta_-(t_3 + t_4) \qquad (|\zeta| \leqq 1)$,

respectively, and hence in no case are the barycentric coordinates all convex.

The argument in the proof of Theorem 2 amounts to determining barycentric coordinates $\lambda_0, \cdots, \lambda_m$ for $H(s_0, \cdots, s_m)$ by first choosing λ_0 as small as possible, then choosing λ_1 as small as possible with this choice of λ_0, etc. We remark in conclusion that if we first choose λ_0 as large as possible, then choose λ_1 as large as possible with this choice of λ_0, etc., we do not in general obtain convex barycentric coordinates; this may be seen by considering the case of a square in R^2.

REFERENCES

1. P. Alexandroff and H. Hopf, *Topologie I*, Berlin, 1935.
2. H. G. Eggleston, *Convexity*, Cambridge, England, 1958.
3. S. Lefschetz, *Algebraic topology*, New York, 1942.

UNIVERSITY OF AUCKLAND

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics
Vol. 11, No. 3 BadMonth, 1961