CONTINUITY AND CONVEXITY OF PROJECTIONS AND BARYCENTRIC COORDINATES IN CONVEX POLYHEDRA

JOHN ARNOLD KALMAN
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If \( s_0, \ldots, s_n \) are linearly independent points of real \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) then each point \( x \) of their convex hull \( S \) has a (unique) representation \( x = \sum_{i=0}^{n} \lambda_i s_i \) with \( \lambda_i(x) \geq 0 \) (\( i = 0, \ldots, n \)) and \( \sum_{i=0}^{n} \lambda_i(x) = 1 \), and the barycentric coordinates \( \lambda_0, \ldots, \lambda_n \) are continuous convex functions on \( S \) (cf. [3, p. 288]). We shall show in this paper that given any finite set \( s_0, \ldots, s_m \) of points of \( \mathbb{R}^n \) we can assign barycentric coordinates \( \lambda_0, \ldots, \lambda_m \) to their convex hull \( S \) in such a way that each coordinate is continuous on \( S \) and that one prescribed coordinate (\( \lambda_0 \) say) is convex on \( S \) (Theorem 2); the author does not know whether it is always possible to make all the coordinates convex simultaneously (cf. Example 3). In proving Theorem 2 we shall use certain “projections” which we now define; these projections are in general distinct from those of [1, p. 614] and [2, p. 12]. Given two distinct points \( s_0 \) and \( s \) of \( \mathbb{R}^n \), let \( s_0s \) be the open half-line consisting of all points \( s_0 + \lambda(s - s_0) \) with \( \lambda > 0 \); given a point \( s_0 \) of \( \mathbb{R}^n \) and a closed subset \( S \) of \( \mathbb{R}^n \) such that \( s_0 \notin S \), let \( C(s_0, S) \) be the “cone” formed by the union of all open half-lines \( s_0s_0 \) with \( s \) in \( S \); and given a point \( x \) in such a cone \( C(s_0, S) \), let \( \pi(x) \) be the (unique) point of \( s_0x \cap S \) which is closest to \( s_0 \). Then we shall call the function \( \pi \) the “projection of \( C(s_0, S) \) on \( S \)”.

Our proof of Theorem 2 depends on the fact that if \( S \) is a convex polyhedron then \( \pi \) is continuous (Theorem 1). This result may appear to be obvious, but it is not immediately obvious how a formal proof should be given; moreover, as we shall show (Examples 1 and 2), the conclusion need not remain true for polyhedra \( S \) which are not convex or for convex sets \( S \) which are not polyhedra.

The author is indebted to the referee for improvements to Lemma 3, Example 1, and Example 2, and for the remark at the end of § 1.

1. Projections. For any subset \( A \) of \( \mathbb{R}^n \) we shall denote by \( H(A) \) the convex hull of \( A \) and by \( L(A) \) the affine subspace of \( \mathbb{R}^n \) spanned by \( A \) (cf. [2, pp. 21, 15]). If \( A = \{ s_1, \ldots, s_p \} \) we shall write \( H(A) = H(s_1, \ldots, s_p) \) and \( L(A) = L(s_1, \ldots, s_p) \). Given two points \( x \) and \( y \) of \( \mathbb{R}^n \) we shall denote by \( (x, y) \) the inner product of \( x \) and \( y \) and by \( |x - y| \) the Euclidean distance \( \sqrt{(x - y, x - y)} \) between \( x \) and \( y \).

**Lemma 1.** Let \( s_0 \) be a point of \( \mathbb{R}^n \), let \( S \) be a closed convex subset of \( \mathbb{R}^n \) such that \( s_0 \notin S \), and let \( \pi \) be the projection of \( C(s_0, S) \) on \( S \). Suppose that points \( x, s_1, \ldots, s_p \) of \( S \) and real numbers \( \lambda_1, \ldots, \lambda_p \) are...
such that \( x = \sum_{i=1}^{p} \lambda_i s_i \), \( \lambda_i > 0 \) \((i = 1, \ldots, p)\), \( \sum_{i=1}^{p} \lambda_i = 1 \), and \( \pi(x) = x \). Then

(i) \( \pi(y) = y \) for all \( y \) in \( H(s_1, \ldots, s_p) \); and

(ii) \( s_0 \notin L(s_1, \ldots, s_p) \).

Proof. (i) Given \( y \) in \( H(s_1, \ldots, s_p) \) we can find nonnegative real numbers \( \mu_1, \ldots, \mu_p \) such that \( y = \sum_{i=1}^{p} \mu_i s_i \) and \( \sum_{i=1}^{p} \mu_i = 1 \). Since each \( \lambda_i > 0 \), there exists \( \alpha \) with \( 0 < \alpha < 1 \) such that \( \lambda_i - \alpha \mu_i > 0 \) for each \( i = 1, \ldots, p \). Let

\[
    z = \frac{x - \alpha y}{1 - \alpha} = \sum_{i=1}^{p} \left( \frac{\lambda_i - \alpha \mu_i}{1 - \alpha} \right) s_i ;
\]

then \( z \in H(s_1, \ldots, s_p) \subseteq S \) and \( x = \alpha y + (1 - \alpha)z \). We now use an indirect argument. Suppose that \( \pi(y) \neq y \); then for some \( \beta \) with \( 0 < \beta < 1 \) we have \( \pi(y) = (1 - \beta)s_0 + \beta y \) and

\[
    \alpha(1 - \beta) s_0 + \beta x = \alpha \pi(y) + \beta(1 - \alpha)z = x',
\]

say. It follows from (1) that \( x' \in s_0 x \cap S \) and that \( |s_0 - x'| < |s_0 - x| \), contradicting the hypothesis that \( \pi(x) = x \). This completes the proof of (i).

(ii) Suppose that \( s_0 \in L(s_1, \ldots, s_p) \). Then we can find real numbers \( \nu_1, \ldots, \nu_p \) such that \( s_0 = \sum_{i=1}^{p} \nu_i s_i \) and \( \sum_{i=1}^{p} \nu_i = 1 \). Since each \( \lambda_i > 0 \), there exists \( \gamma \) with \( 0 < \gamma < 1 \) such that \( \lambda_i - \gamma(\lambda_i - \nu_i) > 0 \) for each \( i = 1, \ldots, p \). But then if

\[
    w = \gamma s_0 + (1 - \gamma)x = \sum_{i=1}^{p} [\lambda_i - \gamma(\lambda_i - \nu_i)] s_i ,
\]

we have \( w \in s_0 x \cap S \) and \( |s_0 - w| < |s_0 - x| \), contradicting the hypothesis that \( \pi(x) = x \). This completes the proof of (ii).

Let \( s_0, S, \) and \( \pi \) be as in Lemma 1. Then we shall call a subset \( A \) of \( S \) "\( \pi \)-admissible" if \( \pi(x) = x \) for all \( x \) in \( H(A) \).

**Lemma 2.** Let \( s_0, S, \) and \( \pi \) be as in Lemma 1, let \( A \) be a finite \( \pi \)-admissible subset of \( S \), and let \( \pi' \) be the projection of \( C(s_0, H(A)) \) on \( H(A) \). Then

(i) \( \pi(x) = \pi'(x) \) for all \( x \) in \( C(s_0, H(A)) \); and

(ii) \( \pi' \) is a continuous mapping of \( C(s_0, H(A)) \) into \( H(A) \).

Proof. (i) Let \( x \) be any point of \( C(s_0, H(A)) \). Then \( \pi(\pi'(x)) = \pi'(x) \) since \( A \) is \( \pi \)-admissible, hence \( \pi'(x) \) is the point of \( s_0 \pi'(x) \cap S = s_0 x \cap S \) which is closest to \( s_0 \), and hence \( \pi(x) = \pi'(x) \).

(ii) Let \( A = \{s_1, \ldots, s_p\} \) and let \( x_0 = \sum_{i=1}^{p} (1/p) s_i \); then \( \pi(x_0) = x_0 \).
since $A$ is $\pi$-admissible, and hence $s_0 \notin L(A)$ by Lemma 1. It follows
that if $s_*$ is the point of $L(A)$ which is closest to $s_0$, and $x$ is any point
of $C(s_0, H(A))$, then
\[ \pi'(x) = \frac{x - \lambda(x)s_0}{1 - \lambda(x)}, \quad \text{where} \quad \lambda(x) = \frac{(x - s_*, s_0 - s_*)}{|s_0 - s_*|^2}. \]
Hence $\pi'$ is continuous.

**Lemma 3.** Let $s_0$ be a point of $R^n$ and let $T$ be a closed bounded
subset of $R^n$ such that $s_0 \notin T$. Then \{s_0\} U $C(s_0, T)$ is a closed subset
of $R^n$.

With the help of the Bolzano-Weierstrass theorem it is not difficult
to prove Lemma 3.

**Theorem 1.** Let $s_0, s_1, \cdots, s_m$ be points of $R^n$ such that $s_0 \notin H(s_1, \cdots, s_m) = S$ say, and let $\pi$ be the projection of $C(s_0, S)$ on $S$. Then $\pi$ is a continuous mapping of $C(s_0, S)$ into $S$.

**Proof.** Let $A_1, \cdots, A_q$ be the subsets of $\{s_1, \cdots, s_m\}$ which are $\pi$-admissible subsets of $S$. Then each $x$ in $C(s_0, S)$ belongs to at least one
$C(s_0, H(A_j))$ (1 $\leq j \leq q$); indeed, given $x$ in $C(s_0, S)$, there exist positive integers $x(1), \cdots, x(p)$ and positive real numbers $\lambda_1, \cdots, \lambda_p$ such that
\[ \pi(x) = \sum_{i=1}^{p} \lambda_i s_{x(i)} \text{ and } \sum_{i=1}^{p} \lambda_i = 1, \text{ and then } A = \{s_{x(1)}, \cdots, s_{x(p)}\} \text{ is } \pi\text{-admissible by Lemma 1(i), and } x \in C(s_0, H(A)). \]
For each $j = 1, 2, \cdots, q$ let $\pi_j$ be the projection of $C(s_0, H(A_j))$ on $H(A_j)$.

To prove the theorem it will be enough to show that, if $x, x_1, x_2, \cdots$ in $C(s_0, S)$ are such that $x = \lim_k x_k$, then it follows that $\pi(x) = \lim_k \pi(x_k)$. Let $J$ be the set of all $j$ (1 $\leq j \leq q$) such that $x_k \in C(s_0, H(A_j))$ for infinitely
many values of $k$, and for each $j$ in $J$ let $j(1) < j(2) < \cdots$ be the values
of $k$ such that $x_k \in C(s_0, H(A_j))$. Now, for each $j$ in $J$, $x \in C(s_0, H(A_j))$ by Lemma 3, and hence, by Lemma 2, $\pi(x) = \pi_j(x) = \lim_l \pi_j(x_{j(l)}) = \lim_l \pi(x_{j(l)})$. Since all but a finite number of the positive integers are of the form $j(l)$ for some $j$ in $J$ and some $l = 1, 2, \cdots$, it follows that
$\pi(x) = \lim_k \pi(x_k)$, as we wished to prove.

The following example shows that if $S$ is a non-convex polyhedron
in $R^2$, and $s_0 \notin S$, then the projection of $C(s_0, S)$ on $S$ need not be con-
tinuous.

**Example 1.** Let $s_0 = (0, 2), s_1 = (0, 1), s_2 = (0, 0), \text{ and } s_3 = (1, 0)$; and let $S = H(s_1, s_2) \cup H(s_2, s_3)$. Then the projection of $C(s_0, S)$ on $S$ is not continuous at $s_1$.

The following example shows that if $S$ is a closed convex set in $R^3$, and $s_0 \notin S$, then the projection of $C(s_0, S)$ on $S$ need not be continuous.
EXAMPLE 2. Let $s_0 = (0, 0, 2)$, let $s_\lambda = (0, 0, 1)$, let $K$ be the circle consisting of all points $(\xi, \eta, \zeta)$ in $R^3$ such that $(\xi - 1)^2 + \eta^2 = 1$ and $\zeta = 0$, let $S = H([s_0] \cup K)$, and let $\pi$ be the projection of $C(s_0, S)$ on $S$. Then if we set $x_k = (1 - \cos k^{-1}, \sin k^{-1}, 0)$ $(k = 1, 2, \cdots)$ we have $x_k \in C(s_0, S)$ and $\pi(x_k) = x_k$ $(k = 1, 2, \cdots)$. When $k \to \infty$, $x_k \to (0, 0, 0) = s_2$ say, and $\pi(x_k) \to s_2$; since $\pi(s_2) = s_1$, this shows that $\pi$ is not continuous at $s_2$.

REMARK. Theorem 1 is valid for each closed convex set $S \subseteq R$, and for each strictly convex closed set $S \subseteq R^n$.

2. Barycentric coordinates. Let $s_0$ be a point of $R^n$, let $S$ be a closed convex subset of $R^n$ such that $s_0 \not\in S$, and let $D(s_0, S)$ be the union of all segments $H(s_0, s)$ joining $s_0$ to points $s$ of $S$; then $D(s_0, S)$ is a convex set. Define a real-valued function $\lambda_0$ on $D(s_0, S)$ as follows: let $\lambda_0(s_0) = 1$, let $\lambda_0(x) = 0$ if $x \in S$, and if $x \not\in s_0$ and $x \not\in S$ let $\lambda_0(x)$ be defined by the equation $x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]\pi(x)$, where $\pi$ is the projection of $C(s_0, S)$ on $S$; then each $x$ in $D(s_0, S)$ has a representation of the form

$$x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]s,$$

with $s$ in $S$. We shall call $\lambda_0$ the "barycentric function of $D(s_0, S)$.''

LEMMA 4. Let $s_0$ be a point of $R^n$, let $S$ be a closed convex subset of $R^n$ such that $s_0 \not\in S$, and let $\lambda_0$ be the barycentric function of $D(s_0, S)$. Then $0 \leq \lambda_0(x) \leq 1$ for all $x$ in $D(s_0, S)$ and $\lambda_0$ is a convex function on $D(s_0, S)$. If $S$ is a polyhedron then $\lambda_0$ is continuous on $D(s_0, S)$.

Proof. It is clear that $\lambda_0(x) \leq 1$ for all $x$ in $D(s_0, S)$; the proof that $\lambda_0(x) \geq 0$ for all $x$ in $D(s_0, S)$ depends on the convexity of $S$, and will be left to the reader. To prove that $\lambda_0$ is convex on $D(s_0, S)$ we show that if $x, x' \in D(s_0, S)$ and $0 < \alpha < 1$ then

$$\lambda_0(\alpha x + (1 - \alpha)x') \leq \alpha\lambda_0(x) + (1 - \alpha)\lambda_0(x').$$

Let $x^* = \alpha x + (1 - \alpha)x'$ and let $\beta = \alpha\lambda_0(x) + (1 - \alpha)\lambda_0(x')$; we may assume that $\beta < 1$ since otherwise (3) is trivial. Then if $\gamma = \alpha[1 - \lambda_0(x)](1 - \beta)^{-1}$, and $s, s'$ in $S$ are such that

$$x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]s, \quad x' = \lambda_0(x')s_0 + [1 - \lambda_0(x')]s', \quad (\text{cf. (2)})$$

we have

$$\gamma s + (1 - \gamma)s' = -\beta(1 - \beta)^{-1}s_0 + (1 - \beta)^{-1}x^*, \quad (4)$$

and $\gamma s + (1 - \gamma)s' \in S$ since $S$ is convex. It follows from (4) that $x^* \neq s_0$. 
If \( x^* \not\in S \) and \( \pi \) is the projection of \( C(s_0, S) \) on \( S \) then
\[
\pi(x^*) = -\lambda_0(x^*)[1 - \lambda_0(x^*)]^{-1}s_0 + [1 - \lambda_0(x^*)]^{-1}x^*,
\]
and hence by (4) and the definition of \( \pi, \lambda_0(x^*) \leq \beta \), as asserted by (3).
If \( x^* \in S \) then (3) is trivial. This completes the proof that \( \lambda_0 \) is convex on \( D(s_0, S) \).

We next show that \( \lambda_0 \) is continuous at \( s_0 \). Given \( \varepsilon \) with \( 0 < \varepsilon < 1 \), let \( \delta = M\varepsilon \), where \( M > 0 \) is the shortest distance from \( s_0 \) to \( S \). Then if \( x \in D(s_0, S) \) and \( 0 < |x - s_0| < \delta \) we have \( x \neq s_0, \ x \notin S \), and
\[
\lambda_0(x) = 1 - |x - s_0|/|\pi(x) - s_0|.
\]
It follows that \( \lambda_0(x) \leq 0 \) if \( x \in S \), and that \( \mu_0(x) = \lambda_0(x) > 0 \) if \( x \in D(s_0, S) \), \( x \neq s_0 \), and \( x \notin S \); thus
\[
\lambda_0(x) = \max \{\mu_0(x), 0\} \quad (x \in D(s_0, S), \ x \neq s_0).
\]
If \( S \) is a polyhedron then \( \mu_0 \) is continuous on \( C(s_0, S) \) by Theorem 1 and (5), and hence \( \lambda_0 \) is continuous on \( D(s_0, S) - \{s_0\} \) by (6). This completes the proof of the lemma.

**Theorem 2.** Let \( s_0, \ldots, s_m \) be points of \( R^n \), and let \( S = H(s_0, \ldots, s_m) \). Then there exist nonnegative real-valued continuous functions \( \lambda_0, \ldots, \lambda_m \) on \( S \), with \( \lambda_0 \) a convex function, such that, for each \( x \) in \( S \),
\[
x = \sum_{i=0}^{m} \lambda_i(x)s_i, \quad \text{and} \quad \sum_{i=0}^{m} \lambda_i(x) = 1.
\]

**Proof.** We use induction on \( m \). The case \( m = 0 \) is trivial. We assume the theorem to have been proved for \( m = M - 1 \) and deduce it for \( m = M \). Let \( T = H(s_0, \ldots, s_M) \). If \( s_0 \in T \) we may set \( \lambda_0(x) = 0 \) for all \( x \) in \( S \), and deduce the existence of \( \lambda_1, \ldots, \lambda_M \) directly from the induction hypothesis; we therefore assume that \( s_0 \notin T \). By the induction hypothesis there exist nonnegative real-valued continuous functions \( \mu_1, \ldots, \mu_M \) on \( T \) such that, for each \( y \) in \( T \),
\[
y = \sum_{i=1}^{M} \mu_i(y)s_i, \quad \text{and} \quad \sum_{i=1}^{M} \mu_i(y) = 1.
\]
Let \( \lambda_0 \) be the barycentric function of \( D(s_0, T) \). Then each \( x \) in \( S = D(s_0, T) \) has a representation of the form \( x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]s_x \) with \( s_x \) in \( T \) (cf. (2)), and if we now set \( \lambda_i(x) = \mu_i(s_x)[1 - \lambda_0(x)] \) \( (x \in S; \ i = 1, \ldots, M) \) then it follows that the \( \lambda_i \) \( (i = 1, \ldots, M) \) are well-defined.
functions on \( S \), and, by Lemma 4, that the functions \( \lambda_0, \ldots, \lambda_m \) satisfy all the conditions in the statement of the theorem.

To show that the functions \( \lambda_i \) defined in the proof of Theorem 2 need not all be convex we can let \( s_0, s_1, s_2, \) and \( s_3 \) be the points \((0, 2), (1, 0), (-1, 0), \) and \((0, 1)\) respectively of \( R^2 \) and let \( S = H(s_0, s_1, s_2, s_3) \); however in this example we obtain convex barycentric coordinates if we interchange the roles of \( s_0 \) and \( s_3 \). In the following example some of the barycentric coordinates determined as in the proof of Theorem 2 fail to be convex no matter how \( s_0 \) is chosen.

**Example 3.** Define \( t_0, \ldots, t_4 \) in \( R^3 \) as follows: \( t_0 = (0, 0, 1) \), \( t_1 = (0, 1, 0) \), \( t_2 = (0, -1, 0) \), \( t_3 = (1, 0, -1) \), and \( t_4 = (-1, 0, -1) \); let \( S = H(t_0, \ldots, t_4) \); and let barycentric coordinates be defined for \( S \) as in the proof of Theorem 2, with

(i) \( t_0 \),

(ii) \( t_1 \) or \( t_2 \), and

(iii) \( t_3 \) or \( t_4 \) playing the role of \( s_0 \). Then if we write \( \theta \pm \) for \( \max \left[ \pm \theta, 0 \right] \) (\( \theta \) real) we obtain

(i) \( (\xi, 0, 0) = |\xi| t_0 + \left( \frac{1}{2} - |\xi| \right)(t_1 + t_2) + \xi t_3 + \xi t_4 \quad (|\xi| \leq \frac{1}{2}) \),

(ii) \( (0, \eta, 0) = \frac{1}{2}(1 - |\eta|)t_0 + \eta t_1 + \eta t_2 + \frac{1}{4}(1 - |\eta|)(t_3 + t_4) \quad (|\eta| \leq 1) \), and

(iii) \( (0, 0, \zeta) = \zeta t_0 + \frac{1}{2}(1 - |\zeta|)(t_1 + t_2) + \frac{1}{2} \zeta(t_3 + t_4) \quad (|\zeta| \leq 1) \),

respectively, and hence in no case are the barycentric coordinates all convex.

The argument in the proof of Theorem 2 amounts to determining barycentric coordinates \( \lambda_0, \ldots, \lambda_m \) for \( H(s_0, \ldots, s_m) \) by first choosing \( \lambda_0 \) as small as possible, then choosing \( \lambda_1 \) as small as possible with this choice of \( \lambda_0 \), etc. We remark in conclusion that if we first choose \( \lambda_0 \) as large as possible, then choose \( \lambda_1 \) as large as possible with this choice of \( \lambda_0 \), etc., we do not in general obtain convex barycentric coordinates; this may be seen by considering the case of a square in \( R^2 \).

**References**

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