

Pacific Journal of Mathematics

**ON THE DIFFERENCE AND SUM OF BASIC SETS OF
POLYNOMIALS**

M. N. MIKHAIL AND M. NASSIF

ON THE DIFFERENCE AND SUM OF BASIC SETS OF POLYNOMIALS

N. N. MIKHAIL AND M. NASSIF

1. If the two functions $f(z)$ and $g(z)$ are connected by the relation

$$g(z + 1) - g(z) = f(z),$$

then $f(z)$ is called the difference of $g(z)$ and $g(z)$ is the sum of $f(z)$. These relations are denoted by

$$f(z) = \Delta g(z); \quad g(z) = \mathcal{S}f(z),$$

and it is obvious that any function of period unity can be added to the sum of a given function. The authors considered recently [1] the difference set $\{u_n(z)\}$ and the sum set $\{v_n(z)\}$ of a given simple set of polynomials¹ $\{p_n(z)\}$. These sets are the simple sets defined by

$$(1.1) \quad u_n(z) = \Delta p_{n+1}(z); \quad (n \geq 0),$$

$$(1.2) \quad v_0(z) = 1; \quad v_n(z) = \mathcal{S} p_{n-1}(z); \quad (n \geq 1),$$

and the indetermination in the sum set is removed by supposing that

$$(1.3) \quad v_n(0) = 0; \quad (n \geq 1).$$

The main result of the above mentioned work concerns the order δ and σ of the difference and sum sets respectively of a simple set of a given order ω . In fact, it has been shown that

$$(1.4) \quad \delta \leq \max(1, \omega),$$

and

$$(1.5) \quad \sigma \leq \omega + 1.$$

Our aim in the present paper is to generalise these results for more general classes of basic sets of polynomials. It will be here shown that, with suitable modification of the definition of difference sets, the upper bound in (1.4) remains the same for the most general classes of basic sets of polynomials. As for the sum sets, it will be here proved that, in order to get a finite upper bound for the order of the sum set, a limitation on the class of basic sets is inevitable.

2. This section and the following one are devoted to the study of

Received January 9, 1960, and in revised form May 23, 1960.

¹ The reader is supposed to be acquainted with the theory of basic sets of polynomials as given by Whittaker [3].

the difference set $\{u_n(z)\}$ of a general basic set of polynomials $\{p_n(z)\}$. As an introduction we first consider the difference set $\{\vartheta_n(z)\}$ of the unit set (z^n) . According to (1.1), this set is given by

$$(2.1) \quad \vartheta_n(z) = (z + 1)^{n+1} - z^{n+1} ; \quad (n \geq 0) .$$

It has been shown [1; formula (2.14)] that this set admits the representation

$$(2.2) \quad z^n = \sum_{k=0}^n \lambda_{n,k} \vartheta_k(z) ,$$

where²

$$(2.3) \quad |\lambda_{n,k}| < \frac{Kn!}{(k + 1)!(2\pi)^{n-k}} ; \quad (0 \leq k \leq n) .$$

Considering now the difference set $\{u_n(z)\}$ of a general basic set $\{p_n(z)\}$, the definition (1.1) has to be slightly modified in order to avoid a linear dependence among the polynomials of the resulting set. In fact, suppose that z^n admits the representation

$$(2.4) \quad z^n = \sum_k \pi_{n,k} p_k(z) ; \quad (n \geq 0) ,$$

and in particular,

$$(2.5) \quad 1 = \sum_k \pi_{0,k} p_k(z) .$$

A process of differencing, operating on this last relation, yields a linear relation between the involved differenced polynomials. For this reason a polynomial of the set $\{p_n(z)\}$ has to be eliminated. Suppose, in fact, that $\pi_{0,\mu}$ is the first non-zero coefficient in (2.5) then the difference set $\{u_n(z)\}$ is not defined by

$$(2.6) \quad \begin{cases} u_n(z) = \Delta p_n(z) & (0 \leq n \leq \mu - 1) \\ u_n(z) = \Delta p_{n+1}(z) & (n \geq \mu) . \end{cases}$$

Thus the polynomial $p_\mu(z)$ is eliminated. In case of simple sets $\mu = 0$ and the definition (2.6) reduces to that in (1.1).

We first show³ that the set of polynomials $\{u_n(z)\}$, as defined by (2.6) is basic. To this end we define the set $\{p_n^\dagger(z)\}$ by

$$(2.7) \quad \begin{aligned} p_n^\dagger(z) &= (1/z)\{p_n(z) - p_n(0)\} ; & (0 \leq n \leq \mu - 1) , \\ p_n^\dagger(z) &= (1/z)\{p_{n+1}(z) - p_{n+1}(0)\} ; & (n \geq \mu) . \end{aligned}$$

It can be easily verified from the definitions (2.1), (2.6) and (2.7) that the set $\{u_n(z)\}$ is the product set

² In our notation K denotes positive finite numbers independent of n that not necessarily of the same value at different occurrences.

³ The following discussion is due to Newns in his study on the derived sets [2; p. 465].

$$(2.8) \quad \{u_n(z)\} = \{p_n^\dagger(z)\}\{\vartheta_n(z)\} .$$

Hence, in order to prove that $\{u_n(z)\}$ is basic it is sufficient to show that the set $\{p_n^\dagger(z)\}$ is basic. In fact, rewriting (2.5) and using (2.7) we get

$$1 = \sum_k \pi_{0,k}(0) + z \left\{ \pi_{0,\mu} \mathcal{P}(z) + \sum_{k \geq \mu} \pi_{0,k+1} p_k^\dagger(z) \right\} ,$$

where $\mathcal{P}(z) = (1/z)\{p_\mu(z) - p_\mu(0)\}$. Since the first sum is equal to 1 it follows that

$$(2.9) \quad \mathcal{P}(z) = - \sum_{k \geq \mu} \frac{\pi_{0,k+1}}{\pi_{0,\mu}} p_k^\dagger(z) .$$

Inserting (2.7) and (2.9) in (2.4), written for z^{n+1} we obtain the unique representation

$$z^{n+1} = \sum_k \pi_{n+1,k}(0) + z \left\{ \sum_{k=0}^{\mu-1} \pi_{n+1,k} p_k^\dagger(z) + \sum_{k \geq \mu} \left(\pi_{n+1,k+1} - \pi_{n+1,\mu} \frac{\pi_{0,k+1}}{\pi_{0,\mu}} \right) p_k^\dagger(z) \right\} .$$

As the first sum is zero it follows that the set $\{p_n^\dagger(z)\}$ is a basic set that admits the unique representation

$$(2.10) \quad z^n = \sum_k \pi_{n,k}^\dagger p_k^\dagger(z) ,$$

where

$$(2.11) \quad \begin{cases} \pi_{n,k}^\dagger = \pi_{n+1,k} & (0 \leq k \leq \mu - 1) , \\ \pi_{n,k}^\dagger = \pi_{n+1,k+1} - \pi_{n+1,\mu} \frac{\pi_{0,k+1}}{\pi_{0,\mu}} & (k \geq \mu, n \geq 0) . \end{cases}$$

Hence the set $\{u_n(z)\}$ is the basic, as required.

3. We propose here to establish an upper bound for the order of the difference set of a given basic set of polynomials, and the following theorem shows that the bound given in (1.4) remains the same for general basic sets.

THEOREM 1. *Let $\{p_n(z)$ be a general basic set of polynomials of order ω ; then the difference set $\{u_n(z)\}$, defined by (2.6), will be of order δ , where*

$$(3.1) \quad \delta \leq \max(1, \omega) .$$

Proof. Let $(\gamma_{n,k})$ be the coefficients corresponding to those in (2.4) for the set $\{u_n(z)\}$. Hence, in view of (2.2), (2.8) and (2.10) it follows that

$$(3.2) \quad \gamma_{n,k} = \sum_{j=0}^n \lambda_{n,j} \tau_{j,k}^\dagger .$$

Adopting the usual notations for basic sets we write⁴

$$F_n(p; R) = \max_{i, j} \max_{|z|=R} \left| \sum_{k=1}^j \pi_{n, k} p_k(z) \right| ,$$

and $F_n(u; R)$ for the corresponding expression for the set $\{u_n(z)\}$. Since the set $\{p_n(z)\}$ is of order ω , then given any finite number $\omega_1 > \omega$ we shall have

$$(3.3) \quad F_n(p; R) < Kn^{\omega, n} ; \quad (n \geq 1) .$$

Choose the integers s_n and t_n for the set $\{u_n(z)\}$ such that

$$F_n(u; R) = \max_{|z|=R} \left| \sum_{k=s_n}^{t_n} \gamma_{n, k} u_k(z) \right| ,$$

hence (3.2) implies that

$$(3.4) \quad F_n(u; R) = \max_{|z|=R} \left| \sum_{j=0}^n \lambda_{n, j} f_j(z) \right| ,$$

where

$$f_j(z) = \sum_{k=s_n}^{t_n} \pi_{j, k}^\dagger u_k(z) .$$

Writing $g_j(z) = \sum_{k=s_n}^{t_n} \pi_{j, k}^\dagger p_k^\dagger(z)$, then it follows from (2.8) that, if

$$(3.5) \quad g_j(z) = \sum_k g_{j, k} z^k ,$$

then

$$(3.6) \quad f_j(z) = \sum_k g_{j, k} \vartheta_k(z) .$$

Applying the relations (2.7) and (2.11) in $g_j(z)$ we easily obtain

$$(3.7) \quad g_j(z) = (1/z) \left[\sum_{k=s_n}^{\mu-1} \pi_{j+1, k} \{p_k(z) - p_k(0)\} + \sum_{k=\mu+1}^{t_n+1} \left\{ \pi_{j+1, k} - \frac{\pi_{j+1, \mu} \pi_{0, k}}{\pi_{0, \mu}} \right\} \{p_k(z) - p_k(0)\} \right] .$$

Putting

$$L_j(R) = \max_{|z|=R} |f_j(z)| ; \quad \mathcal{M}_j(R) = \max_{|z|=R} |g_j(z)| ;$$

$$M(p_n; R) = \max_{|z|=R} |p_n(z)| ,$$

and observing that

⁴ Because of the variety of sets considered here the dependence of the entity on the particular set is explicitly written, whenever it is possible; thus we shall, for example, write $M(p_n; R) = \max_{|z|=R} |p_n(z)|$.

$$\left| \sum_{k=m}^n \pi_{j+1,k} p_k(0) \right| \leq \max_{|z|=R} \left| \sum_{k=m}^n \pi_{j+1,k} p_k(z) \right| \leq F_{j+1}(p; R); \quad (R > 0),$$

then, for any positive number ρ , (3.7) yields

$$\begin{aligned} (3.8) \quad \omega_j(\rho) &\leq (1/\rho) \left[4F_{j+1}(p; \rho) + \frac{|\pi_{j+1,\mu}| M(p_\mu; \rho)}{|\pi_{0,\mu}| M(p_\mu; \rho)} 2F_0(p; \rho) \right] \\ &\leq (2/\rho) F_{j+1}(p; \rho) \left[2 + \frac{F_0(p; \rho)}{|\pi_{0,\mu}| M(p_\mu; \rho)} \right] \\ &< KF_{j+1}(p; \rho). \end{aligned}$$

Suppose now that $\rho > R + 1$ and apply Cauchy's inequality for $g_j(z)$ as given in (3.5). Hence inserting (2.1) and (3.8) in (3.6) we obtain

$$\begin{aligned} (3.9) \quad L_j(R) &< KF_{j+1}(p; \rho) \sum_k \frac{M(\partial_k; R)}{\rho^k} \\ &< KF_{j+1}(p; \rho) \sum_k \frac{(R+1)^k}{\rho^k} < KF_{j+1}(p; \rho). \end{aligned}$$

Substitution from (2.3), (3.3) and (3.9) in (3.4) now gives

$$\begin{aligned} F_n(u; R) &< K \sum_{j=0}^n |\lambda_{n,j}| F_{j+1}(p; \rho) \\ &< \frac{Kn!}{(2\pi)^{n+1}} \sum_{j=1}^{n+1} \frac{(2\pi)^j j^{\omega_1 j}}{j!}, \end{aligned}$$

and since $j! > (j/e)^j$; ($j \geq 1$), the above inequality yields

$$(3.10) \quad F_n(u; R) < \frac{Kn!}{(2\pi)^{n+1}} \sum_{j=1}^{n+1} (2e\pi)^j j^{j(\omega_1-1)}.$$

Suppose that $\omega \geq 1$, then $\omega_1 > 1$ and (3.10) implies that

$$F_n(u; R) < K(n+1)! e^{n+1} (n+1)^{(n+1)(\omega_1-1)},$$

which ensures the order δ of the set $\{u_n(z)\}$ cannot exceed ω_1 ; and since ω_1 can be taken as near to ω as we please we deduce that $\delta \leq \omega$.

Suppose that $\omega < 1$; then by suitable choice we shall have $\omega_1 \leq 1$ and hence (3.10) gives

$$F_n(u; R) < K(n+1)! e^{n+1},$$

and thus $\delta \leq 1$. We thus conclude that $\delta \leq \max(1, \omega)$ and the proof of Theorem 1 is complete.

4. In the remaining sections the sum set $\{v_n(z)\}$ of a given set $\{p_n(z)\}$ is considered. As in the case of difference set we introduce by considering the sum set $\{\phi_n(z)\}$ of the unit set (z^n) .

This set, according to the definitions (1.2) and (1.3), is given by

$$(4.1) \quad \begin{cases} \phi_0(z) = 1; & \phi_n(z + 1) - \phi_n(z) = z^{n-1} \\ & \phi_n(0) = 0; \end{cases} \quad (n \geq 1).$$

It has been shown [1; formula (4.7)] that this set admits the representation

$$(4.2) \quad z^n = \sum_{k=1}^n \binom{n}{k-1} \phi_k(z),$$

and that it accords to the inequalities [1; formulae (2.5), (2.6)]

$$(4.3) \quad \frac{(n-1)!}{(2\pi)^n} < M(\phi; R) < (n-1)!e^R; \quad (n \geq 1; R > 1).$$

In case of sum sets the definitions (1.2) and (1.3) remain valid for general basic sets of polynomials. In fact, the sum set $\{v(z)\}$ of the set $\{p_n(z)\}$ is the basic set given by

$$(4.4) \quad v_0(z) = 1; \quad v_n(z) = \sum_k p_{n-1,k} \phi_{k+1}(z); \quad (n \geq 1),$$

where

$$(4.5) \quad p_n(z) = \sum_k p_{n,k} z^k,$$

and it admits the unique representation

$$(4.6) \quad z^n = \sum_k \varpi_{n,k} v_k(z),$$

where

$$(4.7) \quad \begin{cases} \varpi_{0,0} = 1, & \varpi_{0,n} = 0; (n > 0), & \varpi_{n,0} = 0, (n \geq 1), \\ \varpi_{n,k} = \sum_{j=1}^n \binom{n}{j-1} \pi_{j-1,k-1} & (k \geq 1, n \geq 1). \end{cases}$$

It should be noted that, if the class of the basic set $\{p_n(z)\}$ is not restricted, the order of the sum set $\{v_n(z)\}$ may be infinite, even if the order of the set $\{p_n(z)\}$ is zero. This fact is illustrated by the following example⁵, which also suggests that, in order to ensure a finite upper bound for the order of the sum set, the basic set $\{p_n(z)\}$ should accord to the restriction that $D_n = 0(n)$, where D_n is the degree of the polynomial of highest degree in the representation

$$z_n = \sum_k \pi_{n,k} p_k(z).$$

EXAMPLE. Let (ν_n) be an increasing sequence of even integers such that $\nu_n > 2n$ for all large n and $\lim_{n \rightarrow \infty} \nu_n / (n \log n) = 0$. Consider the

⁵ This generalised example was suggested by the referee of this paper, as a substitute of two, originally given, particular examples.

basic set $\{p_n(z)\}$ given by

$$p_{2n}(z) = z^{2n}$$

$$p_{2n+1}(z) = z^{2n+1} + \{(2n + 1)!\}^\omega z^{\nu_n} ,$$

where ω is any nonnegative number.

It is easily seen that this set is of order ω . Forming the sum set $\{v_n(z)\}$, (4.4) gives

$$(4.8) \quad \begin{aligned} v_0(z) &= 1 , \\ v_{2n+1}(z) &= \phi_{2n+1}(z) , \\ v_{2n+2}(z) &= \phi_{2n+2}(z) + \{(2n + 1)!\}^\omega \phi_{\nu_{n+1}}(z) . \end{aligned}$$

In view of (4.2), it is easily verified that

$$(4.9) \quad z^{2n} = \sum_{j=1}^{2n} \binom{2n}{j-1} v_j(z) - \sum_{j=0}^{n-1} \binom{2n}{2j+1} \{(2j+1)!\}^\omega v_{\nu_{j+1}}(z) .$$

Writing, in view of (4.6),

$$(4.10) \quad \omega_n(v; R) = \sum_k |\varpi_{n,k}| M(v_k; R) ,$$

and applying (4.3), equations (4.8) and (4.9) yield

$$\begin{aligned} \omega_{2n}(v; R) &> 2n\{(2n - 1)!\}^\omega M_{\nu_{n-1}+1}; R\} \\ &> \{(2n - 1)!\}^\omega (\nu_{n-1})! / (2\pi)^{\nu_{n-1}} . \end{aligned}$$

Hence, the order of the sum set $\{v_n(z)\}$ is

$$\begin{aligned} \sigma &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\log \omega_n(v; R)}{n \log n} \geq \limsup_{n \rightarrow \infty} \frac{\omega \log (2n - 1)! + \log(\nu_{n-1})!}{2n \log 2n} \\ &\geq \omega + \limsup_{n \rightarrow \infty} \nu_n / 2n = \omega + \limsup_{n \rightarrow \infty} D_n / n , \end{aligned}$$

since

$$D_{2n} = 2n ; \quad D_{2n+1} = \max(2n + 1, \nu_n) .$$

The possibilities $\limsup_{n \rightarrow \infty} D_n/n = 1, \alpha, \infty$, where $1 < \alpha < \infty$, can be covered by choosing $\nu_n = 2n + 2, 2[(n + \frac{1}{2})\alpha], 2(2n + 1)[\sqrt{\log(2n + 1)}]$ respectively, where $[x]$ has its usual meaning of ‘‘the integral part of x ’’.

5. The above example shows also that the result formulated in the following theorem is best possible.

THEOREM 2. *Let ω be the order of the basic sets $\{p_n(z)\}$ which satisfies the condition that*

$$(5.1) \quad \limsup_{n \rightarrow \infty} D_n/n = \alpha < \infty ,$$

then the sum set $\{v_n(z)\}$ will be of order $\sigma \leq \omega + \alpha$.

Proof. Since the set $\{p_n(z)\}$ is of order ω , then for any finite number $\omega_1 > \omega$ we have

$$(5.2) \quad \omega_n(p; R) < Kn^{\omega_1 n}, \quad (n \geq 1),$$

where $\omega_n(p; R)$ is the Cannon sum for the set $\{p_n(z)\}$, given by

$$\omega_n(p; R) = \sum_k |\pi_{n,k}| M(p_k; R).$$

Also, given any finite number $\alpha' > \alpha$, there exists, in view of (5.1), a positive integer n_0 such that

$$D_n < \alpha'n, \quad (n > n_0),$$

so that

$$(5.3) \quad (D_n)! < \Gamma(\alpha'n + 1) < K(\alpha'n)^{\alpha'n + \frac{1}{2}}, \quad (n > n_0).$$

Now, let d_n be the degree of the polynomial $p_n(z)$; then combining (4.3) and (4.4) we obtain

$$(5.4) \quad \begin{aligned} M(v_n; R) &\leq \sum_{k=0}^{d_{n-1}} |p_{n-1,k}| \{ M(\phi_{k+1}; R) \\ &< K(d_{n-1} + 1)! M(p_{n-1}; R); \quad (n \geq 1, R > 1). \end{aligned}$$

Applying (4.7) we get the familiar inequality

$$(5.5) \quad \omega_n(v; R) \leq \sum_{k=0}^{n-1} \binom{n}{k} \sum_j |\pi_{k,j}| M(v_{j+1}; R).$$

Inserting (5.2), (5.3) and (5.4) in (5.5) and mindful of the definition of the number D_n , it follows, for $n > n_0$, that

$$\begin{aligned} \omega_n(v; R) &< K \sum_{k=0}^{n-1} \binom{n}{k} (D_k + 1)! \omega_k(p; R) \\ &< K \left[1 + \sum_{k=1}^{n_0} \binom{n}{k} (D_k + 1)! k^{\omega_1 k} + \sum_{k=n_0+1}^n \binom{n}{k} (\alpha'k + 1)^{\alpha'k + (3/2)k} k^{k\omega_1} \right] \\ &< K(\alpha'n + 1)^{\alpha'n + (3/2)n} n^{\omega_1 n} \sum_{k=0}^n \binom{n}{k} = K2^n (\alpha'n + 1)^{\alpha'n + (3/2)n} n^{\omega_1 n}. \end{aligned}$$

This relation implies that the order σ of the sum set $\{v_n(z)\}$ does not exceed $\omega_1 + \alpha'$. Since ω_1 and α' can be arbitrarily chosen near to ω and α respectively, we conclude that $\sigma \leq \omega + \alpha$; and Theorem 2 is therefore established.

Finally, the authors wish to express their thanks to the referee of this paper for his helpful comments and constructive suggestions.

REFERENCES

1. N. N. Mikhail, and M. Nassif, *On the difference and sum of simple sets of polynomials*, Assiut University Bulletin of Science and Technology, (in press).
2. W. F. Newns, *On the representation of analytic functions by infinite series*, Phil. Trans. of the Roy. Soc. of London, Ser. A, **245** (1953), 429-468.
3. J. M. Whittaker, *Sur les Séries de Base de Polynomes Quelconques*, Gauthier—Villars, Paris (1949).

ASSIUT UNIVERSITY,
ASSIUT, EGYPT, U. A. R.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS

Stanford University
Stanford, California

F. H. BROWNELL

University of Washington
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California
Los Angeles 7, California

L. J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
T. M. CHERRY

D. DERRY
M. OHTSUKA

H. L. ROYDEN
E. SPANIER

E. G. STRAUS
F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 11, No. 3

BadMonth, 1961

Errett Albert Bishop, <i>A generalization of the Stone-Weierstrass theorem</i>	777
Hugh D. Brunk, <i>Best fit to a random variable by a random variable measurable with respect to a σ-lattice</i>	785
D. S. Carter, <i>Existence of a class of steady plane gravity flows</i>	803
Frank Sydney Cater, <i>On the theory of spatial invariants</i>	821
S. Chowla, Marguerite Elizabeth Dunton and Donald John Lewis, <i>Linear recurrences of order two</i>	833
Paul Civin and Bertram Yood, <i>The second conjugate space of a Banach algebra as an algebra</i>	847
William J. Coles, <i>Wirtinger-type integral inequalities</i>	871
Shaul Foguel, <i>Strongly continuous Markov processes</i>	879
David James Foulis, <i>Conditions for the modularity of an orthomodular lattice</i>	889
Jerzy Górski, <i>The Sochocki-Plemelj formula for the functions of two complex variables</i>	897
John Walker Gray, <i>Extensions of sheaves of associative algebras by non-trivial kernels</i>	909
Maurice Hanan, <i>Oscillation criteria for third-order linear differential equations</i>	919
Haim Hanani and Marian Reichaw-Reichbach, <i>Some characterizations of a class of unavoidable compact sets in the game of Banach and Mazur</i>	945
John Grover Harvey, III, <i>Complete holomorphs</i>	961
Joseph Hersch, <i>Physical interpretation and strengthening of M. Protter's method for vibrating nonhomogeneous membranes; its analogue for Schrödinger's equation</i>	971
James Grady Horne, Jr., <i>Real commutative semigroups on the plane</i>	981
Nai-Chao Hsu, <i>The group of automorphisms of the holomorph of a group</i>	999
F. Burton Jones, <i>The cyclic connectivity of plane continua</i>	1013
John Arnold Kalman, <i>Continuity and convexity of projections and barycentric coordinates in convex polyhedra</i>	1017
Samuel Karlin, Frank Proschan and Richard Eugene Barlow, <i>Moment inequalities of Pólya frequency functions</i>	1023
Tilla Weinstein, <i>Imbedding compact Riemann surfaces in 3-space</i>	1035
Azriel Lévy and Robert Lawson Vaught, <i>Principles of partial reflection in the set theories of Zermelo and Ackermann</i>	1045
Donald John Lewis, <i>Two classes of Diophantine equations</i>	1063
Daniel C. Lewis, <i>Reversible transformations</i>	1077
Gerald Otis Losey and Hans Schneider, <i>Group membership in rings and semigroups</i>	1089
M. N. Mikhail and M. Nassif, <i>On the difference and sum of basic sets of polynomials</i>	1099
Alex I. Rosenberg and Daniel Zelinsky, <i>Automorphisms of separable algebras</i>	1109
Robert Steinberg, <i>Automorphisms of classical Lie algebras</i>	1119
Ju-Kwei Wang, <i>Multipliers of commutative Banach algebras</i>	1131
Neal Zierler, <i>Axioms for non-relativistic quantum mechanics</i>	1151