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ALEX I. ROSENBERG AND DANIEL ZELINSKY

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1. Introduction. In this note we begin by noticing that for any commutative ring C, the isomorphism classes of finitely generated, projective C-modules of rank one (for the definition, see $\S 2$) form an abelian group $\mathcal{J}(C)$ which reduces to the ordinary ideal class group if C is a Dedekind domain. In [2], Auslander and Goldman proved that if $\mathcal{J}(C)$ contains only one element then every automorphism of every central separable C-algebra is inner. Using similar techniques, we prove that for general C and for any central separable C-algebra A, $\mathcal{J}(C)$ contains a subgroup isomorphic to the group of automorphisms of A modulo inner ones. We characterize both this subgroup and the factor group. For example, in the case of an integral domain or a noetherian ring, the subgroup is the set of classes of projective ideals in C which become principal in A (i.e., Ker β in Theorem 7). If C is a Dedekind ring and A is the (split) algebra of endomorphisms of a projective C-module of rank n, the subgroup is the set of classes of ideals whose nth powers are principal.

2. Generalization of the ideal class group. Let C be a commutative ring¹ and let J be a projective C-module. Then for every maximal ideal M in C, the module² $J \otimes C_M$ is a projective, hence free, C_M -module. Following [7, § 3] we say J has rank one if for all $M, J \otimes C_M$ is free on one generator,³ i.e. $J \otimes C_M \cong C_M$ as C_M -modules.

DEFINITION. $\mathcal{J}(C)$ will denote the set of isomorphism classes of finitely generated, projective, rank one *C*-modules. If J is a finitely generated, projective, rank one *C*-module, $\{J\}$ will denote the isomorphism class of J.

We note that if $\{J\} \in \mathscr{J}(C)$ then J is faithful, for if an ideal I annihilates J then $0 = I(J \otimes C_{\mathfrak{M}}) \cong IC_{\mathfrak{M}} \cong I \otimes C_{\mathfrak{M}}$ for every M, and so I = 0 [4, Chap. VII, Ex. 11].

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¹ All rings will be assumed to have units, all modules will be unitary, and if R is a subring of S then R will contain the unit element of S. A homomorphism of rings will preserve unit elements.

² The unadorned \otimes always means tensor product over C. $C_{\mathcal{M}}$ denotes the ring of quotients of C with respect to the maximal ideal M.

³ $J \otimes C_M \cong C_M$ for all M does not imply that J is either finitely generated or projective. For example, let C be the ring of integers and $J = \bigcup_n C p_1^{-1} \cdots p_n^{-1}$ where p_i is the *i*th prime.

LEMMA 1. $\mathcal{J}(C)$ is an abelian group under the operation \otimes . The identity element is C.

Proof. The only nontrivial item is the existence of inverses. If $\{J\} \in \mathcal{J}(C)$, let $J^* = \operatorname{Hom}_{\mathcal{O}}(J, C)$. Since Hom distributes over direct sums and since J is a direct summand in a finite direct sum of C's we see that J^* has the same property. Furthermore $J^* \otimes C_{\mathfrak{M}} \cong \operatorname{Hom}_{\mathcal{O}_{\mathfrak{M}}}(J \otimes C_{\mathfrak{M}}, C_{\mathfrak{M}}) \cong C_{\mathfrak{M}}$ [4, Chap. VI, Ex. 11] so that $J^* \in \mathcal{J}(C)$. Since J is faithful, the mapping $J \otimes J^* \to C$ defined by $x \otimes f \to f(x)$ is known to be an epimorphism [1, Prop. A. 3]. If its kernel is K then $K \otimes C_{\mathfrak{M}} = \operatorname{Ker}(J \otimes J^* \otimes C_{\mathfrak{M}} \to C \otimes C_{\mathfrak{M}}) = 0$ for each M. Thus K = 0, $J \otimes J^* \cong C$ and J^* is the inverse of J.

If C is semisimple (with minimum condition) then $\mathcal{J}(C) = 1$. Using [6, Lemma 3.14] it is easily seen that if N is a radical ideal in a ring C and $\mathcal{J}(C/N) = 1$ then $\mathcal{J}(C) = 1$; therefore $\mathcal{J}(C) = 1$ whenever C is semilocal (i.e. C has only finitely many maximal ideals, but is not necessarily noetherian). This fact also follows from Serre's theorem on the structure of projective modules over semilocal rings [7, Prop. 6 and 6, Lemma 3.15].

When C is an integral domain, $\mathcal{J}(C)$ is the ordinary group of (projective) ideal classes. We proceed to (prove and) generalize this statement by considering the functorial properties of $\mathcal{J}(C)$.

If C and D are commutative rings and $C \to D$ is a ring homomorphism, there is a corresponding homomorphism $\mathscr{J}(C) \to \mathscr{J}(D)$ given by $\{J\} \to \{J \otimes D\}$: Clearly $J \otimes D$ is a finitely generated projective D-module. To prove that it is of rank one, let N be any maximal ideal of D and let M be any maximal ideal of C containing the kernel of the composite homomorphism $C \to D \to D/N$. Since every element of C not in M maps into a unit of D_N , we have a homomorphism $C_M \to D_N$. Thus $(J \otimes D) \otimes_D D_N \cong$ $J \otimes D_N \cong (J \otimes C_M) \otimes_{\mathcal{O}_M} D_N \cong D_N$.

If S is a multiplicatively closed subset of C containing no zerodivisors we define an analog, $\mathscr{I}(C, S)$ of the ideal class group of an integral domain as follows:

Two ideals I and I' of C are equivalent if I' = uI for some unit u in the ring of quotients C_s . Then $\mathscr{I}(C, S)$ is the set of equivalence classes of projective ideals of C which meet S.⁴ Multiplication of ideals induces a product in $\mathscr{I}(C, S)$. Among other things, the following lemma shows that $\mathscr{I}(C, S)$ is a group.

Lemma 2. $\mathscr{I}(C, S) \cong \operatorname{Ker} (\mathscr{J}(C) \longrightarrow \mathscr{J}(C_s))$.

Proof. Let the class of I belong to $\mathscr{I}(C, S)$. Then some element ⁴ The same proof as in [4, Chap. VII, Prop. 3.3] shows that such an ideal is finitely

generated.

of I is not a zero divisor, and consequently $I \otimes C_M \neq 0$ for each M. Hence $I \otimes C_M$ is a nonzero projective (hence free) ideal of C_M and so $I \otimes C_M \cong C_M$. Moreover, if I' is in the same class as I then I and I' are isomorphic C-modules. Conversely, if δ is an isomorphism of I with I', then $\delta \otimes 1$ gives an isomorphism of $C_s = I \otimes C_s$ with $C_s = I' \otimes C_s$. Thus $\delta \otimes 1$ is simply multiplication by a unit u of C_s , and so I' = Iu. Finally, by [4, Chap. VI, Ex. 19] $I \otimes I' \cong II'$, and therefore mapping the class of I in $\mathcal{J}(C, S)$ to $\{I\}$ in $\mathcal{J}(C)$ yields a monomorphism of $\mathcal{J}(C)$.

As we already noted $I \otimes C_s = IC_s = C_s$ so that the image of I lies in Ker $(\mathcal{J}(C) \to \mathcal{J}(C_s))$. On the other hand, if $\{J\} \in \mathcal{J}(C)$ lies in this kernel, $J \otimes C_s \cong C_s$ and so J is isomorphic to a C-submodule of C_s . Since J is finitely generated, it is isomorphic to an ideal I of C and $I \otimes C_s = IC_s = C_s$. Hence $I \cap S \neq \phi$ and the class of I lies in $\mathcal{J}(C, S)$.

COROLLARY 3. If C is an integral domain or a noetherian ring and S is the complement of the set of zero divisors then $\mathcal{J}(C_s) = 1$, and hence $\mathcal{J}(C, S) \cong \mathcal{J}(C)$.

Proof. If C is an integral domain this is now clear. If C is noetherian, S is the complement of the union of the primes of zero. Since there are only finitely many of these, the standard theorems concerning the relation of ideals in C and C_s show that C_s is a semilocal ring and so by the remarks following Lemma 1, $\mathcal{J}(C_s) = 1$.

3. Separable algebras. If A is an algebra over the commutative ring C, A is said to be separable over C if the left A^{e} -module⁵ A is projective. Central separable C-algebras are a natural generalization of central simple algebras, and their basic theory has been given in [2] and [3]. In particular, we single out the following results which we use several times:

PROPOSITION 4. Let A be a central separable C-algebra and X a left A^{e} -module. Then $X \cong A \otimes Y$ as A^{e} -modules where the C-module $Y = \{x \in X \mid ax = xa \text{ for all } a \text{ in } A\} \cong \operatorname{Hom}_{A^{e}}(A, X)$. The C-module Y is unique: If $X \cong A \otimes Y'$ as A^{e} -modules then $Y' \cong Y$ as C-modules. The following three statements are equivalent:

- (a) X is a finitely generated projective C-module
- (b) Y is a finitely generated projective C-module
- (c) X is a finitely generated projective A^{e} -module.

Proof. The first assertion is [2, Theorem 3.1]. As for the uniqueness: $Y \cong \operatorname{Hom}_{A^e}(A, A \otimes Y') \cong \operatorname{Hom}_{A^e}(A, A) \otimes Y' \cong C \otimes Y' \cong Y'$ where

⁵ The algebra A^e is the tensor product over C of A and its opposite.

all the isomorphisms are C-isomorphisms; the second isomorphism follows from the statement " φ_3 is an isomorphism" on p. 210 of [4] if Y' is identified with $\operatorname{Hom}_{\sigma}(C, Y')$; the third isomorphism follows from $\operatorname{Hom}_{\mathfrak{s}}(A, A) \cong C$ which is the condition that A is central.

For the rest, we prove the implications $a \Rightarrow b \Rightarrow c \Rightarrow a$. Since A is a finitely generated projective A^e -module, the C-module $Y \cong \operatorname{Hom}_{A^e}(A, X)$ is a direct summand in a finite direct sum of copies of X. Thus if X is a finitely generated projective C-module, Y is also. If Y is finitely generated and C-projective then $X \cong A \otimes Y$ is an A^e -direct summand in a finite direct sum of copies of A's and thus a finitely generated projective C-module. Finally, since A is a finitely generated projective C-module [2, Theorem 2.1], A^e is also, and so if X is finitely generated and projective as an A^e -module it has the same properties as C-module.

As is usual in the study of simple algebras, for any central separable C-algebra A and a pair of C-algebra automorphisms σ, τ of A we make A into a new A^{e} -module, ${}_{\sigma}A_{\tau}$ by defining

$$(x \otimes y)(a) = \sigma(x)a\tau(y)$$
 for $x \otimes y \in A^e$, $a \in A$.

Of course as a C-module, ${}_{\sigma}A_{\tau} \cong A$ and so is finitely generated and projective.

Since ${}_{\sigma}A_{\tau}$ is A^{e} -isomorphic to ${}_{\rho\sigma}A_{\rho\tau}$ by the mapping $a \to \rho(a)$, we need only be concerned about ${}_{\sigma}A_{\tau}$ with $\tau = 1$. Proposition 4 shows that ${}_{\sigma}A_{\tau}$ is isomorphic to $A \otimes J_{\sigma}$ with $J_{\sigma} = \{a \in A \mid \sigma(x)a = ax \text{ for all } x \text{ in } A\}$, a finitely generated projective C-module. Moreover the chain of C_{M} -module isomorphisms

$$A\otimes C_{\mathtt{M}}\cong {}_{\sigma}A_{1}\otimes C_{\mathtt{M}}\cong (A\otimes C_{\mathtt{M}})\otimes {}_{\sigma_{\mathtt{M}}}(J_{\sigma}\otimes C_{\mathtt{M}}),$$

together with the fact that $A \otimes C_{\mathfrak{M}}$ is a finitely generated free $C_{\mathfrak{M}}$ -module, shows that $J_{\sigma} \otimes C_{\mathfrak{M}} \cong C_{\mathfrak{M}}$ and so $\{J_{\sigma}\} \in \mathscr{J}(C)$.

LEMMA 5. $J_{\sigma} \cong C$ if and only if σ is an inner automorphism. Also $J_{\sigma} \otimes J_{\tau} \cong J_{\tau\sigma}$.

Proof. The first part of the Lemma is essentially [2, Theorem 3.6] (cf. also [5, p. 143]): If $\sigma(x) = uxu^{-1}$ then $J_{\sigma} = \{a \in A \mid uxu^{-1}a = ax \text{ for} all x \text{ in } A\} = \{a \in A \mid x(u^{-1}a) = (u^{-1}a)x \text{ for all } x \text{ in } A\} = uC \cong C$. Conversely, suppose J_{σ} is a free C-module on one generator, u. Since the isomorphism $A \otimes J_{\sigma} \cong_{\sigma} A_1$ is defined by $a \otimes j \to ja$, we have $A = J_{\sigma} A = uA = \sigma(A)u = Au$. Thus u is a unit in A lying in J_{σ} . The definition of J_{σ} then shows that $\sigma(x) = uxu^{-1}$ for all x in A.

By Proposition 4 and the remark following the definition of ${}_{\sigma}A_{\tau}$ we have the following chain of A^{e} -isomorphisms:

$$A\otimes J_{\tau}\otimes J_{\sigma}\cong ({}_{\sigma}A_{1})\otimes {}_{{}_{A}}({}_{\tau}A_{1})\cong ({}_{\tau}A_{1})\otimes {}_{{}_{A}}({}_{1}A_{\tau}{}^{-1})\cong {}_{\sigma}A_{\tau}{}^{-1}\cong {}_{\tau\sigma}A_{1}\cong A\otimes J_{\tau\sigma}\;.$$

The uniqueness statement in Proposition 4 then asserts $J_{\sigma} \otimes J_{\tau} \cong J_{\tau\sigma}$.

DEFINITION. If A is a central separable C-algebra, $\mathcal{O}(A)$ denotes the group of automorphisms of A modulo inner ones.

By Lemma 5 the mapping $\sigma \to J_{\sigma}$ induces a group monomorphism $\alpha: \mathscr{O}(A) \to \mathscr{J}(C).$

COROLLARY 6. $\mathcal{O}(A)$ is an abelian group.

We next obtain a description of $Im \alpha$.

DEFINITION. $\mathcal{J}(A)$ is the set of left A-isomorphism classes of left A^{e} -modules P with the properties

- (i) P is C-projective and finitely generated
- (ii) $P \otimes C_{\mathcal{M}} \cong A \otimes C_{\mathcal{M}}$ as $C_{\mathcal{M}}$ -modules for all M.

For the same reasons that ${}_{\sigma}A_1 \cong A \otimes J_{\sigma}$ with $\{J_{\sigma}\} \in \mathscr{J}(C)$, we have that if $\{P\} \in \mathscr{J}(A)$ then $P \cong A \otimes J$ as A^{e} -modules with $J \in \mathscr{J}(C)$. Since, conversely, $\{J \otimes A\} \in \mathscr{J}(A)$ whenever $\{J\} \in \mathscr{J}(C)$, we see that $\mathscr{J}(A)$ is just the set of left A-isomorphism classes of A^{e} -modules $A \otimes J$ with $\{J\} \in \mathscr{J}(C)$.

Note that if we had defined $\mathcal{J}(A)$ to be the set of A^e -isomorphism classes instead of left A-isomorphism classes, we would have had a set in one-to-one correspondence with $\mathcal{J}(C)$, according to Proposition 4. See also the remark after Theorem 7.

There is a natural multiplication in $\mathcal{J}(A)$: $(P_1, P_2) \to P_1 \otimes {}_{A}P_2$. If $P_1 = A \otimes J_1$ and $P_2 = A \otimes J_2$ then $P_1 \otimes {}_{A}P_2 \cong A \otimes (J_1 \otimes J_2)$. Thus the mapping β : $\mathcal{J}(C) \to \mathcal{J}(A)$ defined by $\{J\} \to \{A \otimes J\}$ is an epimorphism and so $\mathcal{J}(A)$ is a group.

THEOREM 7. The sequence

$$1 \longrightarrow \mathcal{O}(A) \stackrel{\alpha}{\longrightarrow} \mathcal{J}(C) \stackrel{\beta}{\longrightarrow} \mathcal{J}(A) \longrightarrow 1$$

is exact.

Proof. The only thing that still needs to be shown is that $Im \alpha = \text{Ker }\beta$. If $\{J\} \in Im \alpha$ then $J = J_{\sigma}$ and $\beta\{J\} = \{A \otimes J_{\sigma}\} = \{{}_{\sigma}A_{1}\} = \{{}_{1}A_{\sigma^{-1}}\} = \{A\}$ which is the unit element of $\mathcal{J}(A)$. Thus $Im \alpha \subset \text{Ker }\beta$. Conversely, if $\{J\} \in \text{Ker }\beta$ then $P = A \otimes J$ is left A-isomorphic to A. That is, each element of P is of the form p = aw for some fixed w in P and for suitable a in A, uniquely determined by p. Since P is an A^e-module, $wa \in P$ for every a in A. Hence $wa = \sigma(a)w$ where σ is a well defined mapping of A to A. It is trivial to verify that σ is a C-algebra endomorphism. Now by [2, Theorem 3.5] σ is an automorphism and so

 $a \to \sigma(a)w$ is an A^{e} -isomorphism of ${}_{\sigma^{-1}}A_1$ to P. Thus $P \cong A \otimes J_{\sigma^{-1}}$. By the uniqueness in Proposition 4, $J \cong J_{\sigma^{-1}}$ and $\{J\} \in Im \alpha$.

REMARK. The proof of Theorem 7 shows that $A \otimes J \cong A \otimes J'$ as left A-modules if and only if $J \cong J' \otimes J''$ as C-modules with $\{J''\} \in Im \alpha$. Clearly the same proof will show $A \otimes J \cong A \otimes J'$ as right A-modules if and only if $J \cong J' \otimes J''$ with $\{J''\} \in Im \alpha$. Thus, given two A^e -modules P and P', satisfying conditions (i) and (ii) in the definition of $\mathcal{J}(A)$, the following conditions are equivalent: $P \cong P'$ as left A-modules; $P \cong P'$ as right A-modules; $P \cong P$ both as left and as right A-modules. This means that $\mathcal{J}(A)$ could equally well have been defined as the set of right A-isomorphism classes or as the set of left and right A-isomorphism classes (but not as the set of A^e -isomorphism classes).

In the theory of separable algebras the role of full matrix algebras over fields is played by the *split* algebras, i.e. algebras of the form $\operatorname{Hom}_{\sigma}(V, V)$ with V a finitely generated, faithful, projective C-module. For such algebras we give a fuller description of $\mathcal{J}(A)$.

DEFINITION. For any finitely generated, faithful, projective C-module V, let $\mathcal{J}(V)$ be the set of C-isomorphism classes of finitely generated projective C-modules W such that $\operatorname{Hom}_{\sigma}(W, W)$ is a C-algebra isomorphic to $\operatorname{Hom}_{\sigma}(V, V)$.

LEMMA 9. $W \in \mathcal{J}(V)$ if and only if $W \cong V \otimes J$ as C-modules, for some C-module J with $\{J\} \in \mathcal{J}(C)$.

Proof. Let $\{J\} \in \mathcal{J}(C)$. By [4, p. 210] there is a natural isomorphism as C-modules, and so also as C-algebras, $\operatorname{Hom}_{\mathcal{O}}(V \otimes J, V \otimes J) \cong$ Hom_{\mathcal{O}} $(V, V) \otimes$ Hom_{\mathcal{O}} (J, J). But Hom_{\mathcal{O}} (J, J) = C since C is embedded in Hom_{\mathcal{O}} (J, J) by its action on the module J, and since, for every M, $C_{M} \cong \operatorname{Hom}_{\mathcal{O}_{M}}(J \otimes C_{M}, J \otimes C_{M}) \cong \operatorname{Hom}_{\mathcal{O}}(J, J) \otimes C_{M}$ [4, Chap. VI, Ex. 11]. Thus $\{V \otimes J\} \in \mathcal{J}(V)$ for $\{J\} \in \mathcal{J}(C)$. Conversely, if $\{W\} \in \mathcal{J}(V)$ then W is a module over $A = \operatorname{Hom}_{\mathcal{O}}(V, V)$. By [1, Prop. A. 3 and Prop. A. 6] there is a C-isomorphism $W \cong V \otimes J$ with $J = \operatorname{Hom}_{\mathcal{A}}(V, W)$. Since V is projective and finitely generated as A-module, $J = \operatorname{Hom}_{\mathcal{A}}(V, W)$ is projective and finitely generated as a C-module. Since both $\operatorname{Hom}_{\mathcal{O}_{M}}(W \otimes C_{M}, W \otimes C_{M}) \cong \operatorname{Hom}_{\mathcal{O}}(W, W) \otimes C_{M}$ and $\operatorname{Hom}_{\mathcal{O}_{M}}(V \otimes C_{M}, V \otimes C_{M}) \cong$ Hom_{\mathcal{O}} $(V, V) \otimes C_{M}$ are matrix rings of the same size over $C_{M}, V \otimes C_{M}$ and $W \otimes C_{M} \cong W \otimes C_{M} \cong (V \otimes C_{M}) \otimes_{\mathcal{O}_{M}}(J \otimes C_{M})$, which forces $J \otimes C_{M} \cong C_{M} \cong C_{M}$, and so $J \in \mathcal{J}(C)$.

By Lemma 9 we can define a multiplication in $\mathcal{J}(V)$ by $(V \otimes J_1, V \otimes J_2) \rightarrow V \otimes J_1 \otimes J_2$. Then $\beta': \mathcal{J}(C) \rightarrow \mathcal{J}(V)$, given by $J \rightarrow V \otimes J$, is an epimorphism, and so $\mathcal{J}(V)$ is an abelian group with unit V.

LEMMA 10. Ker $\beta = \text{Ker } \beta'$. Hence if $A = \text{Hom}_{\sigma}(V, V)$, the mapping $\gamma: \mathcal{J}(A) \to \mathcal{J}(V)$ given by $A \otimes J \to V \otimes J$ is an isomorphism making a commutative diagram:



Proof. If $\{J\} \in \text{Ker } \beta'$, the C-modules $V \otimes J$ and V are isomorphic. Then clearly the left A-modules $A = \text{Hom}_{\sigma}(V, V)$ and $\text{Hom}_{\sigma}(V, V \otimes J)$ are isomorphic. However the latter module is isomorphic to

 $\operatorname{Hom}_{\sigma}(V,V)\otimes\operatorname{Hom}_{\sigma}(C,J)\cong A\otimes J,$

and so $\{J\} \in \operatorname{Ker} \beta$.

Inversely, if $\{J\} \in \operatorname{Ker} \beta$, then by Theorem 7, $J = J_{\sigma}$ for some automorphism σ of A. We prove $V \otimes J_{\sigma} \cong V$ by localizing. We first compute $J_{\sigma} \otimes C_{\mathfrak{M}}$. From the definition of $J_{\sigma}, {}_{\sigma}A_1 \cong A \otimes J_{\sigma}$ and so $A \otimes J_{\sigma} \otimes C_{\mathfrak{M}} \cong {}_{\sigma}A_1 \otimes C_{\mathfrak{M}} = {}_{\sigma\otimes 1}(A \otimes C_{\mathfrak{M}})_{1\otimes 1}$. By the uniqueness part of Proposition 4, $J_{\sigma} \otimes C_{\mathfrak{M}} \cong J_{\sigma\otimes 1}$. Furthermore, since $C_{\mathfrak{M}}$ is local, $\mathscr{J}(C_{\mathfrak{M}}) = 1$ and thus by Theorem 7, $\sigma \otimes 1$ is an inner automorphism of $A \otimes C_{\mathfrak{M}}$. The last part of the proof of Lemma 5 then shows that $J_{\sigma} \otimes C_{\mathfrak{M}} = C_{\mathfrak{M}}u$ with u a unit in $A \otimes C_{\mathfrak{M}}$.

Next, since $J_{\sigma} \subset \operatorname{Hom}_{\sigma}(V, V) = A$, there is a C-module homomorphism $\theta: V \otimes J_{\sigma} \to V$ defined by $\theta(v \otimes j) = vj$. Then $\theta \otimes 1$ maps $V \otimes J_{\sigma} \otimes C_{M}$ to $V \otimes C_{M}$ and, in fact, if we write $J_{\sigma} \otimes C_{M} = C_{M}u$, $(\theta \otimes 1)(v \otimes cu) =$ $(v \otimes c)u$ for v in V, c in C_{M} ; here $(v \otimes c)u$ is defined because $u \in A \otimes C_{M}$ and $V \otimes C_{M}$ is an $(A \otimes C_{M})$ -module. Since u is a unit in $A \otimes C_{M}, \theta \otimes 1$ is an isomorphism. Hence if U and V are the kernel and cokernel of θ respectively $U \otimes C_{M} = V \otimes C_{M} = 0$ for all M. This proves that U =V = 0, and θ is an isomorphism. Hence $\{J\} \in \operatorname{Ker} \beta'$.

THEOREM 11. If $A = \text{Hom}_{\sigma}(V, V)$ with V a faithful, finitely generated, projective C-module, the sequence

$$1 \longrightarrow \mathcal{O}(A) \xrightarrow{\alpha} \mathcal{J}(C) \xrightarrow{\beta'} \mathcal{J}(V) \longrightarrow 1$$

is exact.

COROLLARY 12. If $\mathcal{J}(C) = 1$ then not only is every automorphism of every central separable C-algebra inner (i.e. O(A) = 1 for all A), but also, for every split C-algebra Hom_o(V, V), the module V is uniquely determined (i.e. $\mathcal{J}(V) = 1$ for every V, and, in fact $\mathcal{J}(A) = 1$ for every A). Conversely, if for some central separable C-algebra A [resp.

⁶ We consider V a right A-module, so that $Hom_{\mathcal{O}}(V, X)$ becomes a left A-module.

split C-algebra $A = \operatorname{Hom}_{\sigma}(V, V)$] both $\mathcal{O}(A)$ and $\mathcal{J}(A)$ are trivial [resp., $\mathcal{O}(A) = \mathcal{J}(V) = 1$] then $\mathcal{J}(C) = 1$ and so $\mathcal{O}(A) = \mathcal{J}(A) = 1$ for every A.

If we change the base ring C the exact sequences in Theorems 7 and 11 behave in the expected way: Specifically, if $C \rightarrow D$ is a ring homomorphism and if A is a central separable C-algebra then $A \otimes_{\sigma} D$ is a central separable D-algebra [2, Corollary 1.6] and Theorem 7 yields the exact sequence

(2)
$$1 \longrightarrow \mathcal{O}(A \otimes D) \xrightarrow{\alpha_D} \mathcal{J}(D) \xrightarrow{\beta_D} \mathcal{J}(A \otimes D) \longrightarrow 1$$
.

LEMMA 13. The homomorphism $C \rightarrow D$ gives rise to a homomorphism of complexes (1) \rightarrow (2).

Proof. The mapping $\mathcal{O}(A) \to \mathcal{O}(A \otimes D)$ is of course given by sending each automorphism class $[\sigma]$ in $\mathcal{O}(A)$ to $[\sigma \otimes 1]$ in $\mathcal{O}(A \otimes D)$. That the mapping $\{J\} \to \{J \otimes D\}$ yields a homomorphism $\mathcal{J}(C) \to \mathcal{J}(D)$ was already proved in § 2, and a similar argument shows that for $\{P\} \in \mathcal{J}(A)$ the mapping $P \to P \otimes D$ yields a homomorphism $\mathcal{J}(A) \to \mathcal{J}(A \otimes D)$. The desired commutativity properties of these maps with α, β, α_p and β_p are easily verified.

We remark that if $\mathcal{O}(A \otimes D) = 1$, then by Lemma 13, $\alpha \mathcal{O}(A) \subset \text{Ker}(\mathcal{J}(C) \to \mathcal{J}(D))$. This combined with Lemma 2, Corollary 3, Theorems 7 and 11 and Lemma 13 yields

THEOREM 14. Let C be an integral domain or a noetherian ring, S the complement of the set of zero-divisors (or more generally suppose C is any commutative ring, S a multiplicatively closed subset containing no zero divisors such that $\mathcal{O}(A \otimes C_s) = 1$). Then $\mathcal{O}(A)$ is isomorphic to the subgroup of the ideal class group, $\mathscr{I}(C, S)$, consisting of ideal classes [I] such that $IA \cong A$ as left A-module.

If besides $A \cong \operatorname{Hom}_o(V, V)$ is the algebra of endomorphisms of a finitely generated, faithful, projective C-module V, then $\mathcal{O}(A)$ is also isomorphic to the subgroup of $\mathscr{I}(C, S)$ consisting of those ideal classes [I] with $IV \cong V$ as C-modules.

THEOREM 15. Let C be a Dedekind ring, $A = \text{Hom}_{\sigma}(V, V)$ with V a finitely generated, projective module of rank n. Then $\mathcal{O}(A)$ is isomorphic to the subgroup of the ideal class group of C consisting of the ideal classes whose orders divide n.

Proof. By classical results, [8], V is isomorphic to a direct sum $I_1 \oplus \cdots \oplus I_n$ of ideals with n and the class of $I_1 I_2 \cdots I_n$ uniquely determining the C-isomorphism class of V. Thus $IV \cong V$ if and only if

 $I^n \prod I_i \cong \prod I_i$. Since $\prod I_i$ is an invertible ideal, $IV \cong V$ if and only if $I^n \cong C$, i.e. I^n is principal. Theorem 14 completes the proof.

REMARKS. (1) If C is any integral domain and V is a free Cmodule on n generators, the same proof shows that if $\{I\} \in Im \ \alpha$ then I^n is principal.

(2) In case V is free so that A is a matrix algebra over C, Theorem 15 was also proved by Kaplansky (unpublished).

(3) If C is the ring of integers of an algebraic number field, $\mathscr{F}(C, S)$ is well known to be a finite group. If $\mathscr{F}(C, S) \neq 1$, if n is an integer prime to the order of $\mathscr{F}(C, S)$, and if A is the algebra of $n \times n$ matrices over C, we obtain an example with $\mathscr{O}(A) = 1$ but $\mathscr{F}(C) \cong \mathscr{F}(A) \cong \mathscr{F}(V) \neq 1$. It is an open question whether $\mathscr{O}(A) = 1$ for every A implies $\mathscr{F}(C) = 1$.

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