

# Pacific Journal of Mathematics

**ON THE FIELD OF RATIONAL FUNCTIONS OF ALGEBRAIC  
GROUPS**

A. BIAŁYNICKI-BIRULA

# ON THE FIELD OF RATIONAL FUNCTIONS OF ALGEBRAIC GROUPS

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**O. Introduction.** Let  $K$  be an algebraically closed field of characteristic 0, let  $k$  be a subfield of  $K$  and suppose that  $G$  is a  $(k, K)$  algebraic group, i.e., an algebraic group defined over  $k$  and composed of  $K$ -rational points. Let  $k(G)$  denote the fields of  $k$ -rational functions on  $G$ .  $G_k$  denotes the subgroup of  $G$  composed of all  $k$ -rational points of  $G$ . If  $g \in G_k$  then the regular mapping  $L_g(R_g)$  of  $G$  onto  $G$  defined by  $L_g x = gx$  ( $R_g x = xg$ ) induces an automorphism of  $k(G)$  denoted by  $g_i(g_r)$ . Let  $D_k$  denote the Lie algebra of all  $k$ -derivations of  $k(G)$  (i.e., of all derivations of  $k(G)$  that are trivial on  $k$ ) which commute with  $g_r$ , for every  $g \in G_k$ .

For any subset  $A$  of  $k(G)$  let  $G(A)$  denote the subgroup of  $G$  composed of all elements  $g$  such that  $g_r(f) = f$ , for every  $f \in A$ . In the sequel we shall always assume that  $G_k$  is dense in  $G$ .

The main result of this paper is the following theorem:

**THEOREM 1.** *Let  $F$  be a subfield of  $k(G)$  containing  $k$ . Then the following three conditions are equivalent:*

- (1)  $F$  is  $(G_k)_i$  - stable
- (2)  $F$  is  $D_k$  - stable
- (3)  $F = k(G/G(F))$  and so  $F$  coincides with the field of all elements of  $k(G)$  that are fixed under  $G(F)_r$ .

By means of the theorem, we establish a Galois correspondence between a family of subgroups of  $G$  and the family of  $(G_k)_i$ -stable subalgebras of the algebra of representative functions of  $G$ .

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1. Let  $K$  be an algebraically closed field of characteristic 0, let  $k$  be a subfield of  $K$  and suppose that  $V, W$  are  $(k, K)$  - algebraic varieties. Let  $k(V), k(W)$  denote the fields of  $k$ -rational functions on  $V$  and  $W$ , respectively. If  $A$  is a subset of  $k(V)$  then  $k(A)$  denotes the fields generated by  $k$  and  $A$ .

The following result is known<sup>1</sup>:

(1) Let  $F$  be a rational mapping of  $V$  onto a dense subset of  $W$  and let  $\varphi$  be the cohomomorphism corresponding to  $F$ . Then there exists

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<sup>1</sup> See e.g. [2],

an open subset  $W_1 \subset W$  such that  $F^{-1}(x)$  contains exactly  $[k(V):\varphi(k(W))]$  elements, for every  $x \in W_1$ .

**LEMMA 1.** *Let  $A$  be a subset of  $k(V)$  and suppose that there exists a dense set  $V_1 \subset V$  and an open subset  $V_2 \subset V$  such that for any two distinct points  $x_1, x_2$ , where  $x_1 \in V_1, x_2 \in V_2$ , there exists a function  $f \in A$  which is defined at  $x_1, x_2$  and  $f(x_1) \neq f(x_2)$ . Then  $k(A) = k(V)$ .*

*Proof.* Let  $B$  be a finite subset of  $A$ , say  $B = \{f_1, \dots, f_n\}$ . Then  $F_B$  denotes the rational mapping  $F_B: V \rightarrow K^n$  defined by  $F_B(x) = (f_1(x), \dots, f_n(x))$  and  $W_B = (F_B(V))^- \subset K^n$ . Let  $\Delta(W_B)$  be the diagonal of  $W_B \times W_B$  and  $V_B = ((F_B \times F_B)^{-1}\Delta(W_B))^- \subset V \times V$ . Then there exists a finite subset  $B_0 \subset A$  such that  $V_{B_0} \subset V_B$ , for every finite subset  $B \subset A$  (since  $V \times V$  satisfies the minimal condition for closed sets). Let  $V_0$  be an open subset of  $V$  such that  $F_{B_0}$  is regular on  $V_0$ . We may assume that  $V_0 = V_2 = V$ , since we may replace  $V$  by  $V_0 \cap V_2$ . If  $x_1 \in V_1, x_2 \in V$  and  $x_1 \neq x_2$  then there exists  $f \in A$  such that  $f$  is defined at  $x_1, x_2$  and  $f(x_1) \neq f(x_2)$ . Hence  $(x_1, x_2) \notin V_{\{f\}}$  and so  $(x_1, x_2) \notin V_{B_0}$ . Thus  $F_{B_0}(x_1) \neq F_{B_0}(x_2)$ . Therefore, for every  $x \in F_{B_0}(V_1)$ ,  $F_{B_0}^{-1}(x)$  contains exactly one element. But  $F_{B_0}(V_1)$  is dense in  $W_{B_0}$ . Hence it follows from (i) that  $[k(V):k(B_0)] = 1$ , i.e.,  $k(V) = k(B_0)$ . Thus  $k(V) = k(A)$ .

Let  $G$  be a  $(k, K)$ -algebraic group. Suppose that  $G_k$  is dense in  $G$ . Let  $D$  be the Lie algebra of all derivations of  $K(G)$  commuting with  $g_r$ , for every  $g \in G$ , and let  $D_k$  denote the Lie algebra consisting of all derivations from  $D$  that map  $k(G)$  into  $k(G)$ . Let  $k[D]$  ( $K[D]$ ) denote the  $k$ -algebra ( $K$ -algebra) of transformations generated by the identity map and  $D_k(D)$ .

If  $d \in D_k$  then  $d$  restricted to  $k(G)$  is a  $k$ -derivation commuting with  $g_r$ , for every  $g \in G_k$ . On the other hand if  $d_1$  is a  $k$ -derivation of  $k(G)$  commuting with  $g_r$ , for every  $g \in G_k$ , then there exists a unique extension  $d$  of  $d_1$  to a  $K$ -derivation of  $K(G)$ , and the extension belongs to  $D_k$ . Hence we may identify  $D_k$  and the Lie algebra of all  $k$ -derivations of  $k(G)$  that commute with  $g_r$ , for every  $g \in G_k$ .

(ii)<sup>2</sup> If  $f \in K(G)$  and  $f$  is defined at a point  $g \in G$  then  $df$  is defined at  $g$ , for any  $d \in K[D]$ .

**LEMMA 2.** *Let  $f \in K(G)$  and suppose that  $f$  is defined at  $g \in G_k$ . If  $f \neq 0$  then there exists  $d \in k[D]$  such that  $(df)(g) \neq 0$ .*

*Proof.* Suppose that  $f \neq 0$ . If  $f(g) \neq 0$  then the identity element of  $k[D]$  satisfies the desired condition. Hence we may assume that  $f(g) = 0$ . Let  $\mathcal{O}_k(\mathcal{O}_K)$  denote the local ring of  $g$  in  $k(G)$  ( $K(G)$ ) and let  $m_k(m_K)$  be the maximal ideal of  $\mathcal{O}_k(\mathcal{O}_K)$ . Then  $f \in m_K$ . Let

<sup>2</sup> See [4] p.51.

$x_1, \dots, x_m$  be elements of  $m_k$  such that  $x_1 + m_k^2, \dots, x_m + m_k^2$  is a  $k$ -basis of  $m_k/m_k^2$ . The  $x_1 + m_k^2, \dots, x_m + m_k^2$  is a  $K$ -basis of  $m_K/m_K^2$ . Hence every mapping  $(x_1, \dots, x_m) \rightarrow k$  can be extended to a derivation  $\partial: \mathcal{O}_K \rightarrow K$ . On the other hand  $f \neq 0$  and so there exists an integer  $t$  such the  $f \in m_K^t - m_K^{t+1}$ . Hence  $f = \sum_{i_1+\dots+i_m=t} a_{i_1, \dots, i_m} x_1^{i_1}, \dots, x_m^{i_m} + f_1$ , where  $f_1 \in m_K^{t+1}$ ,  $a_{i_1, \dots, i_m} \in K$  and at least one  $a_{i_1, \dots, i_m}$  is different from zero. Let  $\partial_i$  be the derivation of  $\mathcal{O}_K$  into  $K$  such than  $\partial_i x_j = \delta_{ij}$ , where  $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$  It is known<sup>3</sup>, that there exist  $d_i \in D_k$  such that  $(d_i f)(g) = \partial_i f$  for every  $f \in \mathcal{O}_K$ . Then  $(d_1^{i_1} \dots d_m^{i_m})f(g) = i_1! \dots i_m! a_{i_1, \dots, i_m} \neq 0$  if  $a_{i_1, \dots, i_m} \neq 0$ . Hence the lemma is proved.

If  $A$  is a subset of  $k(G)$  then  $G(A)$  denotes the subgroup of  $G$  composed of all elements  $g$  such that  $g_r$  leaves the elements of  $A$  fixed. For any  $A \subset k(G)$ ,  $G(A)$  is a  $k$ -closed subgroup of  $G$ .

(iii)<sup>4</sup> Let  $G_1$  be a  $k$ -closed subgroup of  $G$ . Then  $G/G_1$  is defined over  $k$ . Let  $\varphi$  be the cohomomorphism of  $k(G/G_1)$  into  $k(G)$  corresponding to the canonical mapping  $G \rightarrow G/G_1$ . Then  $\varphi(k(G/G_1))$  coincides with the subfield of all elements of  $k(G)$  which are fixed under  $g_r$ , for every  $g \in G_1$ . In the sequel we shall identify  $k(G/G_1)$  and  $\varphi(k(G/G_1))$ .

*Proof of the theorem.*

Implications (3)  $\Rightarrow$  (1) and (3)  $\Rightarrow$  (2) are obvious.

(1)  $\Rightarrow$  (3)<sup>5</sup>. Let  $g_1 \in G_k, g_2 \in G$  and  $G(F)g_1 \neq G(F)g_2$ . Then  $g_2g_1^{-1} \notin G(F)$ . Hence there exists  $f_0 \in F$  such that  $(g_2g_1^{-1})_r f_0 \neq f_0$ . Therefore there exists an element  $g \in G_k$  such that  $(g_2g_1^{-1})_r f_0$  and  $f_0$  are defined at  $g$  and  $(g_2g_1^{-1})_r f_0(g) \neq f_0(g)$ , i.e.,  $f_0(g_2g_1^{-1}g) \neq f_0(g)$ ,  $(g_1^{-1}g)_i f_0(g_2) \neq (g_1^{-1}g)_i f_0(g)$ . Let  $f = (g_1^{-1}g)_i f_0$ . Then  $f \in F$  since  $g_1^{-1}g \in G_k$ ;  $f$  is defined at  $g_1$  and  $g_2$ , and  $f(g_1) \neq f(g_2)$ . Thus it follows from Lemma 1 that  $F = k(G/G(F))$ , because  $G(F) \cdot G_k/G(F)$  is dense in  $G/G(F)$ .

(2)  $\Rightarrow$  (3). Let  $f_1, \dots, f_n$  be a set of generators of  $F$  over  $k$ , and let  $V_1$  be an open subset of  $G$  such that  $f_1, \dots, f_n$  are regular on  $V_1$ . We may assume that  $V_1 = G(F)V_1$ . Let  $g_1 \in V_1 \cap G_k, g_2 \in V_1, G(F)g_1 \neq G(F)g_2$ . Then  $g_2g_1^{-1} \notin G(F)$  and so there exists  $f_i$  such that  $(g_2g_1^{-1})_r f_i \neq f_i$ . We know that  $(g_2g_1^{-1})_r f_i$  and  $f_i$  are defined at  $g_1$ . Hence it follows from Lemma 2 that there exists an element  $d \in k[D]$  such that

$$d((g_2g_1^{-1})_r f_i)(g) \neq (df_i)(g), \text{ i.e., } (df_i)(g_1) \neq (df_i)(g_2).$$

Therefore, for any pair of distinct elements  $G(F)g_1, G(F)g_2$  such that

$$G(F)g_1 \in G(F) \cdot G_k \cap V_1/G(F) \text{ and } G(F)g_2 \in V_1/G(F),$$

<sup>3</sup> See [4] p. 51,

<sup>4</sup> See Proposition 2, p. 495 in [5].

<sup>5</sup> This part of the proof is modeled after the proof of Lemma 5.3 p.515 in [3].

there exists an element  $f \in F$  which is defined at  $G(F)g_1, G(F)g_2$  and such that  $f(G(F)g_1) \neq f(G(F)g_2)$ . But  $V_1/G(F)$  is an open subset of  $G/G(F)$ , and  $G(F)G_k \cap V_1/G(F)$  is dense in  $G/G(F)$ . Hence it follows from Lemma 1 that  $F = k(G/G(F))$ .

This completes the proof of the theorem.

**2. Applications.** As a consequence of Lemma 2 one can get the following corollary:

**COROLLARY.** *If  $\alpha$  is an automorphism of  $k(G)$  commuting with  $D_k$  and leaving the elements of  $k$  fixed then there exists  $h \in G_k$  such that  $\alpha = h_r$ .*

*Proof.*  $\alpha$  induces a rational map  $F_\alpha: G \rightarrow G$ . Let  $g \in G_k$  be a point such that  $F_\alpha$  is defined at  $g$  and let  $F_\alpha(g) = h^{-1}g$ . Then  $h \in G_k$  and  $f(g) = (\alpha f)(h^{-1}g)$ , for every  $f \in k(G)$  that is defined at  $g$ . Hence  $(df)g = (\alpha(df))(h^{-1}g)$ , for every  $d \in k[D]$ . But  $(\alpha(df))(h^{-1}g) = (h_r^{-1}(\alpha(df)))(g)$  and  $d$  commutes with  $\alpha$  and  $h_r^{-1}$ . Therefore  $(df)(g) = (d(h_r^{-1}(\alpha f)(g)))$ . Hence it follows from Lemma 2 that  $f = h_r^{-1}(\alpha f)$ . Thus  $h_r f = \alpha f$ , for every  $f$  that is defined at  $g$ . Therefore  $h_r f = \alpha f$ , for every  $f \in k(G)$ .

It follows from the corollary that if  $F$  is a  $D_k$ -stable subfield of  $k(G)$  containing  $k$  then every  $D_k$ -automorphism of  $k(G)$  leaving the elements of  $F$  fixed belongs to  $G(F)_r$ , i.e., the  $D_k$ -Galois group of  $k(G)$  over  $F$  coincides with  $G(F)_r$ . Combining this result and the above theorem we obtain that there exists the usual one to one Galois correspondence between  $D_k$ -stable subfields of  $k(G)$  containing  $k$  and  $k$ -closed subgroups of  $G$ .

Let  $k[G]$  denote the ring of regular (i.e., representative) functions on  $G$ . Let  $\mathcal{R}$  be the family of all  $(G_k)_i$ -stable (or, equivalently,  $D_k$ -stable) subrings  $R$  of  $k[G]$  containing  $k$  and satisfying the following condition if  $f \in R, g \in R$  and  $f/g \in k[G]$  then  $f/g \in R$ . Let  $\mathcal{S}$  denote the family of all  $k$ -closed subgroups  $H$  of  $G$  such that  $G/H$  is isomorphic to an open subset of an affine variety.

**THEOREM 2.** *The mappings  $H \rightarrow k[G] \cap k(G/H)$  and  $R \rightarrow G(R)$  establish a Galois correspondence between  $\mathcal{S}$  and  $\mathcal{R}$ <sup>8</sup>.*

*Proof.*  $H \in \mathcal{S}$  then  $k[G] \cap k(G/H) \in \mathcal{R}$  and  $G(k[G] \cap k(G/H)) = H$ , since  $k(G/H)$  is generated by  $k[G] \cap k(G/H)$ .

Now, if  $R \in \mathcal{R}$  then  $G(R) \in \mathcal{S}$ . In fact, if  $R \in \mathcal{R}$ , then  $k(R)$  is  $(G_k)_i$ -stable and so  $k(R) = k(G/G(R))$ . For every  $f \in R$ ,  $(G_k)_i f$  generates a finite dimensional  $k$ -vector space, Hence there exists a finitely generated over  $k(G_k)_i$ -stable subring  $R_0$  of  $R$  such that  $k(R_0) = k(R)$ . Let  $W$  denote

<sup>8</sup> C.f. [1] p. 324.

the affine variety that has  $R_0$  as its coordinate ring. One can define a structure of a  $G$ -homogeneous space on  $W$ , since  $K[R_0]$  is  $G_i$ -stable. Let  $\eta$  be the canonical mapping of  $G/G(R)$  into  $W$ . Then  $\eta$  commutes with the action of  $G$  and is birational. Hence  $\eta$  is an isomorphism of  $G/G(R)$  onto an open subset  $\eta(G/G(R))$  of  $W$ .

Moreover,  $R = k[G] \cap k(G/G(R))$ , since  $R \in \mathcal{R}$  and  $k(R) = k(G/G(R))$ . This completes the proof of the theorem.

*Added in Proof.* The equivalence (1)  $\iff$  (2) of Theorem 1 in the case where  $k$  is algebraically closed has been proved by E. Abe and T. Kanno (Tohoku Math. Jour. 2nd series 11 (1959), 376-384).

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