EXTENSIONS OF HOMOMORPHISMS

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1. Introduction. A multiplication was introduced by R. Arens [1] [2] into the second conjugate space $B^{**}$ of a Banach algebra, $B$, which made $B^{**}$ into a Banach algebra. The algebra of the second conjugate space was studied by Civin and Yood [3], with particular attention given to the case where $B$ was $L(\mathcal{G})$, the group algebra of the locally compact abelian group $\mathcal{G}$. Among the results they noted was that the algebra $M(\mathcal{G})$ of finite regular Borel measures on $\mathcal{G}$ was isomorphic as an algebra with a quotient algebra of $L^{**}(\mathcal{G})$. With $\mathcal{G}$ also a locally compact abelian group, P. J. Cohen showed [4, p. 220] that any homomorphism of $L(\mathcal{G})$ into $M(\mathcal{G})$ has an extension which was a homomorphism of $M(\mathcal{G})$ into $M(\mathcal{G})$.

In §3 we discuss the extensions of homomorphisms defined on a Banach algebra $A$ into either the second conjugate algebra $B^{**}$ of a Banach algebra $B$ or certain of its quotient algebras. The result of Cohen quoted above is included in Theorem 3.7 when $\mathcal{G}$ and $\mathcal{H}$ are compact groups. In §4 we indicate, for compact $\mathcal{G}$, a class of homomorphisms from $L(\mathcal{G})$ into $M(\mathcal{G})$, which are induced by homomorphisms of $L(\mathcal{G})$ into $L^{**}(\mathcal{G})$.

2. Notation. The notation of Civin and Yood [3] is used throughout. If $A$ is a Banach algebra, $A^*, A^{**}, \ldots$ denote the various conjugate spaces of $A$. For $f \in A^*$, $x \in A$, $\langle f, x \rangle \in A^*$ is defined by $\langle f, x \rangle \langle y \rangle = f(xy)$, $y \in A$. For $F \in A^{**}$, $f \in A^*$, $[F,f] \in A^*$ is defined by $[F,f] \langle x \rangle = F(\langle f, x \rangle)$, $x \in A$. Also for $F \in A^{**}$, $G \in A^{**}$ the multiplication $FG$ is defined in $A^{**}$ by $FG(\langle f \rangle) = F([G,f])$, $f \in A^*$.

For some purposes, Arens [2] also considers a second multiplication $F \cdot G$ defined for $F$ and $G$ in $A^{**}$ in a manner similar to the above, except that at the first stage, $\langle f | x \rangle \in A^*$ is defined by $\langle f | x \rangle \langle y \rangle = f(yx)$, $f \in A^*$, $x, y \in A$. Arens calls the multiplication in $A$ regular provided that $F \cdot G = GF$ for all $F,G \in A^{**}$. Clearly, if $A$ is commutative, then $A^{**}$ is commutative if and only if the multiplication in $A$ is regular. The same notation as above, in terms of bilinear functionals, is used in the sequel with respect to a multiplication in $A^{****}$ which comes from the first of the above multiplications in $A^{**}$.

If $\pi$ is the natural mapping of $A$ into $A^{**}$, we say that a mapping $\varphi$ defined on $A^{**}$ into a set $\mathcal{S}$ is an extension of a mapping $\rho$ defined on $A$ into $\mathcal{S}$ if $\varphi(\pi x) = \rho(x)$ for $x \in A$.

For any subset $\mathcal{Y}$ in $A^*$, we use the notation $\mathcal{Y} \perp$ for $\{ F \in A^{**} | F(f) = 0, f \in \mathcal{Y} \}$.

Received December 12, 1960. This research was supported by the National Science Foundation, grant NSF-G-14, 111.
For a commutative Banach algebra $A$, we let $\mathcal{V}(A)$ denote the closed subspace of $A^*$ generated by the multiplicative linear functionals. If $A = L(G)$, the group algebra of the locally compact group $G$, we write $\mathcal{V}(G)$ in place of $\mathcal{V}(L(G))$.

3. Extension of homomorphisms. We first consider the possibility of extending a bounded homomorphism of the Banach algebra $A$ into the Banach algebra $B^{**}$ to a $w^*$-continuous homomorphism of $A^{**}$ into $B^{**}$. Throughout this section we adopt the notation $\pi$ for the natural mapping of $A$ into $A^{**}$ and $\sigma$ for the natural mapping of $B^*$ into $B^{***}$.

3.1 Theorem. Let $A$ and $B$ be Banach algebras. Let $\varphi$ be a bounded homomorphism of $A$ into the center of $B^{**}$. Then there is a unique $w^*$-continuous homomorphism $\psi$ of $A^{**}$ into $B^{**}$ which is the extension of $\varphi$.

Proof. Let $f \in B^*$, and $x, y \in A$. Then $\langle \varphi^* \sigma f, x \rangle(y) = \varphi^* \sigma f(xy) = \varphi(xy)(y) = \varphi(y) \varphi(x)(f) = \varphi(y)(\varphi(x), f) = \varphi^* \sigma [\varphi(x), f](y)$. Thus $\langle \varphi^* \sigma f, x \rangle = \varphi^* \sigma [\varphi(x), f]$. For any $G \in A^{**}$, $[G, \varphi^* \sigma f](x)G(\varphi^* \sigma f, x) = G(\varphi^* \sigma [\varphi(x), f]) = \varphi^* \sigma [\varphi(x), f] = \varphi^* \sigma [\varphi^* \sigma [\varphi(x), f]] = \varphi^* \sigma [\varphi^* \sigma [\varphi(x), f]]$. Consequently, $[G, \varphi^* \sigma f] = \varphi^* \sigma [\varphi^* \sigma [\varphi(x), f]]$. Therefore for any $F \in A^{**}$, $F(G, \varphi^* \sigma f) = F(\varphi^* \sigma [\varphi^* \sigma [\varphi(x), f]]) = \varphi^* \sigma [\varphi^* \sigma [\varphi(x), f]]$. Hence $\sigma^* \varphi^* [\varphi^* \sigma G, \sigma] = \sigma^* \varphi^* [\varphi^* \sigma [\varphi(x), f]] = \sigma^* \varphi^* [\varphi^* \sigma [\varphi(x), f]]$. Thus $\sigma^* \varphi^*$ is a homomorphism of $A^{**}$ into $B^{**}$.

For $x \in A$, and $f \in B^*$, $\sigma^* \varphi^* (\pi x)(f) = \pi x(\varphi^* \sigma f) = \varphi^* \sigma f(x) = \sigma f(\varphi(x)) = \varphi(x)(f)$. Thus $\sigma^* \varphi^* (\pi x) = \varphi(x)$ and $\sigma^* \varphi^*$ is an extension of $\varphi$.

Let $G \in A^{**}$, $G_a \in A^{**}$ and suppose $G = w^*-\lim G_a$. Then for any $f \in B^*$, $\lim \sigma^* \varphi^* G_a(f) = \lim \sigma^* \varphi^* G_a(f) = \sigma^* \varphi^* G(f)$, and so $\sigma^* \varphi^*$ is $w^*$-continuous.

The assertion of uniqueness follows from the following.

3.2 Lemma. Let $A$ and $B$ be Banach algebras, and let $\varphi$ be any bounded linear transformation of $A$ into $B^{**}$. Then $\sigma^* \varphi^*$ is the only $w^*$-continuous extension of $\varphi$ to a transformation of $A^{**}$ into $B^{**}$.

Proof. That $\sigma^* \varphi^*$ is a $w$-continuous extension was given above. Suppose that $\psi$ is a $w^*$-continuous extension of $\varphi$, so that $\psi(\pi x) = \varphi(x)$ for all $x \in A$. Let $G \in A^{**}$ and let $\{x_a\} \subseteq A$ such that $w^*-\lim x_a = G$. Then for $f \in B^*$, $\psi(G)(f) = \lim \psi(\pi x_a)(f) = \lim \varphi(x_a)(f) = \lim \varphi^* \sigma f(x_a) = \pi x_a(\varphi^* \sigma f) = G(\varphi^* \sigma f) = \sigma^* \varphi^* G(f)$. Hence $\psi(G) = \sigma^* \varphi^* G$.

If $B$ is commutative with a regular multiplication, an alternative proof of Theorem 3.1 may be given on the basis of the following lemma and Theorem 6.1 of [3].
3.3 Lemma. If $B$ is a commutative Banach algebra with a regular multiplication then $\sigma^*$ is a homomorphism of $B^{**}$ into $B^{**}$.

Proof. Since multiplication in $B$ is regular, $B^{**}$ is [2] a commutative algebra. Let $U, V \in B^{**}$. For $f \in B^*$, and $F, G \in B^{**}$, $\langle \sigma f, F \rangle(G) = \sigma f (FG) = FG(f) = GF(f) = G([F, f]) = \sigma [F, f](G)$, and therefore $\langle \sigma f, F \rangle = \sigma [F, f]$. Also $[V, \sigma f](F) = V(\sigma f, F) = \sigma^* V[F, f] = (\sigma^* V) F(f) = F \sigma^* V(f) = F(\sigma^* V, f) = \sigma [\sigma^* V, f](F)$. Thus $[V, \sigma f] = \sigma [\sigma^* V, f]$. Consequently $\sigma^* (U V)(f) = U V (\sigma f) = U(\sigma V, f) = \sigma^* U(\sigma^* V, f) = \sigma^* U \sigma^* V(f)$ and $\sigma^*$ is a homomorphism claimed.

We note that it is impossible in general to conclude that the range of the extension of $\varphi$ is in the center of $B^{**}$ even though the range of $\varphi$ is in the center. For let $A = B$ be a commutative algebra whose multiplication is not regular, and let $\varphi = \pi$. Then the $w^*$-continuous extension of $\pi$ is the identity map and $B^{**}$ is not commutative.

One further example is in order, to see that in general a bounded homomorphism $\varphi$ from $A$ into $B^{**}$ does not admit a $w^*$-continuous extension as a homomorphism from $A^{**}$ into $B^{**}$. For this purpose let $A$ be the group algebra of the integers, $\mathbb{Z}$, and let $B = A$. Let $t_\gamma$, $\gamma \in \mathbb{Z}$ be the translation operator on $A^*$, defined by $t_\gamma f(\alpha) = f(\alpha + \gamma)$, $f \in A^*$, and $\alpha, \gamma \in \mathbb{Z}$. Let $e \in A^*$ correspond to the function identically one on $\mathbb{Z}$. Let $A = \{F \in A^{**} \mid F(t_\gamma f) = F(\gamma f), \text{ for all } \gamma \in \mathbb{Z}, f \in A^*\}$. Then as noted in formula (3.2) of [3],

$$\tag{3.1} GF = G(e) F, \quad F \in A^{**}, \quad G \in A^*.$$ 

In particular any $F \in A^{**}$ with $F(e) = 1$ is an idempotent. As noted in [3], $A$ is a two sided ideal in $A^{**}$ with only zero in common with the center of $A^{**}$. Since $\mathbb{Z}$ is a discrete group $A$ has an identity and thus [3, Lemma 5.4] $A^{**}$ has an identity $E$. Let $F$ be a nonzero idempotent in $A^{**}$. Thus $E - F$ is also an idempotent. Let $\varphi(x) = \pi x(E - F)$. Since $\pi A$ is in the center of $A^{**}$, $\varphi(x)$ is a homomorphism of $A$ into $A^{**}$. If $\varphi$ had a $w^*$-continuous extension as a homomorphism, the extension $\psi$ would have the value $\psi(G) = G(E - F)$, $G \in A^{**}$. We now show that $\psi$ is not a homomorphism. As noted above $F$ is not in the center of $A^{**}$, so we may pick $H \in A^{**}$ such that $HF \neq FH$. Also pick $G \in A^{**}$ such that $G(e) = 1$. Then $\psi(GH) = \pi GH(E - F) = GH - GHF = GH - (GH)(e)F$. Now $e$ is a multiplicative linear functional on $A$, and so by Lemma 3.6 of [3], $(GH)(e) = G(e)H(e) = H(e)$. Thus $\psi (GH) = GH - H(e)F = GH - HF$. On the other hand $\psi(G)\psi(H) = (G - GF)(H - HF) = (G - F)(H - H(e)F) = GH - FH - H(e)GF + H(e)F = GH - FH$. Since $FH \neq HF$, $\psi(GH) \neq \psi(G)\psi(H)$ and $\psi$ is not a homomorphism.

Before turning to other types of extensions we note one further
item on the matter of $w^*$-continuity of homomorphisms.

3.4 LEMMA. If $A$ and $B$ are Banach algebras and $\psi$ is a bounded homomorphism of $A^{**}$ into the center of $B^{**}$, then there is a $w^*$-continuous homomorphism $\rho$ of $A^{**}$ into $B^{**}$ such that $\psi(\pi x) = \rho(\pi x)$ for $x \in A$.

Proof. Since $\psi\pi$ is a homomorphism of $A$ into the center of $B^{**}$, we may take $\rho = \sigma^* \psi\pi^*$ and apply Theorem 3.1.

Homomorphisms of $A^{**}$ into $B^{**}$ which are not $w^*$-continuous exist, as may be seen in the following example. Let $\mathfrak{G}$ be an infinite compact group and let $A = B$ be the group algebra of $\mathfrak{G}$. Then by Lemma 3.8 of [3], $A^{**}$ has a right identity $E$ which is not an identity. Define for $F \in A^{**}$, $\psi(F) = EF$. Then $\psi(FG) = EFG = EFEG = \psi(F)\psi(G)$. However $\psi$ although bounded is not $w^*$-continuous. For let $G \in A^{**}$ and let $(x_\alpha)$ be a net such that $w^* - \lim \pi x_\alpha = G$. Then if $\psi$ were $w^*$-continuous we would have $\psi(G) = \lim \psi(\pi x_\alpha) = \lim E\pi x_\alpha = \lim \pi x_\alpha = G$. However, $\psi(G) = EG$ and $EG \neq G$ for some $G \in A^{**}$.

We next turn to the question of extending homomorphisms from $A$ into certain quotient algebras of $B^{**}$ in the case in which both $A$ and $B$ are commutative. We must first characterize the $w^*$-closed ideals of a second conjugate algebra.

3.5 LEMMA. Let $A$ be a commutative Banach algebra. Let $\mathfrak{X}$ be a $w^*$-closed subspace of $A^{**}$ and let $\mathfrak{X}_0 = \{f \in A^* | F(f) = 0, F \in \mathfrak{X}\}$. Then $\mathfrak{X}$ is an ideal of $A^{**}$ if and only if $[G, f] \in \mathfrak{X}_0$ for all $G \in A^{**}, f \in \mathfrak{X}_0$.

Proof. Since $\mathfrak{X}$ is $w^*$-closed, $\mathfrak{X} = \mathfrak{X}_0^\perp$. Suppose $\mathfrak{X}$ is an ideal of $A^{**}$. For any $F \in \mathfrak{X}, G \in A^{**}$, and $f \in \mathfrak{X}_0$, $FG \in \mathfrak{X}$ and $FG(f) = 0$. Therefore $F([G, f]) = 0$ for all $F \in \mathfrak{X}$, and so by definition $[G, f] \in \mathfrak{X}_0$. Suppose next that the stated condition holds. Let $F \in \mathfrak{X}$ and $G \in A^{**}$. For any $f \in \mathfrak{X}_0, [G, f] \in \mathfrak{X}_0$ and thus $FG(f) = F([G, f]) = 0$. Consequently $FG \in \mathfrak{X}_0^\perp = \mathfrak{X}$ and $\mathfrak{X}$ is a right ideal. For any $x \in A, \pi x$ is in the center of $A^{**}$, hence if $F \in \mathfrak{X}$, $\pi xF = F\pi x \in \mathfrak{X}$. Since $\pi A$ is $w^*$-dense in $A^{**}$ and left multiplication is $w^*$-continuous [2], we see that $GF \in \mathfrak{X}$ for any $G \in A^{**}$, and thus $\mathfrak{X}$ is an ideal of $A^{**}$.

3.6 THEOREM. Let $A$ and $B$ be commutative Banach algebras. Let $\mathfrak{Y}$ be a $w^*$-closed ideal of $B^{**}$. Suppose that $\varphi$ is a bounded homomorphism of $A$ into the center of $B^{**}/\mathfrak{Y}$. Then there exists a $w^*$-closed ideal $\mathfrak{Y}'$ of $A^{**}$ and a homomorphism $\psi$ of $A^{**}/\mathfrak{Y}'$ into $B^{**}/\mathfrak{Y}$ such that if $\pi$ is the natural embedding of $A$ into $A^{**}$, then $\psi(\pi x + \mathfrak{Y}') = \varphi(x), x \in A$. 

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Proof. Since $\mathcal{Z}$ is $w^*$-closed, $\mathcal{Z} = \mathcal{Z}^\perp_0$ where $\mathcal{Z}_0 = \{f \in B^* | F(f) = 0 \text{ for all } F \in \mathcal{Z}\}$. Let $\beta$ be the linear space isometric isomorphism of $\mathcal{Z}_0^*$ onto $B^{**}/\mathcal{Z}$ defined for $F \in \mathcal{Z}_0^*$ by $\beta F_0 = F + \mathcal{Z}$ where $F \in B^{**}$ is an arbitrary extension of $F_0$. Define multiplication in $\mathcal{Z}_0^*$ so that $\beta$ (and thus $\beta^{-1}$) is an algebra isomorphism. For $f \in \mathcal{Z}_0$, define $\varphi_* f$ by $\varphi_* f(x) = (\beta^{-1}\varphi(x))(f)$, $x \in A$. Then $\varphi_* f$ is linear and since $\varphi$ is bounded \[||\varphi_* f(x)|| \leq ||\varphi|| \cdot ||x|| \cdot ||f||,\] and $\varphi_* f \in A^*$.

Let $\mathcal{Z}_0'$ be the $w^*$-closure of the range of $\varphi_*$, and let $\mathcal{Z}' = \mathcal{Z}_0'^\perp$. Clearly $\mathcal{Z}'$ is $w^*$-closed. We next show that $\mathcal{Z}'$ is an ideal of $A^{**}$. Let $f \in \mathcal{Z}_0'$. Then for any $x, y \in A$, $\langle\varphi_* f, x \rangle(y) = \varphi_* f(xy) = (\beta^{-1}\varphi(xy))(f) = (\beta^{-1}\varphi(x))(\beta^{-1}\varphi(y))(f)$, since the range of $\varphi$ is commutative. Suppose that $\varphi(y) = U + \mathcal{Z}$, and $\varphi(x) = V + \mathcal{Z}$ so that $\varphi(yx) = UV + \mathcal{Z}$. Then $(\beta^{-1}\varphi(xy))(f) = UV(f) = U([V, f])$. Since $f \in \mathcal{Z}_0'$, and $\mathcal{Z}' = \mathcal{Z}_0'^\perp$ is an ideal, $g = [V, f] \in \mathcal{Z}_0'$ by Lemma 3.5. Hence $(\beta^{-1}\varphi(xy))(f) = U(g) = (\beta^{-1}\varphi(y))(g) = \varphi_*(g)$, for all $y \in A$. We therefore have $\langle\varphi_* f, x \rangle = \varphi_*(g)$ and so $\langle\varphi_* f, x \rangle \in \mathcal{Z}_0'$ for any $x \in A$ and $f \in \mathcal{Z}_0'$. Suppose next that $g \in \mathcal{Z}_0'$, and $x \in A$. Say $g = w^*$-lim $\varphi_* f_\alpha$ with $f_\alpha \in \mathcal{Z}_0$. Then for $y \in A$, $\langle g, x \rangle(y) = g(xy) = \lim \varphi_* f_\alpha(x) = \lim \langle f_\alpha, x \rangle(y)$, and hence $\langle g, x \rangle = w^* - \lim \langle f_\alpha, x \rangle$. However, by the above, $\langle f_\alpha, x \rangle \in \mathcal{Z}_0'$, and $\mathcal{Z}_0'$ is $w^*$-closed. Consequently, by Lemma 3.5, $\mathcal{Z}_0' = \mathcal{Z}_0'^\perp$ is a $w^*$-closed ideal of $A^{**}$.

For $F \in A^{**}$, define $\gamma F(f) = F(\varphi_* f)$ for $f \in \mathcal{Z}_0$. Clearly $\gamma F$ is a bounded linear functional on $\mathcal{Z}_0$, and so has an extension of the same norm which is an element of $B^{**}$. We again denote the extension by $\gamma F$. Thus $\gamma$ is a bounded linear map from $A^{**}$ into $B^{**}$. Note that if $F_1 - F_2 \in \mathcal{Z}_0'$ and $f \in \mathcal{Z}_0$, then $\gamma(F_1 - F_2)(f) = (F_1 - F_2)(\varphi_* f)(0)$, and thus $\gamma F_1 - \gamma F_2 \in \mathcal{Z}_0'$. Thus for any $F \in F_0 + \mathcal{Z}$, $||\gamma F_0 + \mathcal{Z}|| = ||\gamma F + \mathcal{Z}|| \leq ||\gamma F|| \leq ||F|| ||\varphi_*||$, and hence $||\gamma F_0 + \mathcal{Z}|| \leq ||F_0 + \mathcal{Z}|| ||\varphi_*||$.

Define $\psi$ on $A^{**}/\mathcal{Z}_0'$ by $\psi(F + \mathcal{Z}_0') = \gamma F + \mathcal{Z}$. By the above, we see that $\psi$ is a bounded linear mapping of $A^{**}/\mathcal{Z}_0'$ into $B^{**}/\mathcal{Z}$. Also for $x \in A$, $\psi(\pi x + \mathcal{Z}_0') = \pi x + \mathcal{Z}$. Since $\pi x(f) = \pi x(\varphi_* f) = \varphi_* f(x) = (\beta^{-1}\varphi(x))(f)$ for $f \in \mathcal{Z}_0$, $\pi x - \beta^{-1}\varphi(x) \in \mathcal{Z}_0'$, and $\psi(\pi x + \mathcal{Z}_0') = \varphi(x)$.

Thus all that remains is to see that $\psi$ satisfies the required multiplicative property of a homomorphism. Let $F, G \in A^{**}$. To see that $\psi(FG) = \psi(F)\psi(G)$, we must show that for $f \in \mathcal{Z}_0$, $\langle \gamma(FG)(f), \varphi(f) \rangle = 0$. Since $\langle \gamma(FG)(f), \varphi(f) \rangle = \gamma(F)(\varphi(f)) - FG(\varphi f) = F(\varphi f) - [G, \varphi f]$, it suffices if we show that $\varphi f$ is an algebra isomorphism. Let $x, y \in A$ and suppose that $\varphi(x) = U + \mathcal{Z}$, $\varphi(y) = V + \mathcal{Z}$, and thus $\varphi(xy) = \varphi(yx) = UV + \mathcal{Z}$. It follows that $\langle \varphi f, x \rangle(y) = \varphi f(xy) = UV(f) = V(U, f)$. Now, since $f \in \mathcal{Z}_0$, $[U, f] \in \mathcal{Z}_0$ by Lemma 3.5. We therefore have $\langle \varphi f, x \rangle(y) =$.
\[ \varphi_*[U,f](y) \] for all \( y \in A \), and consequently \( \langle \varphi_* f, x \rangle = \varphi_*[U,f] \). Thus \( [G, \varphi_* f](x) = G(\langle \varphi_* f, x \rangle) = G(\varphi_*[U,f]) = \gamma G([U,f]) = (\gamma G)U(f) \). On the other hand, \( \varphi_*[\gamma G, f](x) = U(\langle \gamma G, f \rangle) = U\gamma G(f) \). Since under our hypothesis \( \varphi(x) = U + \mathcal{Z} \) is in the center of \( B^{**}/\mathcal{Z} \), \( U\gamma G(f) = (\gamma G)U(f) \) for \( f \in \mathcal{Z} \), and we have the desired result.

It should be noted that the ideal \( \mathcal{Z}' \) in general is dependent on the homomorphism \( \varphi \). Two instances should be noted where this is not the case. The first, when \( \mathcal{Z}' = 0 \), has already been treated in the discussion of \( w^* \)-continuous extensions of homomorphisms of \( A \) into the center of \( B^{**} \). The other is the following.

3.7 THEOREM. Let \( A \) and \( B \) be commutative Banach algebras. Let \( \varphi \) be a homomorphism of \( A \) into \( B^{**}/\mathcal{Z} \). Then there is a homomorphism \( \psi \) of \( A^{**}/\mathcal{Z} \) such that \( \psi(\pi x + \mathcal{Z}) = \varphi(x) \).

Proof. If in the proof of Theorem 3.6, \( \mathcal{Z}_0 = \mathcal{Z}(B) \), it follows from Lemma 3.6 of [3] that for any \( f \in \mathcal{Z}_0 \) which is a multiplicative linear functional on \( B \), that \( \varphi_* f \) is a multiplicative linear functional on \( A \). Hence, the norm closure of the range of \( \varphi_* \) is contained in \( \mathcal{Z}(A) \). In view of Lemma 3.6 of [3], the subspace \( \mathcal{Z} \) of \( A^{**} \) is a \( w^* \)-closed ideal of \( A^{**} \), and if used in the role of \( \mathcal{Z}' \) affords the same conclusion. Note that the homomorphism \( \varphi \) is not postulated to be bounded or with range in the center of \( B^{**}/\mathcal{Z} \). This is legitimate since in view of Theorem 3.7 of [3], \( B^{**}/\mathcal{Z} \) is automatically commutative and semi-simple, and thus \( \varphi \) is automatically bounded.

If \( A \) and \( B \) are the group algebras of the compact groups \( \mathcal{G} \) and \( \mathcal{F} \), then \( A^{**}/\mathcal{Z} \) and \( B^{**}/\mathcal{Z} \) may be identified with the measure algebras \( M(\mathcal{G}) \) and \( M(\mathcal{F}) \) respectively by Theorem 3.18 of [3]. Thus Theorem 3.7 includes in the case of compact groups, the result of P. J. Cohen [4] quoted in the introduction.

4. Group algebras. Let \( \mathcal{G} \) be a locally compact abelian group. As in §3, we denote the group algebra of \( \mathcal{G} \) by \( L(\mathcal{G}) \) and the algebra of finite regular Borel measures on \( \mathcal{G} \) by \( M(\mathcal{G}) \). For notational purposes, it is also convenient to identify the character group \( \mathcal{G} \) of \( \mathcal{G} \) with the subset of \( L^*(\mathcal{G}) \) consisting of the nonzero multiplicative linear functional on \( L(\mathcal{G}) \). The topology of \( \mathcal{G} \) is then in agreement with the \( w^* \)-topology of \( \mathcal{G} \) as a subset of \( L^*(\mathcal{G}) \).

Suppose that \( \mathcal{H} \) is a locally compact abelian group. A continuous homomorphism \( \nu \) of \( \mathcal{G} \) into \( \mathcal{H} \) is called nonsingular if for every Borel set \( E \) in \( \mathcal{H} \) with zero Haar measure, \( \nu^{-1}(E) \) is of zero Haar measure in \( \mathcal{G} \).

A complete characterization of all homomorphisms \( \varphi \) of \( L(\mathcal{G}) \) into \( M(\mathcal{H}) \) was given by P. J. Cohen [4]. He utilized the function \( \varphi_\ast \) from \( \mathcal{G} \) into \( \{\mathcal{G}, 0\} \) defined by \( \varphi_\ast f(x) = \varphi(x)(f) \), \( x \in L(\mathcal{G}) \), \( f \in \mathcal{H} \).
4.1 Theorem. (P. J. Cohen) Let $\mathbb{G}$ and $\mathbb{H}$ be locally compact abelian groups, $\varphi$ a homomorphism of $L(\mathbb{G})$ into $M(\mathbb{H})$, $\varphi^*$ the induced map of $\mathcal{H}$ into, $\{\mathbb{G}, 0\}$. Then there are a finite number of sets $\mathcal{R}_i$, which are cosets of open subgroups of $\mathbb{G}$, and continuous maps $\psi_i: \mathcal{R}_i \rightarrow \mathbb{G}$, such that

\begin{equation}
\psi_i(x + y - z) = \psi_i(x) + \psi_i(y) - \psi_i(z)
\end{equation}

for all $x, y$ and $z$ in $\mathcal{R}_i$, with the following property: There is a decomposition of $\mathcal{H}$ into the disjoint union of sets $\mathcal{S}_j$, each lying in the Boolean ring generated by the sets $\mathcal{R}_i$, such that on each $\mathcal{S}_j$, $\varphi^*$ is either identically zero or agrees with some $\psi_i$, where $\mathcal{S}_j \subset \mathcal{R}_i$.

Conversely, for any map of $\mathcal{H}$ into $\{\mathbb{G}, 0\}$, there is a homomorphism of $L(\mathbb{G})$ into $M(\mathbb{H})$ which induces it. The map $\varphi$ carries $L(\mathbb{G})$ into $L(\mathcal{H})$ if and only if $\varphi^{-1}$ of every compact subset of $\mathcal{H}$ is compact.

Suppose that the sets $\mathcal{R}_i$ are cosets of the subgroups $\mathcal{U}_i$ of $\mathbb{G}$. There is a closed subgroup $\mathcal{U}_i$ of $\mathcal{H}$, $\mathcal{U}_i = \{h \in \mathcal{H} | (h, \hat{h}) = 1, \hat{h} \in \mathcal{U}_i\}$, such that $\mathcal{U}_i$ may be viewed [6, p. 130] as the character group of $\mathcal{G}/\mathcal{G}_i$. Let $a_i \in \mathcal{R}_i$, and define $\psi_i': \mathcal{U}_i \rightarrow \mathbb{G}$ by

\begin{equation}
\psi_i'(x) = \psi_i(a_i + x) - \psi_i(a_i), \quad x \in \mathcal{U}_i.
\end{equation}

The condition (4.1) on $\psi_i$ is then equivalent to the assertion that $\psi_i'$ is a homomorphism of $\mathcal{U}_i$ into $\mathcal{G}$, and $\psi_i'$ is continuous along with $\psi_i$. We may also consider the dual homomorphism $\rho_i: \mathcal{G} \rightarrow \mathcal{U}_i = \mathcal{H}/\mathcal{U}_i$, defined by

\begin{equation}
(\psi_i'(x), g) = (x, \beta_i(g)), \quad x \in \mathcal{U}_i = (\mathcal{G}/\mathcal{G}_i)^\wedge, \quad g \in \mathcal{G}.
\end{equation}

In view of the Cohen theorem, the homomorphism $\psi$ is determined by the sets $\mathcal{R}_i, \mathcal{S}_j$ and the functions $\beta_i$. The notation introduced above will be used in the sequel without further comment. We also use the notation $\rho_*$ as the mapping of $L^*(\mathbb{G})$ into $L^*(\mathcal{H})$ which is defined by $\rho_*(f(x) = \rho(x)(f)$, $x \in L(\mathbb{G})$, $f \in L^*(\mathcal{H})$, whenever $\rho$ is a bounded linear map of $L(\mathbb{G})$ into $L**(\mathcal{H})$.

4.2 Lemma. Let $\lambda$ be a nonsingular homomorphism of $\mathbb{G}$ into a locally compact abelian group $\mathbb{K}$. Then $\lambda$ induces a homomorphism $\rho$ of $L(\mathbb{G})$ into $L**(\mathbb{K})$ such that for $f \in \mathbb{K}$, $\rho_*(f) = f \circ \lambda$.

Proof. For $k \in L^*(\mathbb{K})$, define $\lambda_*(k)$ by

$$\lambda_*(k)(\alpha) = k \circ \lambda(\alpha), \quad \alpha \in G.$$

We first must show that $\lambda_*$ is a well-defined bounded linear mapping of $L^*(\mathbb{K})$ into $L^*(\mathbb{G})$. Suppose that $K_1$ and $K_2$ are two bounded Borel measurable functions on $\mathbb{K}$ such that $K_1(\beta) = K_2(\beta)$ for almost all $\beta$ in $\mathbb{K}$. Let $\mathcal{C} = \{\alpha \in \mathbb{G} | k_1(\lambda(\alpha)) \neq k_2(\lambda(\alpha))\}$. Then $\mathcal{C} = \lambda^{-1}(\mathcal{C}(\mathbb{G}))$ and by the hypothesis
of non-singularity $\mathcal{G}$ has measure zero in $\mathfrak{G}$. Since it is now immediate that $|\lambda_*(k)(\alpha)| \leq ||k||$ for almost all $\alpha$ in $\mathfrak{G}$, it follows that $\lambda_*$ is a bounded linear map of $L^*(\mathfrak{F})$ into $L^*(\mathfrak{G})$.

For $x \in L(\mathfrak{G})$, define $\rho(x)$ on $L^*(\mathfrak{F})$ by

$$\rho(x)(f) = \lambda_* f(x), \quad f \in L^*(\mathfrak{F}).$$

Clearly $\rho(x) \in L^{**}(\mathfrak{F})$, and $\rho$ is a bounded linear mapping from $L(\mathfrak{G})$ into $L^{**}(\mathfrak{F})$, and $\rho_* f = f \circ \lambda$.

We next show that $\rho$ satisfies the multiplicative condition for a homomorphism. Let $x, y \in L(\mathfrak{G})$ and $f \in L^*(\mathfrak{F})$. Then

$$\rho(xy)(f) = \lambda_* f(xy) = \int_{\mathfrak{G}} \lambda_*(f(\alpha)) \int_{\mathfrak{G}} x(\beta)y(\alpha - \beta) \, d\beta \, d\alpha$$

$$= \int_{\mathfrak{G}} \int_{\mathfrak{G}} f(\lambda(\alpha))x(\beta)y(\alpha - \beta) \, d\beta \, d\alpha$$

$$= \int_{\mathfrak{G}} \int_{\mathfrak{G}} f(\lambda(\alpha + \beta))x(\beta)y(\alpha) \, d\beta \, d\alpha.$$

For any $z \in L(\mathfrak{F})$, and $\delta \in \mathfrak{F}$, it is easily seen [3] that $\langle f, z \rangle(\delta) = \int_{\mathfrak{F}} f(z + \delta) z(\gamma) \, d\gamma$. Therefore,

$$[\rho(y), f](\gamma) = \rho(y)(\langle f, z \rangle) = \lambda_* \langle f, z \rangle(\alpha)y(\alpha) \, d\alpha$$

$$= \int_{\mathfrak{G}} \langle f, z \rangle(\lambda(\alpha))y(\alpha) \, d\alpha = \int_{\mathfrak{G}} \int_{\mathfrak{F}} f(\gamma + \lambda(\alpha)) z(\gamma)y(\alpha) \, d\gamma \, d\alpha.$$

Since the order of integration may be reversed, we see that for $\gamma \in \mathfrak{F}$,

$$[\rho(y), f](\gamma) = \int_{\mathfrak{G}} f(\gamma + \lambda(\beta)) y(\beta) \, d\beta.$$

Hence,

$$\rho(x)\rho(y)(f) = \rho(x)([\rho(y), f]) = \lambda_* [\rho(y), f](x) = \int_{\mathfrak{G}} \lambda_* [\rho(y), f](\alpha) x(\alpha) \, d\alpha$$

$$= \int_{\mathfrak{G}} [\rho(y), f](\lambda(\alpha)) x(\alpha) \, d\alpha = \int_{\mathfrak{G}} \int_{\mathfrak{G}} f(\lambda(\alpha) + \lambda(\beta)) y(\beta) x(\alpha) \, d\beta \, d\alpha.$$

Since we thus have $\rho(xy)(f) = \rho(x)\rho(y)(f)$, for all $f \in L^*(K)$, $\rho$ is a homomorphism.

4.3 Theorem. Let $\mathfrak{G}$ and $\mathfrak{F}$ be locally compact abelian groups, with $\mathfrak{G}$ compact. Let $\varphi$ be a homomorphism of $L(\mathfrak{G})$ into $M(\mathfrak{G})$. Let $M(\mathfrak{F})$ be regarded as $L^{**}(\mathfrak{F})/\mathfrak{F}^\perp(\mathfrak{F})$, and let $\theta$ be the natural mapping of $L^{**}(\mathfrak{G})$ onto $L^{**}(\mathfrak{G})/\mathfrak{F}^\perp(\mathfrak{G})$. Then if each homomorphism $\beta_i$, determined by $\varphi$, is nonsingular, there is a homomorphism $\rho$ of $L(\mathfrak{G})$ into $L^{**}(\mathfrak{F})$ such that $\varphi = \theta \circ \rho$.

Proof. The justification for considering $M(\mathfrak{F})$ as $L^{**}(\mathfrak{F})/\mathfrak{F}^\perp(\mathfrak{F})$ is
Theorem 3.18 of [3].

If \( \varphi_*(f) = 0 \) for all \( f \in \mathfrak{S}_j \), define \( \rho_j : L(\mathfrak{S}) \to L^{**}(\mathfrak{S}) \) by \( \rho_j(x) = 0 \), \( x \in L(\mathfrak{S}) \).

Suppose that \( \mathfrak{S}_j \subset \mathfrak{R}_i \subset \hat{\mathfrak{S}}_i \), and \( \varphi_*(f) = \varphi_j(f) \) for \( f \in \mathfrak{S}_j \). In view of (4.1), the homomorphism \( \varphi_j' \) of \( U_i \) into \( \hat{G} \) may be defined by \( \varphi_j'(k) = \varphi_i(k + k_i) - \varphi_i(k_i) \) for an arbitrary \( k_i \in \mathfrak{S}_j \). The dual homomorphism \( \beta_i \) of \( \mathfrak{S} \) into \( \mathfrak{S}/\mathfrak{S}_i \) is by hypothesis nonsingular. Thus by Lemma 4.2, there is a homomorphism \( \rho_j^* \) of \( L(G) \) into \( L^{**}(\mathfrak{S}/\mathfrak{S}_i) \) such that \( \rho_j^*(\kappa) = k \circ \beta_i \), for \( k \in (\mathfrak{S}/\mathfrak{S}_i)^* = U_i \).

For \( f \in L(\mathfrak{S}/\mathfrak{S}_i) \) define \( \theta_i(f) \) on \( \mathfrak{S}_i \) by \( \theta_i(f)(\beta) = f(\beta + \mathfrak{S}_i) \). Suppose that the Haar measure on \( \mathfrak{S}_i \) is normalized so that the measure of \( \mathfrak{S}_i \) is one. The formula relating integration on a group with that on a quotient group shows that \( \theta_i \) is an isometric isomorphism of \( L(\mathfrak{S}/\mathfrak{S}_i) \) into \( L(\mathfrak{S}_i) \). Thus by Theorem 6.1 of [3], \( \theta_i^{**} \) is a homomorphism of \( L^{**}(\mathfrak{S}/\mathfrak{S}_i) \) into \( L^{**}(\mathfrak{S}) \). Also for any \( u \in L(\mathfrak{S}/\mathfrak{S}_i) \), and \( f \in L^*(\hat{\mathfrak{S}}_i) \),

\[
\theta_i^*f(u) = f(\theta_iu) = \int_{\mathfrak{S}_i} f(\beta)\theta_i(u)(\beta) \, d\beta
= \int_{\mathfrak{S}_i} \int_{\mathfrak{S}_i} f(\beta + \mathfrak{S}_i)\theta_i(u)(\beta + \mathfrak{S}_i) \, d\gamma \, d\hat{\beta},
\]

where \( d\hat{\beta} \) is the Haar measure on \( \mathfrak{S}/\mathfrak{S}_i \). Thus

\[
\theta_i^*f(u) = \int_{\mathfrak{S}_i} u(\hat{\beta}) \int_{\mathfrak{S}_i} f(\beta + \mathfrak{S}_i) \, d\gamma \, d\hat{\beta} ,
\]

and we conclude that \( \theta_i^*f(\hat{\beta}) = \int_{\mathfrak{S}_i} f(\beta + \mathfrak{S}_i) \, d\gamma \).

It is well known that in a group algebra the pointwise multiplication by a character is an automorphism of the algebra. We next show that the same situation prevails in the second conjugate algebra of a group algebra. Let \( \mathfrak{X} \) be a locally compact abelian group and define, for \( \gamma \in \hat{\mathfrak{X}}, \varphi_\gamma g \) and \( \gamma \circ g \) by pointwise multiplication on \( \mathfrak{X} \) if \( x \in L(\mathfrak{X}) \) and \( g \in L^*(\mathfrak{X}) \). Define \( \gamma \circ G(g) = G(\gamma \circ g) \) for \( G \in L^{**}(\mathfrak{X}) \). Clearly the map \( G \to \gamma \circ G \) is a one-to-one bounded linear map of \( L^{**}(\mathfrak{X}) \) onto itself.

Let \( F, G \in L^{**}(\mathfrak{X}) \) and \( \varphi \in L^*(\mathfrak{X}) \). It remains for us to show that \( (\varphi \circ F)(\varphi \circ G) = (\varphi \circ F)(\varphi \circ G) \) for \( x \in L(\mathfrak{X}) \), and \( g \in L^*(\mathfrak{X}) \). Since \( (\varphi \circ F)(\varphi \circ G)(g) = \gamma \circ F(\varphi \circ G)(g) = F(\gamma \circ [\varphi \circ G, g]) \), while \( \gamma \circ [\varphi \circ G, g] = F(\varphi \circ G)(g) = F(\varphi \circ G)(g) \), it suffices if we show that for all \( x \in L(\mathfrak{X}), \gamma \circ [\varphi \circ G, g](x) = G(\gamma \circ g, x) \). Now \( \gamma \circ [\varphi \circ G, g](x) = \gamma \circ G(\varphi \circ G(x), x) = \gamma \circ G(\varphi \circ g, x) \), while \( [\gamma \circ g, x] = G(\gamma \circ g, x) \), so it suffices if we show that for all \( y \in L(\mathfrak{X}) \), \( \gamma \circ [\varphi \circ G, g](x) = G(\gamma \circ g, x) \). Since \( \gamma \circ g \circ x \circ y = \gamma \circ g \circ x \circ y = \gamma \circ g \circ x \circ y = \gamma \circ g \circ x \circ y \), the original assertion follows.

Define the mapping \( \rho_j \) by

\[
(4.4) \quad \rho_j(x) = k_i^{-1} \circ \theta_i^{**} \rho_j^*(\varphi_j(k_i) \circ x) , \quad x \in L(\mathfrak{S}),
\]
where the dot at each occurrence indicates multiplication of the appropriate functions. Since $k_i \in \hat{\mathfrak{H}}$, and $\psi_i(k_i) \in \hat{\mathfrak{H}}$, $\rho_j$ is a composite of four homomorphisms and is thus a homomorphism of $L(\mathfrak{H})$ and $L^{**}(\hat{\mathfrak{H}})$.

Suppose that $f \in \mathfrak{S}_j \subset \mathfrak{S}_i$, so that $\varphi_* f = \psi_i f$. Since $\mathfrak{H}_i$ is a coset of $\mathfrak{H}_i$, there is a $k \in \mathfrak{H}_i$ such that $f = k_i + k$. We use the same notation for $k$ when it is viewed as a member of $\mathfrak{S}_i$. For any $x \in L(\mathfrak{S})$, 

$$\rho_j \ast f(x) = \rho_j(x)(f) = k^{-1} \ast \vartheta_i \ast \rho_j(\psi_i(k_i) \ast x)(f) = \theta_i \ast \rho_j(\psi_i(k_i) \ast x)(k) = \rho_j(\psi_i(k_i) \ast x) \theta_i \ast (k).$$

From the formula obtained earlier for $\theta_i \ast$, it is immediate that $\theta_i \ast$ simply transfers $k$ from being viewed as a member of $\mathfrak{S}_i \subset \mathfrak{H}$, to being viewed as a member of $((\mathfrak{S} \mid \mathfrak{S}_i) \subset L^*(\mathfrak{S} \mid \mathfrak{S}_i)$. Thus

$$\rho_j \ast f(x) = \rho_j(\psi_i(k_i) \ast x)(k) = \int_\Theta \rho_j'(k)(\alpha) \psi_i(k_i)(\alpha) x(\alpha) d\alpha$$

$$= \int_\Theta (k, \beta(\alpha)) \psi_i(k_i)(\alpha) x(\alpha) d\alpha = \int_\Theta (\psi_i(k), \alpha) \psi_i(k_i)(\alpha) x(\alpha) d\alpha,$$

by use of (4.3). Thus by use of the definition of $\psi'$ in terms of $k_i$, we have

$$\rho_j \ast f(x) = \int_\Theta (\psi_i(k + k_i) - \psi_i(k_i), \alpha) \psi_i(k_i)(\alpha) x(\alpha) d\alpha$$

$$= \int_\Theta (\psi_i(f), \alpha) x(\alpha) d\alpha = \int_\Theta \varphi_\ast f(x)(\alpha) d\alpha.$$

We therefore conclude that $\rho_j f(x) = \varphi_\ast f(x)$ for all $x \in L(\mathfrak{S})$ or that $\rho_j f = \varphi_\ast f$ for $f \in \mathfrak{S}_i$.

Now, by the Cohen theorem, $\hat{\mathfrak{S}}$ is the disjoint union of the sets $\mathfrak{S}_j$. The characteristic function of $\mathfrak{S}_j$ is then the Fourier transform of an idempotent measure in $M(\hat{\mathfrak{S}}) = L^{**}(\mathfrak{S})/\mathfrak{S}(\mathfrak{S})$. Let $F_j$ be any member of $L^{**}(\mathfrak{S})$ such that $\vartheta F_j$ is the Fourier transform of the characteristic function of $\mathfrak{S}_j$. Then $F_j^2 - F_j \in \mathfrak{S}(\mathfrak{S})$. Now, Theorem 3.15 of [3] states that $\mathfrak{S}(\mathfrak{S})$ is the radical of $L^{**}(\mathfrak{S})$, and therefore Theorem 2.3.9 of [5] yields $E_j \in L^{**}(\mathfrak{S})$ such that $E_j^2 = E_j$ and $\vartheta E_j = \vartheta F_j$.

We next show that if $i \neq j$, then $E_i F_j E_j = 0$ for any $F \in L^{**}(\mathfrak{S})$. Suppose that $f \in \hat{\mathfrak{S}}$, then Lemma 3.6 of [3] yields

$$E_i F_i E_j (f) = E_i (f) F(f) E_j (f).$$

For $f \in \hat{\mathfrak{S}}$, $E_k(f) = F_k(f) = \chi(\mathfrak{S}_k)(f)$, where $\chi(\mathfrak{S}_k)$ is the characteristic function of $\mathfrak{S}_k$. Thus since $S_i$ and $S_j$ are disjoint $E_i F_i E_j (f) = 0$. Hence $E_i F_i E_j \in \mathfrak{S}(\mathfrak{S})$, the radical of $L^{**}(\mathfrak{S})$. For a compact group $\mathfrak{S}$, the radical is also the right annihilator of $L^{**}(\mathfrak{S})$ by Theorem 3.5 of [3]. Thus since $E_i = E_i^2$, $E_i F_i E_j = E_i (E_i F_i E_j) = 0$.

Let $\rho$ be defined on $L(\mathfrak{S})$ by

$$\rho(x) = E_i \rho_i(x) E_i + \cdots + E_r \rho_r(x) E_r,$$

where $\hat{\mathfrak{S}} = \mathfrak{S}_1 \cup \cdots \cup \mathfrak{S}_r$. Clearly $\rho$ is a bounded linear transformation of $L(\mathfrak{S})$ into $L^{**}(\mathfrak{S})$, and to see that $\rho$ is a homomorphism it suffices if
we show that $E_i\rho_i(xy)E_i = E_i\rho_i(x)E_i\rho_i(y)E_i$. The latter equality is established by an identical argument to that used above to show $E_iFE_j = 0$ for $i \neq j$. Thus $\rho$ is a homomorphism of $L(\mathfrak{B})$ into $L^{**}(\mathfrak{B})$.

To see that $\theta \circ \rho = \varphi$, it suffices if we show that $\varphi_\ast(f) = (\theta \circ \rho)_\ast(f)$ for $f \in \mathfrak{B}$. Suppose that $f \in \mathfrak{B}_k$. Then for $x \in L(\mathfrak{B})$, $(\theta \circ \rho)_\ast(f)(x) = \theta \circ \rho(x)(f) = E_k\rho_k(x)E_k(f)$, since $E_i(f) = 0$ if $i \neq k$. Thus $(\theta \circ \rho)_\ast(f)(x) = \rho_k(x)(f) = \varphi_\ast f$ as was shown earlier.

REFERENCES

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The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
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