MEAN CROSS-SECTION MEASURES OF HARMONIC MEANS OF CONVEX BODIES

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1. In [2] the notion of \( p \)-dot means of two convex bodies in Euclidean \( n \)-space was introduced and certain properties of these means investigated. For \( p = 1 \), the mean is more appropriately called the harmonic mean; here we restrict the discussion to this case. The harmonic mean of two convex bodies \( K_0 \) and \( K_1 \), which will always be assumed to share a common interior point \( Q \), is defined as follows. Let \( \hat{K} \) denote the polar reciprocal of \( K \) with respect to the unit sphere \( E \) centred at \( Q \); let \( (1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1 \), with \( 0 \leq \vartheta \leq 1 \), be the usual arithmetic or Minkowski mean of \( \hat{K}_0 \) and \( \hat{K}_1 \). The harmonic mean of \( K_0, K_1 \) is the convex body \( [(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^{-1} \). In more analytic terms, if \( F_i(x) \) are the distance functions with respect to \( Q \) of \( K_i \), for \( i = 0, 1 \), then the body whose distance function with respect to \( Q \) is \( (1 - \vartheta)F_0(x) + \vartheta F_1(x) \) is the harmonic mean of \( K_0 \) and \( K_1 \).

In the paper mentioned, a dual Brunn-Minkowski theorem was established, namely

\[
V^{1/n}([(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^{-1}) \leq 1\sqrt{\left[ \frac{(1 - \vartheta)}{V^{1/n}(K_0)} + \frac{\vartheta}{V^{1/n}(K_1)} \right]}
\]

where \( V(K) \) means the volume of \( K \). There is equality if and only if \( K_0 \) and \( K_1 \) are homothetic with the centre of magnification at \( Q \).

Here we develop a more inclusive theorem regarding the behaviour of each mean cross-section measure, ("Quermassintegral") \( W_\nu(K) \), \( \nu = 0, 1, \ldots, n - 1 \), cf. [1]. The result is

\[
W^{(\nu+\nu')/n}([(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^{-1}) \leq 1\sqrt{\left[ \frac{(1 - \vartheta)}{W^{(\nu+\nu')/n}(K_0)} + \frac{\vartheta}{W^{(\nu+\nu')/n}(K_1)} \right]}
\]

The cases of equality are just those of the dual Brunn-Minkowski theorem, \( (\nu = 0) \).

2. We first list some preliminary items used in the proof of (2).

We shall use Minkowski’s inequality in the form

\[
\int [(1 - \vartheta)f_0^p + \vartheta f_1^p]^{1/p}dx \leq [(1 - \vartheta)(\int f_0^pdx)^{1/p} + \vartheta(\int f_1^pdx)^{1/p}]^{1/p}.
\]

Here the functions \( f_i \) are assumed to be positive and continuous over the closed and bounded domain of integration common to all the integrals.

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and, for our purposes, $p$ satisfies $-1 \leq p < 0$. There is equality if and only if $f_\lambda(x) \equiv \lambda f(x)$ for some constant $\lambda$. See [3], Theorem 201, coupled with the remark preceding Theorem 200.

Our second tool, which we shall refer to as the projection lemma, was established in [2]. Let $K^*$ denote the projection of $K$ onto a fixed, $m$-dimensional, linear subspace $E_m$ through $Q$ for $1 \leq m < n$. We have

$$[(1 - \partial)\hat{K}^* + \partial \hat{K}^*] = [(1 - \partial)\hat{K} + \partial \hat{K}]^*.$$

Since $E_m$ contains $Q$ and the polar reciprocation is with respect to sphere $E$ centred at $Q$, in forming $\hat{K}^*$ the order of operations is immaterial. This result is proved by a polar reciprocation argument from

$$(1 - \partial)(\hat{K}_0 \cap E_m) + \partial(\hat{K}_1 \cap E_m) \subseteq [(1 - \partial)\hat{K}_0 + \partial \hat{K}_1] \cap E_m.$$

There is equality in either inclusion if $K_0$ and $K_1$ are homothetic with centre of magnification at $Q$.

The dual Brunn-Minkowski theorem (1) will be used.

Finally we shall make use of Kubota's formula and some of its consequences. This material is covered in [1]. An $(n - \nu)$ dimensional cross-section measure ('Quermass') of $K$ is the $(n - \nu)$ dimensional volume of that convex body which is the vertical projection of $K$ onto an $E_{n-\nu}$. The mean cross-section measures are usually defined as the coefficients in Steiner's polynomial which describes $V(K + \lambda E)$, that is

$$V(K + \lambda E) = \sum_{\nu=0}^{n} \binom{n}{\nu} W_\nu(K)\lambda^\nu.$$

If we denote the $(\nu - 1)\nu$ mean cross-section measure of the projection of $K$ onto that $E_{n-\nu}$ through $Q$ which is orthogonal to the vector $u_1$ by $W'_{\nu-1}(K, u_1)$, then Kubota's formula is

$$W_\nu(K) = \frac{1}{\kappa_{n-1}} \int_{\Omega_n} W'_{\nu-1}(K, u) d\omega_n, \quad \nu = 1, 2, \ldots, n - 2.$$

Here the integration with respect to the direction $u_1$ is extended over the surface $\Omega_n$ of $E$, $d\omega_n$ is the element of surface area on $\Omega_n$ and $\kappa_{n-1}$ is the volume of the $n - 1$ dimensional unit sphere.

Kubota's formula can be applied to the mean cross-section measure $W'_{\nu-1}(K, u)$ for fixed $u_1$:

$$W''_{\nu-1}(K, u) = \frac{1}{\kappa_{n-2}} \int_{\Omega_{n-1}} W''_{\nu-2}(K, u_1, u_2) d\omega_{n-1}$$

where $W''_{\nu-2}$ is the $(\nu - 2)$th mean cross-section measure of the projection of $K$ onto the $E_{n-2}$ through $Q$ orthogonal to $u_1$ and $u_2$ with $u_2$ orthogonal to $u_1$. After $\nu$ such steps we have as the extended form of Kubota's formula:
Each vector $u_p$ is orthogonal to $u_q$ for $q < p$ and $W^{(v)}_0(K, u_1, u_2, \ldots, u_\nu)$ is the 0th mean cross-section measure of the projection of $K$ onto that $E_{n-v}$ through $Q$ which is the orthogonal complement of the subspace spanned by $u_1, u_2, \ldots, u_\nu$.

Steiner’s formula (5) with $\lambda = 0$ shows that $W_0(K)$ is the volume of $K$ and so $W^{(v)}_0$ is an $(n - \nu)$ dimensional cross-section measure of $K$. Thus, to within a numerical factor depending on $n$ and $\nu$, $W_0(K)$ is the arithmetic mean of the $(n - \nu)$ dimensional cross-section measures.

In § 3 we shall use the following abbreviations: for $d\omega_{n-\nu} \cdots d\omega_{n-1} d\omega_n$ we write $d\omega$ with sign of integration and omit reference to the domains of integration; for one $1/\kappa_{n-1} \cdots \kappa_{n-\nu}$ we write $k$; finally for $W^{(v)}_0(K, u_1, u_2, \ldots, u_\nu)$ we write $\sigma(K^*)$. In this notation the extended Kubota formula reads

$$W(K) = k \int \sigma(K^*) d\omega.$$ 

3. We now prove (2). By the extended form of Kubota’s formula

$$W^{(n-\nu)}_0(((1 - \vartheta)\hat{K}_0 + \vartheta \hat{K}_1)^*),
\begin{equation}
\tag{6}
\leq \left[ k \int \sigma(((1 - \vartheta)\hat{K}_0 + \vartheta \hat{K}_1)^*) d\omega \right]^{1/(n-\nu)}
\end{equation}$$

in virtue of the projection lemma and the set monotonicity of $\sigma$ i.e., $\sigma(K^*) \leq \sigma(\bar{K}^*)$ if $K^* \subseteq \bar{K}^*$ with equality in the latter relation implying that in the former. We now apply (1), in $E_{n-\nu}$, to the integrand to obtain

$$\sigma(((1 - \vartheta)\hat{K}_0^* + \vartheta \hat{K}_1^*)^*) \leq \left\{ 1 \left[ \frac{(1 - \vartheta)}{\sigma^{1/(n-\nu)}(K_0^*)} + \frac{\vartheta}{\sigma^{1/(n-\nu)}(K_1^*)} \right] \right\}^{(n-\nu)}.$$

Here we take advantage of the fact that

$$(\hat{K})^* = (K^*)^*.$$ 

This gives

$$W^{(n-\nu)}_0(((1 - \vartheta)\hat{K}_0 + \vartheta \hat{K}_1)^*)
\leq \left[ k \int \left\{ 1 \left[ \frac{(1 - \vartheta)}{\sigma^{1/(n-\nu)}(K_0^*)} + \frac{\vartheta}{\sigma^{1/(n-\nu)}(K_1^*)} \right] \right\}^{(n-\nu)} d\omega \right]^{1/(n-\nu)}.$$

There is equality if and only if all the projections $K_0^*$ and $K_1^*$ are homothetic with the centre of magnification at $Q$. This condition is
sufficient for equality in (6); it is necessary and sufficient for (7).

We now use Minkowski’s inequality (3) with $p = -1/n-\nu$. This yields

\[ W_{\nu}^{1/(n-\nu)}((1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1) \]

\[ \leq 1\left[\frac{\sigma(K_0)}{W_{\nu}^{1/(n-\nu)}(K_0)} + \frac{\vartheta}{W_{\nu}^{1/(n-\nu)}(K_1)}\right] \]

\[ = 1\left[\frac{\sigma(K_0)}{W_{\nu}^{1/(n-\nu)}(K_0)} + \frac{\vartheta}{W_{\nu}^{1/(n-\nu)}(K_1)}\right]. \]

The necessary and sufficient conditions for equality in (7) are sufficient for equality in (3) since $K_0 = \lambda K_1$ implies $\sigma(K_0^*) = \lambda^{n-\nu}\sigma(K_1^*)$. This establishes (2).

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