Pacific Journal of Mathematics

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Vol. 11, No. 4 1961

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1. In [2] the notion of p-dot means of two convex bodies in Euclidean n-space was introduced and certain properties of these means investigated. For $p = 1$, the mean is more appropriately called the harmonic mean; here we restrict the discussion to this case. The harmonic mean of two convex bodies K_0 and K_1, which will always be assumed to share a common interior point Q, is defined as follows. Let $\hat{K}$ denote the polar reciprocal of K with respect to the unit sphere E centred at Q; let $(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1$, with $0 \leqq \vartheta \leqq 1$, be the usual arithmetic or Minkowski mean of $\hat{K}_0$ and $\hat{K}_1$. The harmonic mean of K_0, K_1 is the convex body $[(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge$. In more analytic terms, if $F_i(x)$ are the distance functions with respect to Q of K_i, for $i = 0, 1$, then the body whose distance function with respect to Q is $(1-\vartheta)F_0(x) + \vartheta F_1(x)$ is the harmonic mean of K_0 and K_1.

In the paper mentioned, a dual Brunn-Minkowski theorem was established, namely

$$(1) \qquad V^{1/n}([(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) \leqq 1\Big/\left[\frac{(1 - \vartheta)}{V^{1/n}(K_0)} + \frac{\vartheta}{V^{1/n}(K_1)}\right]$$

where $V(K)$ means the volume of K. There is equality if and only if K_0 and K_1 are homothetic with the centre of magnification at Q.

Here we develop a more inclusive theorem regarding the behaviour of each mean cross-section measure, ("Quermassintegral") $W_\nu(K)$, $\nu = 0, 1, \cdots, n - 1$, cf. [1]. The result is

$$(2) \qquad W_\nu^{1/(n-\nu)}([(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) \leqq 1\Big/\left[\frac{(1 - \vartheta)}{W_\nu^{1/(n-\nu)}(K_0)} + \frac{\vartheta}{W_\nu^{1/(n-\nu)}(K_1)}\right].$$

The cases of equality are just those of the dual Brunn-Minkowski theorem, $(\nu = 0)$.

2. We first list some preliminary items used in the proof of (2). We shall use Minkowski's inequality in the form

$$(3) \qquad \int[(1 - \vartheta)f_0^p + \vartheta f_1^p]^{1/p}dx \leqq \left[(1 - \vartheta)\left(\int f_0 dx\right)^p + \vartheta\left(\int f_1 dx\right)^p\right]^{1/p}.$$

Here the functions f_i are assumed to be positive and continuous over the closed and bounded domain of integration common to all the integrals,

Received September 29, 1960.

and, for our puposes, p satisfies $-1 \leqq p < 0$. There is equality if and only if $f_0(x) \equiv \lambda f_1(x)$ for some constant λ. See [3], Theorem 201, coupled with the remark preceding Theorem 200.

Our second tool, which we shall refer to as the projection lemma, was established in [2]. Let K^* denote the projection of K onto a fixed, m-dimensional, linear subspace E_m through Q for $1 \leqq m < n$. We have

$$(4) \qquad [(1-\vartheta)\hat{K}_0^* + \vartheta\hat{K}_1^*]^\wedge \supseteqq \{[(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge\}^* .$$

Since E_m contains Q and the polar reciprocation is with respect to sphere E centred at Q, in forming $\hat{K}^*$ the order of operations is immaterial. This result is proved by a polar reciprocation argument from

$$(1-\vartheta)(\hat{K}_0 \cap E_m) + \vartheta(\hat{K}_1 \cap E_m) \subseteqq [(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1] \cap E_m .$$

There is equality in either inclusion if K_0 and K_1 are homothetic with centre of magnification at Q.

The dual Brunn-Minkowski theorem (1) will be used.

Finally we shall make use of Kubota's formula and some of its consequences. This material is covered in [1]. An $(n-\nu)$ dimensional cross-section measure ("Quermass") of K is the $(n-\nu)$ dimensional volume of that convex body which is the vertical projection of K onto an $E_{n-\nu}$. The mean cross-section measures are usually defined as the coefficients in Steiner's polynomial which describes $V(K+\lambda E)$, that is

$$(5) \qquad V(K+\lambda E) = \sum_{\nu=0}^{n} \binom{n}{\nu} W_\nu(K)\lambda^\nu .$$

If we denote the $(\nu-1)^{\text{th}}$ mean cross-section measure of the projection of K onto that E_{n-1} through Q which is orthogonal to the vector u_1 by $W'_{\nu-1}(K, u_1)$, then Kubota's formula is

$$W_\nu(K) = \frac{1}{\kappa_{n-1}} \int_{\Omega_n} W'_{\nu-1}(K, u_1) d\omega_n , \qquad \nu = 1, 2, \cdots, \nu - 1 .$$

Here the integration with respect to the direction u_1 is extended over the surface Ω_n of E, $d\omega_n$ is the element of surface area on Ω_n and κ_{n-1} is the volume of the $n-1$ dimensional unit sphere.

Kubota's formula can be applied to the mean cross-section measure $W'_{\nu-1}(K, u_1)$ for fixed u_1:

$$W'_{\nu-1}(K, u_1) = \frac{1}{\kappa_{n-2}} \int_{\Omega_{n-1}} W''_{\nu-2}(K, u_1, u_2) d\omega_{n-1}$$

where $W''_{\nu-2}$ is the $(\nu-2)$th mean cross-section measure of the projection of κ onto the E_{n-2} through Q orthogonal to u_1 and u_2 with u_2 orthogonal to u_1. After ν such steps we have as the extended form of Kubota's formula:

$$W_\nu(K) = \frac{1}{\kappa_{n-1}\kappa_{n-2}\cdots\kappa_{n-\nu}}\int_{\Omega_n}\int_{\Omega_{n-1}}\cdots\int_{\Omega_{n-\nu}} W_0^{(\nu)}(K, u_1, u_2, \cdots, u_\nu)d\omega_{n-\nu}\cdots d\omega_{n-1}d\omega_n .$$

Each vector u_p is orthogonal to u_q for $q < p$ and $W_0^{(\nu)}(K, u_1, u_2, \cdots, u_\nu)$ is the 0th mean cross-section measure of the projection of K onto that $E_{n-\nu}$ through Q which is the orthogonal complement of the subspace spanned by $u_1, u_2, \cdots, u_\nu$.

Steiner's formula (5) with $\lambda = 0$ shows that $W_0(K)$ is the volume of K and so $W_0^{(\nu)}$ is an $(n-\nu)$ dimensional cross-section measure of K. Thus, to within a numerical factor depending on n and ν, $W_\nu(K)$ is the arithmetic mean of the $(n-\nu)$ dimensional cross-section measures.

In § 3 we shall use the following abbreviations: for $d\omega_{n-\nu}\cdots d\omega_{n-1}d\omega_n$ we write $d\bar{\omega}$ with sign of integration and omit reference to the domains of integration; for one $1/\kappa_{n-1}\kappa_{n-2}\cdots\kappa_{n-\nu}$ we write k; finally for $W_0^{(\nu)}(K, u_1, u_2, \cdots, u_\nu)$ we write $\sigma(K^*)$. In this notation the extended Kubota formula reads

$$W(K) = k\int\sigma(K^*)d\bar{\omega} .$$

3. We now prove (2). By the extended form of Kubota's formula

$$\begin{aligned} W_\nu^{1/(n-\nu)}([(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) &= \left[k\int\sigma(\{[(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge\}^*)d\bar{\omega}\right]^{1/(n-\nu)} \\ &\leqq \left[k\int\sigma([(1-\vartheta)\hat{K}_0^* + \vartheta\hat{K}_1^*]^\wedge)d\bar{\omega}\right]^{1/(n-\nu)} \end{aligned} \tag{6}$$

in virtue of the projection lemma and the set monotonicity of σ i.e., $\sigma(K^*) \leqq \sigma(\bar{K}^*)$ if $K^* \subseteqq \bar{K}^*$ with equality in the latter relation implying that in the former. We now apply (1), in $E_{n-\nu}$, to the integrand to obtain

$$\sigma([(1-\vartheta)\hat{K}_0^* + \vartheta\hat{K}_1^*]^\wedge) \leqq \left\{1\Big/\left[\frac{(1-\vartheta)}{\sigma^{1/(n-\nu)}(K_0^*)} + \frac{\vartheta}{\sigma^{1/(n-\nu)}(K_1^*)}\right]\right\}^{(n-\nu)} .$$

Here we take advantage of the fact that

$$(\hat{K})^* = (K^*)^\wedge .$$

This gives

$$\begin{aligned} &W_\nu^{1/(n-\nu)}([(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) \\ &\leqq \left[k\int\left\{1\Big/\left[\frac{(1-\vartheta)}{\sigma^{1/(n-\nu)}(K_0^*)} + \frac{\vartheta}{\sigma^{1/(n-\nu)}(K_1^*)}\right]\right\}^{(n-\nu)} d\bar{\omega}\right]^{1/(n-\nu)} . \end{aligned} \tag{7}$$

There is equality if and only if all the projections K_0^* and K_1^* are homothetic with the centre of magnification at Q. This condition is sufficient for equality in (6); it is necessary and sufficient for (7).

We now use Minkowski's inequality (3) with $p = -1/n-\nu$. This yields

$$\begin{aligned} &W_\nu^{1/(n-\nu)}(([1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1) \\ &\leqq 1\Big/\left[\frac{(1-\vartheta)}{\left(k\int\sigma(K_0^*)d\bar{\omega}\right)^{1/(n-\nu)}} + \frac{\vartheta}{\left(k\int\sigma(K_1^*)d\bar{\omega}\right)^{1/(n-\nu)}}\right] \\ &= 1\Big/\left[\frac{(1-\vartheta)}{W_\nu^{1/(n-\nu)}(K_0)} + \frac{\vartheta}{W_\nu^{1/(n-\nu)}(K_1)}\right]. \end{aligned}$$

The necessary and sufficient conditions for equality in (7) are sufficient for equality in (3) since $K_0 = \lambda K_1$ implies $\sigma(K_0^*) = \lambda^{n-\nu}\sigma(K_1^*)$. This establishes (2).

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Reprinted 1966 in the United States of America

Pacific Journal of Mathematics
Vol. 11, No. 4 , 1961