MEAN CROSS-SECTION MEASURES OF HARMONIC MEANS OF CONVEX BODIES

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1. In [2] the notion of p-dot means of two convex bodies in Euclidean n-space was introduced and certain properties of these means investigated. For \( p = 1 \), the mean is more appropriately called the harmonic mean; here we restrict the discussion to this case. The harmonic mean of two convex bodies \( K_0 \) and \( K_1 \), which will always be assumed to share a common interior point \( Q \), is defined as follows. Let \( \tilde{K} \) denote the polar reciprocal of \( K \) with respect to the unit sphere \( E \) centred at \( Q \); let \( (1 - \vartheta)\tilde{K}_0 + \vartheta\tilde{K}_1 \), with \( 0 \leq \vartheta \leq 1 \), be the usual arithmetic or Minkowski mean of \( \tilde{K}_0 \) and \( \tilde{K}_1 \). The harmonic mean of \( K_0, K_1 \) is the convex body \( [(1 - \vartheta)\tilde{K}_0 + \vartheta\tilde{K}_1]^\ast \). In more analytic terms, if \( F_i(x) \) are the distance functions with respect to \( Q \) of \( K_i \), for \( i = 0,1 \), then the body whose distance function with respect to \( Q \) is \( (1 - \vartheta)F_0(x) + \vartheta F_1(x) \) is the harmonic mean of \( K_0 \) and \( K_1 \).

In the paper mentioned, a dual Brunn-Minkowski theorem was established, namely

\[
V^{1/n}([(1 - \vartheta)\tilde{K}_0 + \vartheta\tilde{K}_1]^\ast) \leq 1\left[ \frac{(1 - \vartheta)}{V^{1/n}(K_0)} + \frac{\vartheta}{V^{1/n}(K_1)} \right]
\]

where \( V(K) \) means the volume of \( K \). There is equality if and only if \( K_0 \) and \( K_1 \) are homothetic with the centre of magnification at \( Q \).

Here we develop a more inclusive theorem regarding the behaviour of each mean cross-section measure, ("Quermassintegral") \( W_{\nu}(K), \nu = 0, 1, \ldots, n - 1 \), cf. [1]. The result is

\[
W^{1/(n-\nu)}_{\nu}([(1 - \vartheta)\tilde{K}_0 + \vartheta\tilde{K}_1]^\ast) \leq 1\left[ \frac{(1 - \vartheta)}{W^{1/(n-\nu)}_{\nu}(K_0)} + \frac{\vartheta}{W^{1/(n-\nu)}_{\nu}(K_1)} \right].
\]

The cases of equality are just those of the dual Brunn-Minkowski theorem, (\( \nu = 0 \)).

2. We first list some preliminary items used in the proof of (2).

We shall use Minkowski's inequality in the form

\[
\left[ (1 - \vartheta)f_0^\vartheta + \vartheta f_1^\vartheta \right]^{1/p} dx \leq (1 - \vartheta) \left( \int f_0 dx \right)^p + \vartheta \left( \int f_1 dx \right)^p \right]^{1/p}.
\]

Here the functions \( f_i \) are assumed to be positive and continuous over the closed and bounded domain of integration common to all the integrals,
and, for our purposes, $p$ satisfies $-1 \leq p < 0$. There is equality if and only if $f_0(x) = \lambda f_1(x)$ for some constant $\lambda$. See [3], Theorem 201, coupled with the remark preceding Theorem 200.

Our second tool, which we shall refer to as the projection lemma, was established in [2]. Let $K^*$ denote the projection of $K$ onto a fixed, $m$-dimensional, linear subspace $E_m$ through $Q$ for $1 \leq m < n$. We have

$$[(1 - \vartheta)\hat{K}_0^* + \vartheta\hat{K}_1^*] \supseteq [(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1] \cap E_m.$$  

Since $E_m$ contains $Q$ and the polar reciprocation is with respect to sphere $E$ centred at $Q$, in forming $\hat{K}^*$ the order of operations is immaterial. This result is proved by a polar reciprocation argument from

$$(1 - \vartheta)(K_0 \cap E_m) + \vartheta(K_1 \cap E_m) \subseteq [(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1] \cap E_m.$$  

There is equality in either inclusion if $K_0$ and $K_1$ are homothetic with centre of magnification at $Q$.

The dual Brunn-Minkowski theorem (1) will be used.

Finally we shall make use of Kubota's formula and some of its consequences. This material is covered in [1]. An $(n - \nu)$ dimensional cross-section measure ("Quermass") of $K$ is the $(n - \nu)$ dimensional volume of that convex body which is the vertical projection of $K$ onto an $E_{n-\nu}$. The mean cross-section measures are usually defined as the coefficients in Steiner's polynomial which describes $V(K + \lambda E)$, that is

$$V(K + \lambda E) = \sum_{\nu=0}^{n} \binom{n}{\nu} W_\nu(K\lambda^\nu).$$

If we denote the $(\nu - 1)^{th}$ mean cross-section measure of the projection of $K$ onto that $E_{n-1}$ through $Q$ which is orthogonal to the vector $u_1$ by $W_{\nu-1}(K, u_1)$, then Kubota's formula is

$$W_\nu(K) = \frac{1}{\kappa_{n-1}} \int_{\Omega_n} W_{\nu-1}(K, u_1)d\omega_n, \quad \nu = 1, 2, \ldots, n - 1.$$  

Here the integration with respect to the direction $u_1$ is extended over the surface $\Omega_n$ of $E$, $d\omega_n$ is the element of surface area on $\Omega_n$ and $\kappa_{n-1}$ is the volume of the $n - 1$ dimensional unit sphere.

Kubota's formula can be applied to the mean cross-section measure $W_{\nu-1}(K, u_1)$ for fixed $u_1$:

$$W'_{\nu-1}(K, u_1) = \frac{1}{\kappa_{n-2}} \int_{\Omega_{n-1}} W''_{\nu-2}(K, u_1, u_2)d\omega_{n-1}$$

where $W''_{\nu-2}$ is the $(\nu - 2)^{th}$ mean cross-section measure of the projection of $K$ onto the $E_{n-2}$ through $Q$ orthogonal to $u_1$ and $u_2$ with $u_2$ orthogonal to $u_1$. After $\nu$ such steps we have as the extended form of Kubota's formula:
Each vector $u_p$ is orthogonal to $u_q$ for $q < p$ and $W^{(v)}(K, u_1, u_2, \ldots, u_v)$ is the $0$th mean cross-section measure of the projection of $K$ onto that $E_{n-v}$ through $Q$ which is the orthogonal complement of the subspace spanned by $u_1, u_2, \ldots, u_v$.

Steiner's formula (5) with $\lambda = 0$ shows that $W_0(K)$ is the volume of $K$ and so $W^{(v)}_0$ is an $(n - v)$ dimensional cross-section measure of $K$. Thus, to within a numerical factor depending on $n$ and $v$, $W_0(K)$ is the arithmetic mean of the $(n - v)$ dimensional cross-section measures.

In § 3 we shall use the following abbreviations: for $d\omega_{n-v} \cdots d\omega_{n-1}d\omega_n$ we write $d\bar{\omega}$ with sign of integration and omit reference to the domains of integration; for one $1/\kappa_{n-1} \kappa_{n-2} \cdots \kappa_{n-v}$ we write $k$; finally for $W^{(v)}(K, u_1, u_2, \ldots, u_v)$ we write $\sigma(K^*)$. In this notation the extended Kubota formula reads

$$W(K) = k\int \sigma(K^*)d\bar{\omega}.$$

3. We now prove (2). By the extended form of Kubota's formula

$$W^{(1/(n-v))}_v([(1 - \partial)\hat{K}_0 + \partial \hat{K}_1]^*) = \left[k \int \sigma([(1 - \partial)\hat{K}_0 + \partial \hat{K}_1]^*)d\bar{\omega}\right]^{1/(n-v)}$$

(6)

in virtue of the projection lemma and the set monotonicity of $\sigma$ i.e., $\sigma(K^*) \leq \sigma(\tilde{K}^*)$ if $K^* \subseteq \tilde{K}^*$ with equality in the latter relation implying that in the former. We now apply (1), in $E_{n-v}$, to the integrand to obtain

$$\sigma([(1 - \partial)\hat{K}_0^* + \partial \hat{K}_1^*]^*) \leq \left\{1/ \left[1 - \frac{(1 - \partial)}{\sigma^{1/(n-v)}(K_0^*)} + \frac{\partial}{\sigma^{1/(n-v)}(K_1^*)}\right]\right\}^{(n-v)}.$$

Here we take advantage of the fact that

$$(\hat{K})^* = (K^*)^*.$$

This gives

$$W^{(1/(n-v))}_v([(1 - \partial)\hat{K}_0 + \partial \hat{K}_1]^*)$$

$$\leq \left[k \int \left\{1/ \left[1 - \frac{(1 - \partial)}{\sigma^{1/(n-v)}(K_0^*)} + \frac{\partial}{\sigma^{1/(n-v)}(K_1^*)}\right]\right\}^{(n-v)}d\bar{\omega}\right]^{1/(n-v)}.$$

(7)

There is equality if and only if all the projections $K_0^*$ and $K_1^*$ are homothetic with the centre of magnification at $Q$. This condition is
sufficient for equality in (6); it is necessary and sufficient for (7).

We now use Minkowski’s inequality (3) with \( p = -1/n - \nu \). This yields

\[
W_\nu^{1/(n-\nu)}((1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1) \\
\leq 1 \left[ \frac{(1 - \vartheta)}{W_\nu^{1/(n-\nu)}(K_0)} + \frac{\vartheta}{W_\nu^{1/(n-\nu)}(K_1)} \right].
\]

The necessary and sufficient conditions for equality in (7) are sufficient for equality in (3) since \( K_0 = \lambda K_1 \) implies \( \sigma(K_0^*) = \lambda^{n-\nu} \sigma(K_1^*) \). This establishes (2).

REFERENCES


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