

# Pacific Journal of Mathematics

**BASES OF TENSOR PRODUCTS OF BANACH SPACES**

BERNARD RUSSEL GELBAUM AND JESUS GIL DE LAMADRID

# BASES OF TENSOR PRODUCTS OF BANACH SPACES

B. R. GELBAUM AND J. GIL DE LAMADRID

**1. Introduction.** In this note we use the conventions and notations of Schatten [4] with the exception that we use  $B'$  to indicate the dual (conjugate) space of a Banach space  $B$  and  $\langle x, x' \rangle$  as the action of an element  $x$  and a functional  $x'$  on each other. Schatten defines the tensor product  $B_1 \otimes_{\alpha} B_2$  as the completion of the algebraic tensor product  $B_1 \otimes B_2$  of two Banach spaces  $B_1$  and  $B_2$ , on which the cross norm  $\alpha$  has been imposed. We discuss the proposition, "If  $B_1$  and  $B_2$  have Schauder bases, then  $B_1 \otimes_{\alpha} B_2$  has a Schauder basis." We prove this for  $\alpha = \gamma$  ( $B_1 \otimes_{\gamma} B_2$  is the trace class of transformations of  $B_1'$  into  $B_2$ ). We also prove it for  $\alpha = \lambda$  ( $B_1 \otimes_{\lambda} B_2$  is the class of all completely continuous linear transformations of  $B_1'$  into  $B_2$ ) in the case in which the bases of  $B_1$  and  $B_2$  satisfy an "isometry condition". This condition is not very restrictive. We know of no instance in which it is not satisfied. Next we show that unconditional bases of  $B_1$  and  $B_2$  do not necessarily yield an unconditional basis for the tensor product, even in the nicest conceivable infinite dimensional case, that in which  $B_1 = B_2 =$  Hilbert space, and the bases are orthonormal and identical.

We recall certain facts about Schauder bases, and set some general notation that we use throughout the paper. We usually work with a biorthogonal set  $\Omega = \{x_i, x'_i\}_i$  associated with a Banach space  $B$ , so that  $\chi = \{x_i\}_i$  is a basis for  $B$  with coefficients supplied by the corresponding sequence of functionals  $\chi' = \{x'_i\}_i$ . We will have to do with the closed linear manifold  $B^{\circ}$  of  $B'$  generated by the elements of  $\chi'$ . Since  $B$  and  $B^{\circ}$  are in duality it is possible to embed  $B$  in  $(B^{\circ})'$  by the same formula that effects the embedding of  $B$  in  $B''$ . We denote by  ${}_n P_m$  the projection of  $B$  defined by  ${}_n P_m x = \sum_{i=n}^m \langle x, x'_i \rangle x_i$ . The double sequence  $\{{}_n P_m\}_{n,m}$  is uniformly bounded. We denote by  $T'$  the transpose of any transformation  $T$ . The following lemma, given without proof, is but a trivial strengthening of [2, p. 18, Theorem 1].

**LEMMA 1.** *Let  $E$  be a dense vector subspace of  $B$ ,  $\Omega$  a biorthogonal set of  $B$  such that  $\chi \subset E$ , the vector space spanned by  $\chi$  is dense in  $E$  and the sequence  $\{{}_n P_m\}_{n,m}$  is uniformly bounded on  $E$ . Then  $\Omega$  defines a basis for  $B$ .*

**2. The tensor product of two biorthogonal sets.** Let  $\Omega_1 = \{x_i, x'_i\}_i$  be a biorthogonal set of  $B_1$  and  $\Omega_2 = \{y_i, y'_i\}_i$  a biorthogonal set of  $B_2$ .

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The elements  $x'_i \otimes y'_j$  can be considered as belonging to  $(B_1 \otimes_{\alpha} B_2)'$  for any cross norm  $\alpha$  [4 p. 43], and  $\{x_i \otimes y_j, x'_i \otimes y'_j\}_{i,j}$  is clearly a biorthogonal set. We enumerate it, not by the diagonal method, i.e., as in the usual proof that the rationals are denumerable, but as follows: In the table

$$\begin{array}{ccccccc}
 x_1 \otimes y_1 & x_1 \otimes y_2 & x_1 \otimes y_3 & \cdots \cdots & & & \\
 x_2 \otimes y_1 & x_2 \otimes y_2 & x_2 \otimes y_3 & \cdots \cdots & & & \\
 x_3 \otimes y_1 & x_3 \otimes y_2 & x_3 \otimes y_3 & \cdots \cdots & & & \\
 \cdots \cdots \cdots & \cdots \cdots \cdots & \cdots \cdots \cdots & \cdots \cdots \cdots & & & 
 \end{array}$$

we simply order the elements by listing the entries on the two inner sides of each successive upper left hand block to obtain  $x_1 \otimes y_1, x_1 \otimes y_2, x_2 \otimes y_2, x_2 \otimes y_1, x_1 \otimes y_3, x_2 \otimes y_3, x_3 \otimes y_3, x_3 \otimes y_2, x_3 \otimes y_1, \dots, x_1 \otimes y_k, x_2 \otimes y_k \cdots x_k \otimes y_k, x_k \otimes y_{k-1}, \dots, x_k \otimes y_2, x_k \otimes y_1, \dots$ . This double sequence with the given order is called the tensor product of  $\chi_1 = \{x_i\}_i$  and  $\chi_2 = \{y_j\}_j$  and is denoted by  $\chi_1 \otimes \chi_2$ . Similarly  $\chi'_1 \otimes \chi'_2$  denotes the set  $\{x'_i \otimes y'_j\}_{i,j}$  with the corresponding order. The biorthogonal set formed by  $\chi_1 \otimes \chi_2$  and  $\chi'_1 \otimes \chi'_2$  is called the tensor product of  $\Omega_1$  and  $\Omega_2$  and denoted by  $\Omega_1 \otimes \Omega_2$ .

**THEOREM 1.** *If  $\Omega_1$  defines a basis for  $B_1$  and  $\Omega_2$  defines a basis for  $B_2$ , then  $\Omega_1 \otimes \Omega_2$  defines a basis for  $B_1 \otimes_{\gamma} B_2$ .*

*Proof.* We show that the vector space spanned by  $\chi_1 \otimes \chi_2$  is dense in  $B_1 \otimes B_2$ . To see this let  ${}_n P_m^i$  be the  ${}_n P_m$  defined in § 1 for  $\Omega_i$ , and define

$$\begin{aligned}
 (1) \quad A_m &= x \otimes y - \sum_{k,j=1}^m \langle x, x'_k \rangle \langle y, y'_j \rangle x_k \otimes y_j = x \otimes y - [{}_1 P_m^1 x] \otimes [{}_1 P_m^2 y] \\
 &= x \otimes [y - {}_1 P_m^2 y] + [x - {}_1 P_m^1 x] \otimes {}_1 P_m^2 y .
 \end{aligned}$$

Then

$$(2) \quad \gamma(A_m) \leq \|x\| \|y - {}_1 P_m^2 y\| + \|x - {}_1 P_m^1 x\| \|{}_1 P_m^2 y\| .$$

The right hand side of (2) tends to zero with  $m^{-1}$ . This argument extends by linearity to sums of elements of the form  $x \otimes y$ .

Let now  $T_q$  be the  ${}_1 P_q$  defined in § 1 corresponding to  $\Omega_1 \otimes \Omega_2$ . It remains to show that  $\{T_q\}_q$  is uniformly bounded. It is easy to show that each  $T_q$  has one of the following three forms:  ${}_1 P_n^1 \otimes {}_1 P_n^2, {}_1 P_n^1 \otimes {}_1 P_n^2 + {}_{n+1} P_{n+1}^1 \otimes {}_1 P_m^2, {}_1 P_m^1 \otimes {}_{n+1} P_{n+1}^2$ . Hence, it suffices to show that  $\{{}_n P_m^1 \otimes {}_q P_r^2\}_{q,r,n,m}$  is uniformly bounded. Let  $M$  be a common bound for all  ${}_n P_m^1$  and  ${}_q P_r^2$ . For  $\Sigma x \otimes y \in B_1 \otimes B_2$

$$\begin{aligned}
 (3) \quad \gamma[{}_n P_m^1 \otimes {}_q P_r^2 (\Sigma x \otimes y)] &= \gamma[\Sigma ({}_n P_m^1 x) \otimes ({}_q P_r^2 y)] \\
 &\leq (\Sigma \|x\| \|y\|) M^2 .
 \end{aligned}$$

Since (3) holds for any representation  $\Sigma x \otimes y$  of a given tensor product element, we may replace in it the sum  $\Sigma \|x\| \|y\|$  by  $\gamma(\Sigma x \otimes y)$ , thereby proving our assertion. From Lemma 1, we can conclude that  $\Omega_1 \otimes \Omega_2$  defines a basis for  $B_1 \otimes_{\gamma} B_2$ .

**3. The space of completely continuous transformations.** We recall that there is a canonical imbedding of  $B$ , with a biorthogonal set  $\Omega$  defining a basis of  $B$ , into  $(B^{\alpha})'$ . The norm of the image of an element  $x \in B$  is less than or equal to  $\|x\|$ . We say that  $\Omega$  satisfies the condition of isometry if the imbedding is actually an isometry. For such an  $\Omega$ ,  $(B^{\alpha})^{\alpha} = B$ , isometrically. We state first the following corollary of Theorem 1.

**COROLLARY 1.** *If  $\Omega_k$  is a biorthogonal set defining a basis for  $B_k, k = 1, 2$ , then  $\Omega_1 \otimes \Omega_2$  defines a basis for  $B_1^{\alpha_1} \otimes_{\lambda} B_2^{\alpha_2}$ .*

*Proof.* Each  $x'_i \otimes y'_j$  is an element of  $B_1^{\alpha_1} \otimes B_2^{\alpha_2}$  which, as a subset of  $B'_1 \otimes_{\lambda} B'_2$ , can be imbedded isometrically in  $(B_1 \otimes_{\gamma} B_2)'$  [4, p. 47, Theorem 3.2]. What is more, the vector space spanned by  $\{x'_i \otimes y'_j\}_{i,j}$  is dense, with respect to  $\lambda$ , in  $B_1^{\alpha_1} \otimes B_2^{\alpha_2}$ , hence in  $B_1^{\alpha_1} \otimes_{\lambda} B_2^{\alpha_2}$ . This is true because

$$(4) \quad \lambda[x' \otimes y' - (\sum_{i=1}^n \langle x_i, x' \rangle x'_i) \otimes (\sum_{j=1}^n \langle y_j, y' \rangle y'_j)] \leq \gamma[x' \otimes y' - (\sum_{i=1}^n \langle x_i, x' \rangle x'_i) \otimes (\sum_{j=1}^n \langle y_j, y' \rangle y'_j)]$$

and the latter quantity tends to 0. Hence  $B_1^{\alpha_1} \otimes_{\lambda} B_2^{\alpha_2} = (B_1 \otimes_{\gamma} B_2)^{\alpha_1 \otimes \alpha_2}$ . Our result is a consequence of this.

The next theorem follows easily from this corollary.

**THEOREM 2.** *If both  $\Omega_1$  and  $\Omega_2$  satisfy the condition of isometry  $\Omega_1 \otimes \Omega_2$  defines a basis for  $B_1 \otimes_{\lambda} B_2$ .*

*Proof.* If in Corollary 1 we replace  $B_1$  by  $B_1^{\alpha_1}$  and  $B_2$  by  $B_2^{\alpha_2}$ , we conclude that  $\Omega_1 \otimes \Omega_2$  defines a basis for  $(B_1^{\alpha_1})^{\alpha_1} \otimes_{\lambda} (B_2^{\alpha_2})^{\alpha_2}$ . When the condition of isometry is satisfied the last tensor product can be identified with  $B_1 \otimes_{\lambda} B_2$ , owing to the relations  $B_k = (B_k^{\alpha_k})^{\alpha_k}$  for  $k = 1, 2$ , and the universal character of  $\lambda$ , [4, p. 35, Lemma 2.12].

Theorem 2 can be considered as a sharpening of the well known fact that if  $B_1$  and  $B_2$  have bases, then every completely continuous linear transformation of  $B'_1$  into  $B_2$  can be uniformly approximated by finite dimensional linear transformations. Our theorem goes further to state that if  $\Omega_1$  and  $\Omega_2$  satisfy the condition of isometry, the space of all completely continuous linear transformations of  $B'_1$  into  $B_2$  has a

basis consisting of one-dimensional linear transformations.

The condition of isometry deserves some explanation. It is satisfied by a large class of bases, which includes every base for which

$$(5) \quad B^a = B' .^{(1)}$$

The equation (5) holds always for reflexive spaces. It also holds for certain bases of non-reflexive spaces.

A non-reflexive example of (5) is exhibited in [2, p.188, Example 1], involving the usual basis of  $c_0$ ,  $x_i = \{\delta_j^i\}_j$ , with  $x'_i = \{\delta_j^i\}_j \in l'$ . An example of the condition of isometry, in the absence (5), is obtained from this first example, by setting [2, p.188, Example 2]  $y_1 = x_1$ , and  $y_i = x_i - x_{i-1} + \dots + (-1)^{i-1}x_1$ , for  $i > 1$ , and  $y'_i = x'_i + x'_{i+1}$ . For  $\Omega = \{y_i, y'_i\}_i$ ,  $x'_i \in B' \setminus B^a$ .  $\Omega$  satisfies the condition of isometry for, if  $x \in c_0$ , then

$$\left\| \sum_{k=1}^n \langle x, y'_k \rangle y_k \right\| = \left\| \sum_{k=1}^n \langle x, x'_k \rangle x_k \right\| \leq \|x\|.$$

The conclusion is now a consequence of the following theorem and its corollary.

**THEOREM 3.** *If for every  $x' \in B'$ ,  $\|{}_1P'_n x'\| \rightarrow \|x'\|$ , then  $\Omega$  satisfies the condition of isometry.*

*Proof.* Let  $x_0 \in B$  and  $x'_0 \in B'$  such that  $\|x'_0\| = 1$  and  $\langle x_0, x'_0 \rangle = \|x_0\|$ . Then

$$\lim_{n \rightarrow \infty} \frac{\langle x_0, {}_1P'_n x'_0 \rangle}{\|{}_1P'_n x'_0\|} = \|x_0\|, \quad \text{Q.E.D.}$$

**COROLLARY 2.** *If  $\|{}_1P_n\| \leq 1$  for every  $n$ , then  $\Omega$  satisfies the condition of isometry.*

*Proof.* We show the above hypothesis implies the hypothesis of Theorem 3. To see this, let  $x'_0 \in B'$ , and  $\varepsilon > 0$ . There is  $x_0 \in B$  so that  $\|x_0\| = 1$  and  $\langle x_0, x'_0 \rangle > \|x_0\| - \varepsilon/2$  and an integer  $N > 0$  so that

$$\begin{aligned} \|x'_0\| &\geq \|{}_1P'_n x'_0\| \geq \langle x_0, {}_1P'_n x'_0 \rangle = \langle {}_1P_n x_0, x'_0 \rangle > \langle x_0, x'_0 \rangle - \varepsilon/2 \\ &> \|x'_0\| - \varepsilon, \end{aligned} \quad \text{Q.E.D.}$$

As we have seen, the two biorthogonal sets described above for  $c_0$  satisfy the hypothesis of Corollary 1.

An example of the isometry condition in which  $B'$  is not separable is furnished by Schauder's basis for  $C([0, 1])$ , given by the biorthogonal system  $\Omega = \{x_i, x'_i\}_i$  described in [1, p. 69]. We consider  $[0, 1]$  imbedded

<sup>1</sup> This equation may be described by saying that  $\{x'_i\}_i$  is a *retrobasis* for  $B'$ , [2, p.188, Definition 1].

in  $B'$  and treat its points as functionals. The space  $B^2$  of this example contains the set  $D$  of all dyadic fractions. Consequently  $\Omega$  satisfies the condition of isometry, since, for  $f \in B$ ,  $\|f\| = \sup_{d \in D} |f(d)|$ .

We know of no biorthogonal set defining a basis which does not satisfy the condition of isometry. Neither do we know if  $B_1 \otimes_a B_2$  has a basis for an arbitrary cross norm  $\alpha$ , even if  $B_1$  and  $B_2$  have bases. It is clear that for any element of  $B_1 \otimes B_2$ , the formal expansion of Theorem 1 converges to that element with respect to  $\alpha$ , since it does with respect to  $\gamma \geq \alpha$ . The difficulty lies in establishing that the set  $\{ {}_p P_q^1 \otimes {}_r P_s^2 \}_{p,q,r,s}$  is uniformly bounded with respect to  $\alpha$ .

**4. Hilbert spaces and unconditional bases.** The problem of approximation of compact operators by finite dimensional operators in a Banach space, can, after elaborate rearrangement, lead to the following question: Can there exist a matrix  $C = (c_{ij})_{i,j=1}^\infty$  satisfying the following conditions:

- (a) For some  $a_i \geq 0$ ,  $\sum_{i=1}^\infty a_i^2 < \infty$ ,  $|c_{ij}| \leq a_i a_j$ ;
- (b)  $C^2 = 0$ ;
- (c)  $\sum_{i=1}^\infty c_{ii} = 1$ ?

Of course, (b) and (c) are incompatible if  $C$  is in the trace class. Thus there arises the question: Does (a) imply that  $C$  is in the trace class? To this we can give a definite negative answer via the following theorems.

**Theorem 4.** *Let  $\Omega = \{x_i, x'_i\}_i$ ,  $x_i = \{\delta_i^i\}_j$ ,  $x'_i = \{\delta_i^j\}_j$  be the canonical orthonormal basis in  $l_2$ . Then  $\Omega \otimes \Omega$  defines an unconditional basis in  $l_2 \otimes_\gamma l_2$  if and only if condition (a) implies  $C$  is in the trace class.*

*Proof.* Let  $\Omega \otimes \Omega$  define an unconditional basis for  $l_2 \otimes_\gamma l_2$ . Then we note that (a) may be rephrased by stating:  $c_{ij} = \varepsilon_{ij} a_i a_j$ ,  $|\varepsilon_{ij}| \leq 1$ . Since  $l_2 \otimes_\gamma l_2$  is precisely the trace class of operators [4] it follows that  $\sum_{i,j=1}^\infty \varepsilon_{ij} a_i a_j (x_i \otimes x_j)$  exists in  $l_2 \otimes_\gamma l_2$  and is therefore in the trace class.

On the other hand, if (a) implies that  $C$  is in the trace class, then for  $a \otimes a$  in  $l_2 \otimes_\gamma l_2$  ( $a = (a_1, a_2, \dots)$ ),  $a \otimes a = \sum_{i,j=1}^\infty a_i a_j (x_i \otimes x_j)$ . If  $B = (\varepsilon_{ij} a_i a_j)$  is in the trace class, then  $B$  has an expansion  $\sum_{i,j=1}^\infty \varepsilon_{ij} a_i a_j (x_i \otimes x_j)$ , which shows  $\Omega \otimes \Omega$  defines an unconditional basis for  $l_2 \otimes_\gamma l_2$ .

**THEOREM 5.**  *$\Omega \otimes \Omega$  does not define an unconditional basis for  $l_2 \otimes_\gamma l_2$ .*

*Proof.* Let  $A_1 = (a_{ij})$  be a  $2 \times 2$  matrix with  $a_{11} = a_{12} = a_{22} = -a_{21} = 1$ , and  $A_n$  the  $2^n \times 2^n$  matrix  $(A_{ij})$   $i, j = 1, 2$ , with  $A_{11} = A_{12} = A_{22} = -A_{21} = A_{n-1}$ . Let  $B$  be the direct sum of the matrices  $\{1/2^{n/2} A_n\}_n$ . Then a direct computation reveals that  $B$  is unitary. Let  $B = (b_{ij})$ , and let

$C = (|b_{ij}|)$ . If  $\Omega \otimes \Omega$  were an unconditional basis for  $l_2 \otimes_{\gamma} l_2$ , then for  $B$ , regarded as a member of  $(l_2 \otimes_{\gamma} l_2)'$  [4, p. 47, Theorem 3.2] and arbitrary  $u \otimes v$  in  $l_2 \otimes_{\gamma} l_2$ ,  $\sum_{i,j=1}^{\infty} u_i v_j \langle x_i, Bx_j \rangle$  would converge unconditionally, i.e.  $\sum_{i,j=1}^{\infty} u_i v_j |b_{ji}|$  would converge. In particular, let  $u = v$ , where  $u$  is given by the vector:  $\sum_{n=1}^{\infty} (1/n)x_n$ ,  $(\sqrt{2^n})x_n = \underbrace{(0, 0, \dots, 0,}_{2^{(2^n-1-1)}} 1, 1, \dots, 1, 0, 0, \dots)$ . A simple verification shows that  $u$  exists in  $l_2$ . On the other hand, more calculation shows  $\sum_{i,j=1}^{\infty} |b_{ij}| u_i u_j = \infty$ . The contradiction implies the theorem.

Theorem 5 remains valid when  $\gamma$  is replaced by  $\lambda$ , since  $l_2 \otimes_{\gamma} l_2 = (l_2 \otimes_{\lambda} l_2)'$ , and unconditionality of  $\Omega \otimes \Omega$  in  $l_2 \otimes_{\lambda} l_2$  implies the same in  $l_2 \otimes_{\gamma} l_2$ .

NOTE. We owe to the referee the remark that a space  $B$  with a biorthogonal set  $\Omega$  which defines a basis for  $B$  can always be renormed, preserving the topology of  $B$  [1, Theorem 1, p. 67], in such a way that  $\Omega$  satisfies the condition of isometry (section 3) with respect to the resulting norm of  $B$  and the corresponding norm of  $B'$ . This makes possible the following completely general form of Theorem 2.

**THEOREM 2'.** *If  $\Omega_i$  defines a basis for  $B_i$ , for  $i = 1, 2$ , then  $\Omega_1 \otimes \Omega_2$  defines a basis for  $B_1 \otimes_{\lambda} B_2$ .*

*Proof.* Renorm  $B_1$  and  $B_2$  as indicated above. Then, if  $\lambda'$  denotes the operator norm with respect to the new norms of  $B_1$  and  $B_2$ ,  $B_1 \otimes_{\lambda'} B_2$  has a basis defined by  $\Omega_1 \otimes \Omega_2$  (Theorem 2). But  $B_1 \otimes_{\lambda'} B_2 = B_1 \otimes_{\lambda} B_2$  both point-set-wise and topologically. Hence our conclusion.

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