

# Pacific Journal of Mathematics

**ON A THEOREM OF FEJÉR**

FU CHENG HSIANG

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1. Let

$$T: (\tau_{n\nu}) \quad (n = 0, 1, 2, \dots; \nu = 0, 1, 2, \dots)$$

be an infinite Toeplitz matrix satisfying the conditions

$$(i) \quad \lim \tau_{n\nu} = 0$$

for every fixed  $\nu$ ,

$$(ii) \quad \lim \sum_{\nu=0}^{\infty} \tau_{n\nu} = 1$$

and

$$(iii) \quad \sum_{\nu=0}^{\infty} |\tau_{n\nu}| \leq K,$$

$K$  being an absolute constant independent of  $n$ .

Given a sequence  $(S_n)$  if

$$\lim \sum_{\nu=0}^{\infty} \tau_{n\nu} S_\nu = S,$$

then we say that the sequence  $(S_n)$  or the series with partial sums  $S_n$  is summable  $(T)$  to the sum  $S$ .

2. Suppose that  $f(x)$  is integrable in the Lebesgue sense and periodic with period  $2\pi$ . Let

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Let

$$\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) = \sum B_n(x)$$

be the derived series of the Fourier series of  $f(x)$ . Fixing  $x$ , we write

$$\psi_x(t) = f(x+t) - f(x-t).$$

Fejér [1] has proved the following

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**THEOREM A.** *If  $f(x)$  is of bounded variation in  $(0, 2\pi)$ , then  $\{B_n(x)\}$  is summable  $(C, r)$  to the jump  $l(x) = \{f(x + 0) - f(x - 0)\}/\pi$  for every  $r > 0$  at each point  $x$ .*

Recently, Siddiqi [3] extended Fejér's result and established the following

**THEOREM B.** *Let  $A: (\lambda_{n\nu})$  be a triangular Toeplitz matrix, i.e.,  $\lambda_{n\nu} = 0$  for  $\nu > n$ . If it satisfies, in addition, the condition*

$$(iv) \quad \sum_{\nu=0}^n |\Delta\lambda_{n\nu}| = o(1)$$

as  $n \rightarrow \infty$ , then  $\{B_n(x)\}$  is summable  $(A)$  to  $l(x)$ .

It is known [2] that a series which is summable by the harmonic means is also summable  $(C, r)$  for every  $r > 0$  but not conversely. We take, for the  $(C, r)$  means,  $\lambda_{n\nu} = A_{n-\nu}^r/A_n^r$ ,

$$A_n^r = \Gamma(n + r + 1)/\Gamma(n + 1)\Gamma(r + 1),$$

and for the harmonic means,  $\lambda_{n\nu} = 1/(n - \nu + 1)$ . Both satisfy (iv). Thus, we infer that Siddiqi's theorem contains Fejér's as a special case.

In this note, we develop Siddiqi's theorem into the following general form for the summability  $(T)$  of  $\{B_n(x)\}$  at a given point.

**THEOREM.** *If  $\psi_x(t)$  is of bounded variation in the neighborhood of  $t = 0$  and absolutely continuous in  $(\eta, \pi)$  for any  $0 < \eta < \pi$ , then  $\{B_n(x)\}$  is summable  $(T)$  to the jump  $l(x)$  at  $x$ .*

3. Let us consider

$$\begin{aligned} \sigma_n(x) &= \sum_{\nu=0}^{\infty} \tau_{n\nu} B_{\nu}(x) \\ &= \frac{1}{\pi} \sum_{\nu=0}^{\infty} \tau_{n\nu} \int_0^{\pi} \psi_x(t) \nu \sin \nu t dt \\ &= l(x) \sum_{\nu=0}^{\infty} \tau_{n\nu} + \frac{1}{\pi} \sum_{\nu=0}^{\infty} \tau_{n\nu} \int_0^{\pi} \cos \nu t d\psi_x(t) \\ &= l(x) + o(1) + \frac{1}{\pi} \sum_{\nu=0}^{\infty} \tau_{n\nu} I_{\nu}. \end{aligned}$$

We are going to prove that  $\sum \tau_{n\nu} I_{\nu} = o(1)$  as  $n \rightarrow \infty$ . Since  $\psi_x(t)$  is of bounded variation in the neighborhood of  $t = 0$ , for a given  $\varepsilon > 0$ , we can choose  $\delta > 0$  such that

$$\int_0^{\delta} |d\psi_x(t)| < \varepsilon.$$

Write

$$\begin{aligned}
 I_\nu &= \left( \int_0^\delta + \int_\delta^\pi \right) \cos \nu t d\psi_x(t) \\
 &= I'_\nu + II''_\nu,
 \end{aligned}$$

say. Then

$$\begin{aligned}
 \left| \sum_{\nu=0}^\infty \tau_{n\nu} I'_\nu \right| &\leq \sum_{\nu=0}^\infty |\tau_{n\nu}| \int_0^\delta |d\psi_x(t)| \\
 &< \varepsilon \sum_{\nu=0}^\infty |\tau_{n\nu}| \\
 &\leq K\varepsilon.
 \end{aligned}$$

Remembering that  $\psi_x(t)$  is absolutely continuous in  $(\delta, \pi)$ , we have

$$\int_\delta^\pi \cos \nu t d\psi_x(t) = \int_\delta^\pi \cos \nu t \psi'_x(t) dt.$$

For the given  $\varepsilon > 0$ , we can find  $\nu_0$  such that

$$\left| \int_\delta^\pi \cos \nu t \psi'_x(t) dt \right| < \varepsilon$$

for  $\nu < \nu_0$  by Riemann-Lebesgue's theorem. Fixing  $\nu_0$ , we can take a positive integer  $n_0$  making  $|\tau_{n\nu}| < \varepsilon/(\nu_0 + 1)$   $0 \leq \nu \leq \nu_0$ ,  $n < n_0$ . If we write

$$\begin{aligned}
 \sum_{\nu=0}^\infty \tau_{n\nu} I''_\nu &= \left( \sum_{\nu=0}^{\nu_0} + \sum_{\nu_0+1}^\infty \right) \tau_{n\nu} \int_\delta^\pi \cos \nu t \psi'_x(t) dt \\
 &= I_1 + I_2,
 \end{aligned}$$

say, then

$$\begin{aligned}
 |I_1| &\leq \sum_{\nu=0}^{\nu_0} |\tau_{n\nu}| \int_\delta^\pi |\psi'_x(t)| dt \\
 &\leq M \sum_{\nu=0}^{\nu_0} |\tau_{n\nu}| \\
 &< M(\nu_0 + 1)/(\nu_0 + 1) \\
 &= M\varepsilon.
 \end{aligned}$$

for  $n > n_0$ , where

$$\begin{aligned}
 M &= \int_0^\pi |\psi'_x(t)| dt. \\
 |I_2| &= \left| \sum_{\nu=\nu_0+1}^\infty \tau_{n\nu} \int_\delta^\pi \cos \nu t \psi'_x(t) dt \right| \\
 &< \varepsilon \sum_{\nu=\nu_0+1}^\infty |\tau_{n\nu}|
 \end{aligned}$$

$$\begin{aligned} &\leq \varepsilon \sum_{\nu=0}^{\infty} |\tau_{n\nu}| \\ &\leq K\varepsilon \end{aligned}$$

by (iii). From the above analysis, it follows that

$$\left| \sum_{\nu=0}^{\infty} \tau_{n\nu} I_{\nu} \right| < (M + 2K)\varepsilon$$

for  $n > n_0$ . Since  $\varepsilon$  is an arbitrary quantity, we obtain  $\sum \tau_{n\nu} I_{\nu} = 0(1)$  as  $n \rightarrow \infty$ . This proves the theorem.

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