ON N-HIGH SUBGROUPS OF ABELIAN GROUPS

John McCormick Irwin and Elbert A. Walker
ON \(N\)-HIGH SUBGROUPS OF ABELIAN GROUPS

JOHN M. IRWIN AND ELBERT A. WALKER

In a recent paper [2] the concept of high subgroups of Abelian groups was discussed. The purpose of this paper is to give further results concerning these high subgroups. All groups considered in this paper are Abelian, and our notation is essentially that of L. Fuchs in [1]. Let \(N\) be a subgroup of a group \(G\). A subgroup \(H\) of \(G\) maximal with respect to disjointness from \(N\) will be called an \(N\)-high subgroup of \(G\), or \(N\)-high in \(G\). When \(N = G'\) (the subgroup of elements of infinite height in \(G\)), \(H\) will be called high in \(G\).

After considering \(N\)-high subgroups in direct sums, we give a characterization (Theorem 3) of \(N\)-high subgroups of \(G\) in terms of a divisible hull of \(G\). Next we show (Theorem 5) that if \(G\) is torsion, \(N \subseteq G'\), and \(H\) is \(N\)-high in \(G\), then \(H\) is pure and (Lemma 7) the primary components of any two \(N\)-high subgroups have the same Ulm invariants (see [3]). These results generalize the results in [2]. The concept of \(\Sigma\)-groups is introduced, and it is shown that any two high subgroups of torsion \(\Sigma\)-groups are isomorphic. Further, torsion \(\Sigma\)-groups are characterized in terms of their basic subgroups. Theorem 3 of [2] is generalized to show that high subgroups of arbitrary Abelian groups are pure. This leads to the solution of a more general version of Problem 4 of L. Fuchs in [1]. Finally, the question of whether any two high subgroups of a torsion group are isomorphic is considered, and a theorem in this direction is proved.

Preliminaries.

**Lemma 1.** Let \(M\) and \(N\) be subgroups of a primary group \(G\) such that \(M\) is neat in \(G\) and \(M[p] \oplus N[p] = G[p]\). Then \(M\) is \(N\)-high in \(G\).

*Proof.* Suppose \(M\) is not \(N\)-high in \(G\). Then there exists an \(N\)-high subgroup \(S\) of \(G\) properly containing \(M\). Let \(0 \neq s + M\) be in \((S/M)[p]\). By the neatness of \(S\) in \(G\) ([1], pg. 92) we may suppose that \(s \in S[p]\). But this contradicts \(M[p] \oplus N[p] = G[p]\), and so \(M\) is \(N\)-high in \(G\).

As a consequence of Lemma 1, we obtain a standard

**Corollary.** ([3], pg. 24). Let \(G\) be a primary group, and \(H\) a pure subgroup containing \(G[p]\). Then \(H = G\).

Received August 10, 1960.

A useful generalization of Lemma 1 to torsion groups is

**Lemma 2.** Let $M$ and $N$ be subgroups of a torsion group $G$ such that $M$ is neat in $G$ and $M[p] \oplus N[p] = G[p]$ for each relevant prime $p$. Then $M$ is $N$-high in $G$.

**Proof.** Use the proof of Lemma 1 with the observation that since $S$ properly contains $M$, $(S/M)[p] \neq 0$ for some relevant prime $p$.

Concerning $N$-high subgroups in a direct sum, we have

**Theorem 1.** Let $G = \Sigma G_{\alpha}$ be an arbitrary direct sum of torsion groups, where $H_{\alpha}$ and $N_{\alpha}$ are subgroups of $G_{\alpha}$, and where $H_{\alpha}$ is $N_{\alpha}$-high in $G_{\alpha}$ for each $\alpha$. Let $N = \Sigma N_{\alpha}$. Then $H = \Sigma H_{\alpha}$ is $N$-high in $G$.

**Proof.** First notice that $H$ is neat in $G$. To see this, use the neatness of $H_{\alpha}$ in $G_{\alpha}$ for each $\alpha$ (see [2] Lemma 10). Next observe that $G[p] = \Sigma G_{\alpha}[p] = \Sigma H_{\alpha}[p] \oplus \Sigma N_{\alpha}[p] = H[p] \oplus N[p]$ for each relevant prime $p$. Now apply Lemma 2 to get $H$ to be $N$-high in $G$.

An interesting result concerning high subgroups (which are our main interest) in a direct sum is a corollary of Theorem 1.

**Theorem 2.** Let $G = \Sigma G_{\alpha}$ be an arbitrary direct sum of torsion groups where $H_{\alpha}$ is a high subgroup of $G_{\alpha}$ for each $\alpha$. Then $H = \Sigma H_{\alpha}$ is high in $G$.

**Proof.** By [2], Lemma 9 we have $G^{1} = \Sigma (G_{\alpha})^{1}$. Now use Theorem 1 and the definition of high subgroup.

**Divisible hulls and high subgroups.** Now we shall discuss the notion of a divisible hull for a group $G$, and the connection of such a hull with high subgroups. A group $E$ minimal among those divisible groups containing $G$ as a subgroup will be called a divisible hull of $G$. We need a few lemmas. The following lemma is almost obvious, and its proof is omitted.

**Lemma 3.** Let $E$ be a divisible hull of a torsion group $G$. Let $E_{\pi} = \Sigma E_{\pi}$ and $G = \Sigma G_{\pi}$. Then $E_{\pi} = E$, and $E_{\pi}$ is the unique divisible hull of $G_{\pi}$ in $E$ for each relevant prime $\pi$.

**Lemma 4.** Let $D$ be a divisible hull of a mixed group $G$, and $E$ be a divisible hull of the torsion subgroup $T$ of $G$ in $D$. Then $D = E \ominus F$ where $E$ is torsion and $F$ is torsion free divisible.
Proof. Since $E$ is divisible, it is a direct summand of $D$. Thus $D = E \oplus F$ for some subgroup $F$ of $D$. That $E$ is torsion follows from Lemma 3. Next we show that the torsion subgroup $T_F$ of $F$ is zero. To see this, consider $T_F \cap G \subset T_F \cap T \subset E \cap F = 0$ to get $T_F \cap G = 0$. Then by Kulikov’s Lemma, ([1], pg. 66) applied to $D$, $T_F = 0$ and $F$ is torsion free. That $F$ is divisible is clear, and the lemma is proved.

The following theorem gives a characterization in terms of divisible hulls of $N$-high subgroups of a torsion group $G$.

THEOREM 3. Let $N$ be any subgroup of a torsion group $G$, and $E$ be a divisible hull of $G$ with $D$ a divisible hull of $N$ in $E$. Then the set of $N$-high subgroups of $G$ is the set of intersections of $G$ with complementary summands of $D$ in $E$.

Proof. Let $H = A \cap G$, where $A \oplus D = E$. Now by [2], Lemma 1, and [1] pg. 67, we have for each relevant prime $p$,


By [1] pg. 92, $H$ is neat in $G$, and finally by Lemma 2, $H$ is $N$-high in $G$. Now for the converse, suppose $H$ is $N$-high in $G$, so that $H \cap N = 0$. Now $H \cap D = 0$. To see this, notice that $(H \cap D) \cap N = H \cap N = 0$, and by Kulikov’s lemma, $H \cap D = 0$. Since $D$ is an absolute direct summand (see [1]), there exists $A$ containing $H$ with $A \oplus D = E$. But $H \subset A \cap G$, and since $(A \cap G) \cap N = 0$, by the maximality of $H$ with respect to $H \cap N = 0$, we have $H = A \cap G$.

The reader will note that in particular Theorem 3 yields a characterization of high subgroups in torsion groups.

In general, a group $G$ may have many high subgroups. It is even possible that $H \cap K = 0$ for two high subgroups $H$ and $K$ of $G$. The following theorem indicates the extent of the non-uniqueness of $N$-high subgroups.

THEOREM 4. Let $G$ be a primary group, let $N$ be a subgroup such that $|N[p]| = |G|$ and such that $[G[p]: N[p]] = |G|$. Then there exist $2^{|G|}$ distinct $N$-high subgroups of $G$. Furthermore, there exists an independent set $\{H_\alpha\}_{\alpha \in \mathcal{R}}$ of $N$-high subgroups of $G$ such that $|\mathcal{R}| = |G|$.

Proof. Let $H$ be an $N$-high subgroup of $G$. By [2], $G[p] = H[p] \oplus N[p]$. Clearly $|H[p]| = |N[p]| = |G|$. Let $H[p] = \sum_{\alpha \in \mathcal{S}} \langle x_\alpha \rangle$ and $N[p] = \sum_{\beta \in \mathcal{T}} \langle y_\beta \rangle$. Then $|S| = |T| = |G|$. There exists $2^{|G|}$ one-to-one mappings of $S$ onto $T$. Let $f$ be such a mapping, and let

$$P_f = \sum_{\alpha \in \mathcal{S}} \langle x_\alpha + y_{f(\alpha)} \rangle.$$
If $g$ is any one-to-one mapping of $S$ onto $T$ such that $f \neq g$, then it is easy to see that $P_f \neq P_g$. Let $H_f$ be an $N$-high subgroup of $G$ containing $P_f$. Then $H_f[p] = P_f$, and since $P_f \neq P_g$, it follows that $H_f \neq H_g$. Hence there exist $2^g$ $N$-high subgroups of $G$. Let $T = \bigcup_{\beta \in R} T_{\beta}$, where $|T_{\beta}| = |T|$, $|R| = |T|$, and $T_{\beta} \cap T_{\delta} = \emptyset$ if $\beta \neq \delta$. That is, partition $T$ into $|T|$ subsets each of cardinal $|T|$. Let $f_{\beta}$ be a one-to-one mapping of $S$ onto $T_{\beta}$, and let $H_{\beta}$ be an $N$-high subgroup containing

$$P_{\beta} = \sum_{a \in S} \langle x_a + y_{f_{\beta}(a)} \rangle.$$

It is straightforward to verify that $\{P_{\beta}\}_{\beta \in R}$ is an independent set of subgroups of $G$, that $H_{\beta}[p] = P_{\beta}$, and hence that $\{H_{\beta}\}_{\beta \in R}$ is an independent set of subgroups such that $|R| = |G|$. This concludes the proof.

It is easy to find examples of reduced primary groups $G$ such that $|G'[p]| = |G|$. That $G'$ is $N$-high in $G$. This concludes the proof.

Purity of $N$-high subgroups of torsion groups. We now state and prove a generalization to $N$-high subgroups of torsion groups of [2] Theorem 3, namely that high subgroups of torsion groups are pure.

**Theorem 5.** Let $N$ be a subgroup of a torsion group $G$ with $N \subseteq G'$, and let $H$ be an $N$-high subgroup of $G$. Then $H$ is pure in $G$.

**Proof.** That it suffices to consider the primary case here follows from the fact $H_{\beta}$ is $N_{\beta}$-high in $G_{\beta}$ (see [2] Lemma 10 and [2] Lemma 11). So let $G$ be primary. Now by [2] Lemma 1, we have $G[p] = H[p] \oplus N[p]$. Since $G'[p] \supseteq N[p]$, then $G'[p] = (H \cap G')[p] \oplus N[p]$. Now let $H_{\beta}$ be an $(H \cap G')[p]$-high subgroup of $H$. Since $N$-high subgroups are neat (see [1] c, d pg. 92) and neatness is transitive, we have that $H_{\beta}$ is neat in $G$. By [2] Lemma 1, $H[p] = H_{\beta}[p] \oplus (H \cap G')[p]$, so that $G[p] = H_{\beta}[p] \oplus G'[p]$. An application of Lemma 1 yields $H_{\beta}$ high in $G$. Finally, by [2], Lemma 8, $H_{\beta}$ contains $B$ basic in $G$, so that by [2] Lemma 2, $H$ is pure in $G$ as stated.

Before stating some corollaries, we would like to pose the following question: characterize all subgroups $T$ of an Abelian group $G$ such that $T$-high subgroups of $G$ are pure. Suitable examples are easy to find which show that just any subgroup $T$ will not do.

A couple of corollaries of Theorem 5 are

**Corollary 1.** Let $N_1$ and $N_2$ be subgroups of a torsion group $G$ with $N_1 \subseteq N_2 \subseteq G'$. Then every $N_2$-high subgroup of $G$ contains an $N_1$-high subgroup, and in particular every $N_1$-high subgroup $K$ contains a subgroup $H$ high in $G$. 
Proof. The proof is similar to the proof of Theorem 5.

**Corollary 2.** Let $N$ be a subgroup of $G^i$ in a torsion group $G$, and let $S$ be an infinite subgroup of $G$ with $S \cap N = 0$. Then there exists a subgroup $K$ pure in $G$ with $|K| = |S|$ and $K \cap N = 0$.

Proof. Substitute $N$ for $G^i$ in the proof of [2], Theorem 2.

This last corollary is a generalization of the solution in [2] of Fuchs' Problem 4.

**Σ-groups.** The following ideas arose from an investigation of the question of whether or not all high subgroups of a given group are isomorphic. A natural query in this direction is: If one of the high subgroups of a given group $G$ is a direct sum of cyclic groups, are all of them direct sums of cyclic groups? The answer for torsion groups is yes. It is this observation that gives rise to so called Σ-groups. Before discussing this notion further, we need a few lemmas.

**Lemma 5.** Let $N$ be a subgroup of a torsion group $G$ with $H$ and $K$ both $N$-high subgroups of $G$. Then $((H \oplus N)/N)[p] = ((K \oplus N)/N)[p]$ for each relevant prime $p$.

Proof. For $h \in H$ we have that $o(h + N) = p$ if and only if $o(h) = p$. Suppose $h \in H[p] \cap (K \cap H)$. Then there exists $k \in K$, $n \in N$ with $h - k = n$, whence $o(k) = p$. Thus $h + N = (k + N) \in ((K \oplus N)/N)[p]$; and since $p$ was arbitrary, we have by symmetry that

$$((H \oplus N)/N)[p] = ((K \oplus N)/N)[p]$$

as stated.

**Lemma 6.** Let $N$ be a subgroup of a torsion group $G$ with $N \leq G^i$. Let $H$ be an $N$-high subgroup of $G$. Then $((H \oplus N)/N)$ is pure in $G/N$.

Proof. Suppose $m(g + N) = h + N$ for some $h \in H$, $g \in G$, $m$ a non-zero integer. Then $mg - n = h$ for some $n \in N$, and since $n \in G^i$ and $H$ is pure (Theorem 5), we have $h = mh_i$ for some $h_i \in H$. Thus $h + N = m(h_i + N)$ and the lemma is proved.

**Corollary.** Let $N$ be a subgroup of a reduced torsion group $G$ with $N \leq G^i \neq 0$, and let $H$ be an $N$-high subgroup of $G$. Then $H$ is not closed.
Proof. This follows easily from a theorem of Kulikov and Papp ([1] pg. 117).

**Lemma 7.** Let $H$ and $K$ be any two $N$-high subgroups of a primary group $G$ with $N$ a subgroup of $G^1$. Then for all positive integers $n$
(a) $p^nH$ is $N$-high in $p^nG$,
(b) $p^nH$ is pure in $p^nG$,
(c) $p^n((H \oplus N)/N)[p] = (p^n((K \oplus N)/N))[p]$,
(d) $H$, $K$, and $G$ have the same $n$th Ulm invariants (see [3]).

**Proof.**
(a) Use the proof of Theorem 5 (e) in [2] and the fact that $N \subseteq p^nG$ for all $n$.
(b) Use (a), $N \subseteq (p^nG)^i$, and Theorem 5.
(c) First notice that $p^n((H \oplus N)/N) = (p^nH \oplus N)/N$. Now (c) follows immediately from Lemma 5 applied to the right sides of this equation and the corresponding one for $K$.
(d) The proof is similar to that of Theorem 6 in [2].

The following theorem is a slight generalization of the fact that any two high subgroups of a countable group $G$ are isomorphic. (See [2], Theorem 5 (u).)

**Theorem 6.** Let $N$ be a subgroup of a countable torsion group $G$ with $N \subseteq G^1$, and $G^1$ elementary. Then any two $N$-high subgroups $H$ and $K$ of $G$ are isomorphic.

**Proof.** Write $G = \Sigma G_p$, $H = \Sigma H_p$, $K = \Sigma K_p$. Now $H_p$ and $K_p$ are both $N_p$ high in $G_p$, and $N_p \subseteq G^1_p$ for each relevant prime $p$. Let $ar{H}_p = (H_p \oplus N_p)/N_p$ and $\bar{K}_p = (K_p \oplus N_p)/N_p$. Using Lemmas 5 and 6 and the fact that $G^1$ is elementary, we get immediately that $(p^n\bar{H}_p)[p] = (p^n\bar{K}_p)[p]$, and that $(p^{n+1}\bar{H}_p)[p] = (p^{n+1}\bar{K}_p)[p] = 0$. Thus for $\alpha \geq \omega$, the $\alpha$th Ulm invariants of $\bar{H}_p$ and $\bar{K}_p$ are the same. Since $\bar{H}_p \cong H_p$ and $\bar{K}_p \cong K_p$, Lemma 7 (d) implies that $H_p$ and $K_p$ have the same Ulm invariants. Since $G^1$ is elementary, $H$ and $K$ are reduced, and Ulm’s theorem yields $H \cong K$.

**Remark.** In Theorem 6, if we take $N$ to be a subgroup of $G^1$ such that $N[p] \neq G^1[p]$, then neither $H$ nor $K$ will be a direct sum of cyclic groups as is easily seen.

A $\Sigma$-group is any group $G$ all of whose high subgroups are direct sums of cyclic groups. This means that in a torsion $\Sigma$-group every high subgroup is basic. This implies further that in $\Sigma$-groups, every high subgroup is an endomorphic image. Examples of $\Sigma$-groups are very easy to find. For instance, direct sums of countable groups turn out to
be $\Sigma$-groups. Also, any group $G$ such that $G/G^1$ is a direct sum of cyclic groups is a $\Sigma$-group. (See the proof of Theorem 7.) For a non-$\Sigma$-group, see [2], Theorem 5(t).

**Theorem 7.** Let $H$ and $K$ be high subgroups of a torsion group $G$. Then if $H$ is a direct sum of cyclic groups, so is $K$. Moreover $H \cong K$.

**Proof.** Let $\tilde{S}$ be the image of $S$ under the natural homomorphism of $G$ onto $G/G^1$. Now $\tilde{H} \cong H$, $\tilde{K} \cong K$, and by Lemma 5 we have $\tilde{K}[p] = \tilde{H}[p]$ for each relevant prime $p$. By [3], Theorem 12, we have that $\tilde{H}[p]$ is the union of a sequence $\tilde{P}_n$ of subgroups of bounded height in $\tilde{H}$. The purity of $\tilde{H}$ (Theorem 5) tells us that $\tilde{P}_n$ has bounded height in $\tilde{K}$ for each $n$. Hence by [3], Theorem 12, each $\tilde{K}_n$ is a direct sum of cyclic groups, so that $\tilde{K}$ is a direct sum of cyclic groups. Thus $K$, which is isomorphic to $\tilde{K}$, is a direct sum of cyclic groups. Since $H$ and $K$ are both basic in $G$, we have $H \cong K$. Thus we have shown that in a torsion group, if one high subgroup is a direct sum of cyclic groups, they all are, and they are all isomorphic.

From Theorem 7 we see that if in a torsion group $G$ there exist two non-isomorphic high subgroups, then no high subgroup is a direct sum of cyclic groups. The next theorem shows that torsion $\Sigma$-groups are closed under direct sums.

**Theorem 8.** For torsion groups it is true that a direct sum of $\Sigma$-groups is a $\Sigma$-group.

**Proof.** By Theorem 2, a direct sum of highs is high. But such a direct sum is basic. An application of Theorem 7 completes the proof.

**Corollary.** A direct sum of countable torsion group is a $\Sigma$-group.

**Proof.** It suffices by Theorem 8 to verify that a countable torsion group is a $\Sigma$-group, and this is very easy.

**Remark.** Examples exist of torsion groups $G$ such that $G/G^1$ is a direct sum of cyclic groups, but such that $G$ is not a direct sum of countable groups. Therefore, we see that the class of torsion $\Sigma$-groups properly contains the class of all torsion groups that are the direct sum of countable groups.

The next theorem gives an interesting characterization of torsion $\Sigma$-groups.

**Theorem 9.** A torsion group $G$ is a $\Sigma$-group if and only if $G$
contains a maximal basic subgroup.

Proof. If $G$ is a $\Sigma$-group, then any high subgroup will be a maximal basic subgroup of $G$. Now suppose $G$ contains a maximal basic subgroup $B$. Let $H$ be a high subgroup containing $B$, and suppose $B \neq H$. By [1] pg. 114, there exists $B_1$ basic in $H$ with $B_1 > B$. Since $H$ is pure and $G/H$ is divisible, $B_1$ is basic in $G$, a contradiction. Therefore $B = H$, and $G$ is a $\Sigma$-group by Theorem 7.

The next theorem is a result concerning the $\Sigma$-groups of a torsion group.

**Theorem 10.** Every torsion group $G$ contains a $\Sigma$-subgroup $R$ pure in $G$ such that $R^1 = G^1$.

**Proof.** First if $G^1 = 0$, put $R = B$ basic in $G$. Also if $G$ is a $\Sigma$-group, but $R \neq G$. So suppose that $G^1 \neq 0$ and $G$ is not a $\Sigma$-group. Let $B$ be a basic subgroup of $G$. Embed $B$ in a high subgroup $H$ of $G$. By Theorem 8 and the assumptions on $G$, $H/B \neq 0$. $B$ is basic in $H$ so that $G/B = H/B \oplus R/B$, where the divisibility of $H/B$ guarantees that $R/B$ may be chosen to contain $(G^1 \oplus B)/B$. Hence $R$ contains $G^1$. The purity of $R/B$ in $G/B$ gives us that $R$ is pure in $G$. Hence $R^1 = G^1$.

Now $H \cap R = B$, so that $G[p] = H[p] \oplus G'[p]$ and

$$R[p] = (R \cap H)[p] \oplus G'[p] = B[p] \oplus R'[p].$$

By Lemma 2, $B$ is high in $R$ so that by Theorem 7, $R$ is a $\Sigma$-group, and the proof is complete.

We do not know whether every subgroup of a $\Sigma$-group is a $\Sigma$-group. However, every pure subgroup of a torsion $\Sigma$-group is a $\Sigma$-group. In fact, we have

**Theorem 11.** Every subgroup $L$ of a torsion $\Sigma$-group $G$ with $L^1 = L \cap G^1$ is a $\Sigma$-group.

**Proof.** Embed a high subgroup $H_L$ of $L$ in a high subgroup $H$ of $G$. Since $G$ is a $\Sigma$-group, $H$ is a direct sum of cyclic groups and hence so is $H_L$. Now apply Theorem 7 to $L$ to get that $L$ is a $\Sigma$-group.

**Corollary.** Every pure subgroup $R$ of a torsion $\Sigma$-group $G$ is a $\Sigma$-group.

**Proof.** $R^1 = R \cap G^1$, and Theorem 11 then yields the desired result.

**Corollary.** Every pure subgroup of a direct sum of countable torsion groups is a $\Sigma$-group.
High subgroups in mixed groups. In this section we will discuss some properties of high subgroups of arbitrary Abelian groups and generalize some of our results for the torsion case. A lemma which is useful is

**Lemma 8.** Let $S$ be any subgroup of an Abelian group $G$ with $S \cap G^t = 0$. Then for any subgroup $T$ with $(T/S) \cap (G/S)^t = 0$, we have $T \cap G^t = 0$.

**Proof.** Suppose $T \cap G^t \neq 0$. Then $(T/S) \cap (G/S)^t \neq 0$.

The following theorem was proved in [2] for the torsion case. The fact that $G^t$ is divisible in the torsion free case makes this case easy, so we proceed directly to the general case.

**Theorem 12.** Let $H$ and $K$ be any two high subgroups of a group $G$. Then

(a) $G/H$ is divisible

(b) $G/H$ is a divisible hull of $(G^t \oplus H)/H \cong G^t$

(c) $G/H \cong G/K$.

**Proof.** (a) Let $T/H$ be the torsion subgroup of $G/H$. Now $T/H$ is divisible, for if not, $T/H$ would have a non-zero cyclic direct summand $L/H$. But $L/H$ would be a direct summand of $G/H$ since $T/H$ is pure in $G/H$. Hence $(L/H) \cap (G/H)^t = 0$, and Lemma 8 gives us that $L \cap G^t = 0$. Consequently $H$ is not high in $G$, a contradiction. Thus we have $G/H = T/H \oplus F/H$, where $F/H$ is torsion free. This means that $(F/H)^t$ is divisible, whence $F/H = (F/H)^t \oplus R/H$. Now clearly, $(R/H) \cap (G/H)^t = 0$, so that by Lemma 8, $R = H$, and $G/H$ is divisible as stated.

(b) As a divisible group, $G/H$ must contain a divisible hull $D/H$ of $(G^t \oplus H)/H$. Put $G/H = D/H \oplus L/H$. Clearly $L \cap G^t = 0$, hence $L/H = 0$ and (b) is proved.

(c) This follows from $(G^t \oplus H)/H \cong G^t \cong (G^t \oplus K)/K$ and the fact that divisible hulls of isomorphic groups are isomorphic. Thus we see that $G/H$ is a structural invariant of $G$.

We shall now discuss a generalization to arbitrary Abelian groups of a theorem proved in [2] for the torsion case. Here again, the torsion free case is easy ($G^t$ is divisible), and for a torsion free group $G$ we see easily that all high subgroups are isomorphic. First we need

**Theorem 13.** Let $T$ be the torsion subgroup of an Abelian group $G$, $H$ be a high subgroup of $G$, and $T_H$ be the torsion subgroup of $H$. Then $T_H$ is high in $T$. 
Proof. We need only consider the case of a mixed group $G$. Let $E$ be a divisible hull of $G$ with $D_H$ and $D_{G'}$ divisible hulls in $E$ of $H$ and $G'$. Then $E = D_H \oplus D_{G'}$ (see [2]). Next let $D_{T_H}$ and $D_{T_{G'}}$ be divisible hulls of $T_H$ and $T_{G'}$ in $D_H$ and $D_{G'}$ respectively. Then

$$E = D_H \oplus D_{G'} = D_{T_H} \oplus D_1 \oplus D_{T_{G'}} \oplus D_2 = D_{T_H} \oplus D_{T_{G'}} \oplus D_1 \oplus D_2.$$  

Applying Lemmas 3 and 4 to $D_T$ and $Z_V$, we have that $T_E = D_{T_H}(£)T_{T_{G'}}$ is the torsion subgroup of $E$, and $D_1 \oplus D_2$ is torsion free. Clearly (Lemma 3) $T_\pi$ is a divisible hull of $T$ in $E$. Now by Theorem 3, it remains to verify that $T \cap D_{T_H} = T_H$. To this end put $L = T \cap D_{T_H}$. That $T_H \subset L$ is clear. By Lemma 4, $H \cap D_{T_H} = T_H$. So suppose there exists $t \in (G \setminus H) \cap L$. Then by the definition of $H$, there exists $h \in H$ with $h + mt = g_1 \neq 0$, where $g_1 \in G'$. But since $L$ is torsion we have that $h \in T_H$. Hence $(h + mt) \in D_{T_H}$, and $h + mt = g_1 \neq 0$ together with $D_{T_H} \subset D_H$ contradict $D_H \cap D_{G'} = 0$, and $T_H$ is high in $T$ as desired.

COROLLARY. Let $H$ be a high subgroup of $G$, and let $T_H$ be the torsion subgroup of $H$. Then $T_H$ is pure in $G$.

Proof. By Theorem 13, $T_H$ is high in $T$, and consequently pure in $T$. Since $T$ is pure in $G$, it follows that $T_H$ is pure in $G$.

THEOREM 14. Let $H$ be a high subgroup of an Abelian group $G$. Then $H$ is pure in $G$.

Proof. Let the notation be the same as in Theorem 13. Now by Theorem 13 $T/T_H$ is divisible, so that $G/T_H = T/T_H \oplus R/T_H$, where $R$ is chosen such that $R/T_H$ contains $H/T_H$. Since $T_H$ is pure in $G$, $R$ is pure in $G$, and since $H$ is neat in $G$, $H/T_H$ is neat in $R/T_H$. But $R/T_H$ is torsion free and since a neat subgroup of a torsion free group is pure we have that $H/T_H$ is pure in $R/T_H$. Thus $H$ is pure in $R$, so that $H$ is pure in $G$, and the proof is complete.

The following embedding theorem is a generalization to arbitrary Abelian groups of the solution to Fuchs’ Problem 4 (see [1]).

THEOREM 15. Let $S$ be any infinite subgroup of an Abelian group $G$ with $S \cap G' = 0$. Then there exists a subgroup $K$ pure in $G$ with $S \subset K$, $K \cap G' = 0$, and $|K| = |S|$.

Proof. Embed $S$ in $H$ high in $G$. By [1] pg. 78 N, there exists a pure subgroup $K$ of $H$ with $S \subset K$ and $|S| = |K|$. The purity of
ON \( \Gamma \)-HIGH SUBGROUPS OF ABELIAN GROUPS

\( \Gamma \) implies the purity of \( K \) in \( G \), and \( K \cap G^i \subset H \cap G^i = 0 \), so that \( K \cap G^i = 0 \), completing the proof.

An unsolved problem. To conclude the present paper we shall make a few remarks concerning the question of whether all high subgroups of an Abelian torsion group are isomorphic. The reader may have observed, from the proof of Theorem 6, that this question is a special case of the more general open question: Given two pure subgroups \( A \) and \( B \) of a primary group \( G \) with \( A[p] = B[p] \), is it true in general that \( A \cong B \)? The authors feel that an affirmative answer to this question would have important consequences in the theory of Abelian torsion groups. A step in this direction is

**Theorem 16.** Let \( A \) and \( B \) be pure subgroups of primary group \( G \) with \( A[p] = B[p] \). Then \( G = A \oplus C \) implies \( G = B \oplus C \) and \( A \cong B \).

**Proof.** Let \( G = A \oplus C \). Then \( G[p] = A[p] \oplus C[p] = B[p] \oplus C[p] \). We will show that \( G = B \oplus C \). First notice that \( A[p] = B[p] \) gives us that \( B \cap C = 0 \). To prove \( G = B \oplus C \), it suffices to verify that \( G \subset B \oplus C \). For this purpose it is sufficient that \( G[p^n] \subset B \oplus C \) for each \( n \). But this is true if and only if \( A[p^n] \subset B \oplus C \) for each \( n \). Now we use induction to show that \( G[p^n] \subset B \oplus C \) for each \( n \). First, \( G[p] \subset B \oplus C \) by hypothesis. Next suppose that \( G[p^n] \subset B \oplus C \), and let \( a \in A \) with \( o(a) = p^{n+1} \). Then \( p^n a = b \in A[p] = B[p] \). By the purity of \( B \), \( p^n a = p^n b_1 \) with \( b_1 \in B \), and \( p^n (a - b_1) = 0 \), so that \( a - b_1 \in B \oplus C \) by the induction hypothesis. Hence \( a \in B \oplus C \), therefore

\[
A[p^{n+1}] \subset B \oplus C,
\]

which means that \( G[p^{n+1}] \subset B \oplus C \). Thus \( G = B \oplus C \). Finally \( A \cong G/C \cong B \), and the proof is complete.

The foregoing theorem suggests the following generalization.

**Theorem 17.** Let \( G \) be a direct sum of torsion groups, \( G = \sum_{\alpha \in A} G_{\alpha} \), and let \( \{T_{\alpha}\}_{\alpha \in A} \) be a family of subgroups pure in \( G \) with \( T_{\alpha}[p] = G_{\alpha}[p] \) for each relevant prime \( p \) and each \( \alpha \in A \). Then for any subfamily \( \{T_{\alpha}\}_{\alpha \in S} \), \( G = \sum_{\alpha \in S} T_{\alpha} \oplus \sum_{\alpha \in S} G_{\alpha} \). In particular, \( G = \sum_{\alpha \in A} T_{\alpha} \) and \( G_{\alpha} \cong T_{\alpha} \) for each \( \alpha \in A \).

**Proof.** Put \( T = \sum_{\alpha \in A} T_{\alpha} \). It suffices to show that \( G = T \). We show as before that for each \( n \) we have \( G[p^n] \subset T \). This is true if for each \( \alpha \in A \) we have for the primary components \( G_{\alpha p} \), that \( G_{\alpha p}[p^n] \subset T \) for each \( n \). This is accomplished as in the proof of Theorem 16. Finally, that \( T_{\alpha} \cong G_{\alpha} \) for each \( \alpha \) follows as before.
REFERENCES


NEW MEXICO STATE UNIVERSITY (UNIVERSITY PARK, N.M.)
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsu-sha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Reprinted 1966 in the United States of America
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. V. Balakrishnan, <em>Prediction theory for Markoff processes</em></td>
<td>1171</td>
</tr>
<tr>
<td>Dallas O. Banks, <em>Upper bounds for the eigenvalues of some vibrating systems</em></td>
<td>1183</td>
</tr>
<tr>
<td>A. Białynicki-Birula, <em>On the field of rational functions of algebraic groups</em></td>
<td>1205</td>
</tr>
<tr>
<td>Thomas Andrew Brown, <em>Simple paths on convex polyhedra</em></td>
<td>1211</td>
</tr>
<tr>
<td>L. Carlitz, <em>Some congruences for the Bell polynomials</em></td>
<td>1215</td>
</tr>
<tr>
<td>Paul Civin, <em>Extensions of homomorphisms</em></td>
<td>1223</td>
</tr>
<tr>
<td>Paul Joseph Cohen and Milton Lees, <em>Asymptotic decay of solutions of differential inequalities</em></td>
<td>1235</td>
</tr>
<tr>
<td>István Fáry, <em>Self-intersection of a sphere on a complex quadric</em></td>
<td>1251</td>
</tr>
<tr>
<td>Walter Feit and John Griggs Thompson, *Groups which have a faithful representation of degree less than ((p - 1/2)) (</td>
<td>\text{asymptotic decay of solutions of differential inequalities})</td>
</tr>
<tr>
<td>William James Firey, <em>Mean cross-section measures of harmonic means of convex bodies</em></td>
<td>1263</td>
</tr>
<tr>
<td>Avner Friedman, <em>The wave equation for differential forms</em></td>
<td>1267</td>
</tr>
<tr>
<td>Bernard Russel Gelbaum and Jesus Gil De Lamadrid, <em>Bases of tensor products of Banach spaces</em></td>
<td>1281</td>
</tr>
<tr>
<td>Ronald Kay Getoor, <em>Infinitely divisible probabilities on the hyperbolic plane</em></td>
<td>1287</td>
</tr>
<tr>
<td>Basil Gordon, <em>Sequences in groups with distinct partial products</em></td>
<td>1309</td>
</tr>
<tr>
<td>Magnus R. Hestenes, <em>Relative self-adjoint operators in Hilbert space</em></td>
<td>1315</td>
</tr>
<tr>
<td>Fu Cheng Hsiang, <em>On a theorem of Fejér</em></td>
<td>1359</td>
</tr>
<tr>
<td>John McCormick Irwin and Elbert A. Walker, <em>On N-high subgroups of Abelian groups</em></td>
<td>1363</td>
</tr>
<tr>
<td>John McCormick Irwin, <em>High subgroups of Abelian torsion groups</em></td>
<td>1375</td>
</tr>
<tr>
<td>R. E. Johnson, <em>Quotient rings of rings with zero singular ideal</em></td>
<td>1385</td>
</tr>
<tr>
<td>David G. Kendall and John Leonard Mott, <em>The asymptotic distribution of the time-to-escape for comets strongly bound to the solar system</em></td>
<td>1393</td>
</tr>
<tr>
<td>Kurt Kreith, <em>The spectrum of singular self-adjoint elliptic operators</em></td>
<td>1401</td>
</tr>
<tr>
<td>Lionello Lombardi, <em>The semicontinuity of the most general integral of the calculus of variations in non-parametric form</em></td>
<td>1407</td>
</tr>
<tr>
<td>Albert W. Marshall and Ingram Olkin, <em>Game theoretic proof that Chebyshev inequalities are sharp</em></td>
<td>1421</td>
</tr>
<tr>
<td>Wallace Smith Martindale, III, <em>Primitive algebras with involution</em></td>
<td>1431</td>
</tr>
<tr>
<td>William H. Mills, <em>Decomposition of homomorphs</em></td>
<td>1443</td>
</tr>
<tr>
<td>James Donald Monk, <em>On the representation theory for cylindric algebras</em></td>
<td>1447</td>
</tr>
<tr>
<td>Shu-Teh Chen Moy, <em>A note on generalizations of Shannon-McMillan theorem</em></td>
<td>1459</td>
</tr>
<tr>
<td>Donald Earl Myers, <em>An imbedding space for Schwartz distributions</em></td>
<td>1467</td>
</tr>
<tr>
<td>John R. Myhill, <em>Category methods in recursion theory</em></td>
<td>1479</td>
</tr>
<tr>
<td>Paul Adrian Nickel, <em>On extremal properties for annular radial and circular slit mappings of bordered Riemann surfaces</em></td>
<td>1487</td>
</tr>
<tr>
<td>Edward Scott O'Keefe, <em>Primal clusters of two-element algebras</em></td>
<td>1505</td>
</tr>
<tr>
<td>Nelson Onuchic, <em>Applications of the topological method of Ważewski to certain problems of asymptotic behavior in ordinary differential equations</em></td>
<td>1511</td>
</tr>
<tr>
<td>Peter Perkins, <em>A theorem on regular matrices</em></td>
<td>1529</td>
</tr>
<tr>
<td>Clinton M. Petty, <em>Centroid surfaces</em></td>
<td>1535</td>
</tr>
<tr>
<td>Charles Andrew Swanson, <em>Asymptotic estimates for limit circle problems</em></td>
<td>1549</td>
</tr>
<tr>
<td>Robert James Thompson, <em>On essential absolute continuity</em></td>
<td>1561</td>
</tr>
<tr>
<td>Harold H. Johnson, <em>Correction to &quot;Terminating prolongation procedures&quot;</em></td>
<td>1571</td>
</tr>
</tbody>
</table>