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THE SEMICONTINUITY OF THE MOST GENERAL INTEGRAL OF THE CALCULUS OF VARIATIONS IN NON-PARAMETRIC FORM.*

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Summary. The positive quasi-regularity of integrals depending upon any number of surfaces in non-parametric form, each with any number of dimensions, is defined. Positive quasi regularity is proved to be sufficient for lower semicontinuity.

1. Let $D_i (i = 1, 2, \cdots, m)$ be a closed bounded set of the n-dimensional euclidean space of the variable vector $x_i \equiv \{x_i^j\} (j = 1, 2, \cdots, n)$, bounded by surfaces which are absolutely continuous in the sense of Tonelli [60, 62, 63], without multiple points, and let D be the cartesian product $\prod_1^m D_i$. Let $y \equiv \{y_i\} (i = 1, 2, \cdots, m)$ denote a vertical m-vector, and let p denote *a mxn* matrix, whose row-vectors are $p_i \equiv \{p_i^j\} (j = 1, 2, \cdots, n)$. Let x be the $m \cdot n$ matrix whose row-vectors are x_i and $\phi[x, y, p]$ a real function, defined for $x_i \in D_i$ $(i = 1, 2, \cdots, m)$ and for any y and p, which is continuous with all its partial derivatives of the types

$$\frac{\partial \phi[x, y, p]}{\partial p_r^s}, \quad \frac{\partial^2 \phi[x, y, p]}{\partial p_r^s \partial p_r^t} \qquad (r = 1, \cdots, m; s, t = 1, \cdots n).$$

Let $q = m$ be a positive integer and let U_q denote a set of distinct positive integers out of the first m; let ζ be an index ranging over U_q, and let $\mu(\delta)$ be a mapping of U_q into the set of the first n integers. It will be assumed throughout that, for every q, every U_q and every $\mu(\zeta)$, all the partial derivatives

$$\frac{\partial^{2q} \phi[x, y, p]}{\prod_1^q \partial x_\zeta^{\mu(\zeta)} \partial p_\zeta^{\mu(\zeta)}} \tag{1.1}$$

exist and are continuous for every $x \in D$ and for every y and p.

Let $y(x) \equiv \{y_i(x_i)\}$ $(i = 1, 2, \cdots, m)$ denote a vector-valued function of the matrix x, such that each *component* $y_i(x_i)$ depends only upon the row vector x_i. We assume that each $y_i(x_i)$ is absolutely continuous, in the sense of Tonelli [63]; we shall call *Variety* V the set of m surfaces represented by $y(x)$.

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We shall say that V is of *class* 1 if all the first partial derivatives of all the $y_i(x_i)$ exist and are continuous; we shall say the V is of *class* 2 if the same is also true for all the partial derivatives of the second order.

Let

$$p_i^j(x) \equiv \frac{\partial y_i(x_i)}{\partial x_i^j} \qquad (i = 1, 2, \cdots m;\ j = 1, 2, \cdots, n)\,,$$

and

$$dx \equiv \prod_1^m {}_i dx_i \equiv \prod_1^m {}_i \prod_1^n {}_j dx_i^j\,.$$

The $m \cdot n$ integral[1]

$$I_V = \int_D \phi[x, y(x), p(x)]dx$$

is called *variety integral in non-parametric form*; all the varieties V where I exists and is finite are called *ordinary*.

REMARK 1.1. *Varieties* V of *class* 1 *and* 2 are ordinary for any function $\phi[x, y, p]$.

Let $\bar{p} \equiv \{\bar{p}_i\} \equiv \{\bar{p}_i^j\}$ denote another variable in the space of the matrix p, $\bar{y} \equiv \{\bar{y}_i\}$ another variable in the space of the vector y, $V \equiv \bar{y}(x) \equiv \{\bar{y}_i(x_i)\}$ another variety V; let

$$\bar{p}_i^j(x_i) = \frac{\partial \bar{y}_i(x_i)}{\partial x_i^j}\,;$$

the distance $\rho(V, \bar{V})$ between V and $\bar{V}$ is defined by the formula

$$\rho(V, \bar{V}) = \sup_{x,\, i} | y_i(x) - \bar{y}_i(x) |\,.$$

Continuity and semicontinuity of the real function I_V will be considered throughout with respect to this *m-uniform* metric.

In one of our previous papers [33] the following theorem was proved:

CONTINUITY THEOREM 1.2. *Necessary and sufficient conditions for the continuity of I_V at every V is that the function $\phi[x, y, p]$ is linear with respect to each one of the vectors p_i.*

REMARK 1.3. As a consequence of Theorem 1.2, the most general function $\phi[x, y, p]$, such that $\int_D \phi[x, y(x), p(x)]\, dx$ is continuous at every

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[1] For the relation between this integral and non local field theories see bibliography [**1, 6, 27, 28, 29, 40, 41, 42, 46, 47, 48, 58**].

V, may be written in the form

$$\sum_{q=1}^{m} \sum_{U_q} \sum_{\mu} \{A_{U_q,\mu}(x, y) \prod_{\zeta \in U_q} p_{\zeta}^{\mu(\zeta)}\} , \tag{1.2}$$

where we assume by convention that, if η is a variable integer ranging over a set S and $\{\alpha_\eta\}$ is a sequence of numbers, then

$$\prod_{\eta \in S} \alpha_\eta = 0, \text{ whenever } S \text{ is empty} .$$

Let $L[x, y, p, \bar{p}]$ denote a polynomial in the indeterminates

$$[\bar{p}_i^{(j)} - p_i^{(j)}] \tag{1.3}$$

of degree not exceeding 1 in any of the vectors $[\bar{p}_i - p_i]$, whose coefficients $W_{U_q,\mu}(x, y, p)$ are functions of (x, y, p) which are continuous together with all their derivatives of the form

$$\frac{\partial^q W_{U_q,\mu}(x, y, p)}{\prod_{\zeta \in U_q} \partial x_{\zeta}^{\mu(\zeta)}} , \tag{1.4}$$

$L[x, y, p, \bar{p}]$ may be written in the form

$$\sum_{q=1}^{m} \sum_{U_q} \sum_{\mu} \{W_{U_q,\mu}(x, y, p) \prod_{\zeta \in U_q} [\bar{p}_{\zeta}^{\mu(\zeta)} - p_{\zeta}^{\mu(\zeta)}]\} . \tag{1.5}$$

Let us define the *generalized Weierstrass function* $E_L[x, y, p, \bar{p}]$ *of* L_V *with respect to* $L[x, y, p, \bar{p}]$, by the formula

$$E_L(x, y, p, \bar{p}) = \phi[x, y, \bar{p}] - L[x, y, p, \bar{p}] . \tag{1.6}$$

The integral $I_V = \int \phi[x, y(x), p(x)]dx$ is said to be *positively quasi-regular with respect to* L (abbreviation: *LPQR*) if both the relations

$$E_L[x, y, p, p] = 0 \tag{1.7}$$

$$E_L[x, y, p, \bar{p}] \geqq 0 \tag{1.8}$$

hold for every $x \in D$ and for every $y, p, \bar{p}$.

REMARK 1.4. Notice that if I_V is *LPQR*, then the element of degree 0 of the polynomial $L[x, y, p, \bar{p}]$ must be $\phi[x, y, p]$, and the vector consisting of the coefficients of the elements of degree 1 is the gradient of $[x, y, p]$ with respect to p: therefore, if $m = 1$, i.e., if I_V is a usual multiple integral [60, 62], the fact that I_V is *LPQR* completely determines the function $L[x, y, p, \bar{p}]$. This does not happen if $m > 1$, as was shown by an appropriate example [30], referring to Fubini-Tonelli integrals, i.e., to the case $(m = 2, n = 1)$.

We say that I_V is *positively quasi-regular* (abbreviation *PQR*) if there exists at least one function $L[x, y, p, \bar{p}]$ such that I_V is *LPQR*.

Remark 1.5. Let us say that I_V is *negatively quasi-regular* with respect to L (abbreviation: *LNQR*) if $\int_D - \phi[x, y(x), p(x)]dx$ is *LPQR*. Then it is easy to prove that, if I_V is both L_1PQR and L_2NQR, then $L_1[x, y, p, \bar{p}] \equiv L_2[x, y, p, \bar{p}]$, and $\phi[x, y, p]$ is a polynomial of degree not exceeding 1 in each p_i; i.e., by Theorem 1.2, I_V is continuous. Theorem 1.2 also implies that every continuous I_V is both *LPQR* and *LNQR* for some $L[x, y, p, \bar{p}]$.

Remark 1.6 In the case $m = 1$, our definition of *positive quasi-regularity* reduces to the one which was given by Tonelli [59, 60] and Cinquini [1] for simple and multiple integrals. In this particular case, the *positive quasi-regularity* of an integral is equivalent to the *lower convexity* of its *figurative*, i.e., of $\phi[x, y, p]$ considered as a function of p only.

In the case $n = 1$, the definition of *positive quasi-regularity* reduces to the one given by this author for the Fubini-Tonelli integrals [30].

Remark 1.7. If I_V is *PQR*, then its value is $+\infty$ at every non-ordinary variety.

2. Let us prove the following

Theorem 2.1. *If I_V is PQR, then it is lower semicontinuous at every variety V of class 2; i.e., if V is of class 2, there exists a positive function $\rho(\varepsilon)$ of the positive variable ε such that, if $\bar{V} \equiv \bar{y}(x)$ is any variety, then*

$$I_{\bar{V}} - I_V > -\varepsilon, \text{ whenever } \rho(V, \bar{V}) < \rho(\varepsilon), \tag{2.1}$$

regardless of whether or not V is of class 2.

Proof. Let $L[x, y, p, \bar{p}]$ be a function, such that I_V is *LPQR*. By (1.6) we may write

$$\begin{aligned} I_{\bar{V}} - I_V = &\int_D E_L[x, \bar{y}(x), p(x), \bar{p}(x)]dx - \int_D E_L[x, y(x), p(x), p(x)]dx \\ &+ \int_D L[x, \bar{y}(x), p(x), \bar{p}(x)]dx - \int_D L[x, y(x), p(x), p(x)]dx\,. \end{aligned} \tag{2.2}$$

Let $V \equiv y(x)$ be a variety of *class* 2;

$$P[x, \bar{y}, \bar{p}] \equiv L[x, \bar{y}, p(x), \bar{p}]$$

is a polynomial of a degree not exceeding 1 in each $\bar{p}$, and all of the derivatives

$$\frac{\partial P[x, \bar{y}, \bar{p}]}{\partial \bar{p}_r^s}, \quad \frac{\partial^2 P[x, \bar{y}, \bar{p}]}{\partial \bar{p}_r^s \partial \bar{p}_r^t}, \quad \frac{\partial^{2q} P[x, \bar{y}, \bar{p}]}{\prod_1^q {}_\zeta \partial x_\zeta^{\mu(\zeta)} \partial \bar{p}_\zeta^{\mu(\zeta)}} \tag{2.3}$$

$$(r = 1, 2, \cdots, m; s, t = 1, 2, \cdots n)$$

exist and are continuous for every $[x, \bar{y}, \bar{p}]$ and for every q, U_q, $\mu(\zeta)$, r, s, t as a consequence of the existence and continuity of the functions (1.4) and of the partial derivatives of the first two orders of the functions $y_r(x)$, $(r = 1, 2, \cdots, m)$.

By the continuity Theorem 1.2,

$$J_V = \int_D P[x, \bar{y}(x), \bar{p}(x)] dx$$

is continuous; hence the difference of the last two integrals on the right side of (2.2) is smaller than any predetermined real positive ε, whenever $\rho(V, \bar{V})$ is less than a certain positive number $\rho(\varepsilon)$. Since the first integral on the right side of (2.2) is not negative by (1.8) and the second vanishes by (1.7), (2.1) holds: the theorem is thus proved.

3. (a) In this section the concept of *asymptotic evaluability of the integral* I_V is defined; the lower semicontinuity on every very variety V of any *positively quasi-regular* and *asymptocally evaluable* integral is proved. The results of this chapter may be regarded as extensions of Tonelli's theorems on usual multiple integrals [59, 60], and of our results on Fubini-Tonelli integrals [30].

(b) Suppose that $I_V = \int_D \phi[x, y(x), p(x)] dx$ is *PQR*, and let $L[x, y, p, \bar{p}]$ be one of the functions, such that I_V is *LPQR*.

The function

$$\bar{\Phi}[x, y, p] \equiv E_L[x, y, \Omega, p], \tag{3.b.1}$$

where Ω is a $m \cdot n$ matrix whose elements are all 0, is never negative. Furthermore,

$$\bar{I}_V = \int_D \bar{\Phi}[x, y(x), p(x)] dx \tag{3.b.2}$$

is $\bar{L}PQR$, where

$$\bar{L}[x, y, p, \bar{p}] = L[x, y, p, \bar{p}] - L[x, y, \Omega, \bar{p}]. \tag{3.b.3}$$

By (1.7), the equation

$$\bar{\Phi}[x, y, \Omega] = 0$$

holds for every $x \in D$ and every y.

Let R denote a positive real number and let $\varphi^R[x, y, p]$ denote a function such that the following conditions are satisfied:

I. $\varphi^R[x, y, \mathrm{p}]$ is continuous with all its partial derivatives of any of the forms

$$\frac{\partial\varphi^R[x, y, p]}{\partial p_r^s}, \quad \frac{\partial^2\varphi^R[x, y, p]}{\partial p_r^s \partial p_r^t}, \quad \frac{\partial^{2q}\varphi^R[x, y, p]}{\prod_1^q {}_\zeta \partial x_\zeta^{\mu(\zeta)} \partial p_\zeta^{\mu(\zeta)}}.$$

II. The integral

$$Y_V = \int_D \varphi^R[x, y, (x), p(x)]dx \tag{3.b.4}$$

is *PQR*.

III. The relation

$$0 \leqq \varphi^R[x, y, p] \leqq \Phi[x, y, p] \tag{3.b.5}$$

holds for every y, p and for every $x \in D$; furthermore

$$\varphi^R[x, y, p] = \Phi[x, y, p], \text{ whenever } \sum_{i=1}^m \sum_{j=1}^n (p_1^j)^2 \leqq R. \tag{3.b.6}$$

IV. There exists at least one function $\Lambda[x, y, p, \bar{p}]$, such that Y_V is ΛPQR, and such that, for each $T > 1$, there exists a number Q, for which the following condition is satisfied:

Let q, U_q, ζ, $\mu(\zeta)$ be defined as they were in § 1; *let $\bar{U}_q$ denote the complement of U_q with respect to the set of the first m positive integers, and let $\bar{\zeta}$ be an index ranging over $\bar{U}_q$. Then the inequality*

$$\left| W^R_{U_q,\mu}[x, y, p] \right| < Q\Bigl(1 + \prod_{\bar{\zeta} \in \bar{U}_q} p_{\bar{\zeta}}^{\mu(\bar{\zeta})} \Bigr|\Bigr)$$

where $W^R_{U_q,\mu}[x, y, p]$ denotes the coefficient of the element

$$\prod_{\zeta \in U_q} [\bar{p}_\zeta^{\mu(\zeta)} - p_\zeta^{\mu(\zeta)}]$$

of the expression $\Lambda[x, y, p, \bar{p}]$, holds for every q, U_q, p, for every $x \in D$ and for every y such that

$$|y_i| < T \qquad (i = 1, 2, \cdots, n).$$

Remark 3.1. In the case of the usual multiple integrals ($m = 1$), Condition IV reduces to the boundedness of the derivatives

$$\frac{\partial \varphi^R[x, y, p]}{\partial p_1^s} \qquad (s = 1, 2, \cdots, n)$$

in any domain where $y(x)$ is bounded; this condition is exactly the one considered by Tonelli [59, 60].
In the case of Fubini-Tonelli integrals ($n = 1$), this condition reduces to the one that this author considered in [30, § 1, page 132].

REMARK 3.2. Y_V exists and is finite on every variety V, i.e., every variety V is ordinary for the integral Y_V.

(c) LEMMA 3.3. *The integral* Y_V *defined by* (3.b.4) *is lower semi-continuous at every variety* V.

Proof. Let $V \equiv y(x) \equiv \{y_i(x_i)\}$ be any variety; and let $1 > \varepsilon > 0$ and $R > 0$ be given, and let $\pi \equiv \pi(x) \equiv \{\pi_i(x_i)\}$ denote a variety of *class* 2, such that

$$\rho(\pi, V) < \varepsilon \tag{3.c.1}$$

Let $T = \sup_{x_i, i} | y_i(x_i) | + 2$.

Let $\pi'(x) \equiv || \pi'^j_i(x) || \equiv \left|\left| \dfrac{\partial \pi_i(x_i)}{\partial x_i^j} \right|\right|$, $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$,

and let $\bar{D}_i \subset D_i$ denote set of the points x_i, such that, for some j, either $p_i^j(x_i)$ does not exist or it is such that

$$| \pi'^j_i(x_i) - p_i^j(x_i) | \geqq \varepsilon . \tag{3.c.2}$$

Suppose further that, for each i $(i = 1, 2, \cdots, m)$,

$$\int_{\bar{D}_i} \sum_1^n{}_j [| \pi'^j_i(x_i) | + | p_i^j(x_i) |] dx_i < \varepsilon . \tag{3.c.3}$$

The construction of such a variety π is possible for any V [68]. If $\bar{V} = \bar{y}(x) \equiv \{\bar{y}_i(x)\}$ is any other variety, we may write

$$\begin{aligned} Y_{\bar{V}} - Y_V = & \int_D E^{\varphi}_{\Delta}[x, \bar{y}(x), \pi'(x), \bar{p}(x)]dx \\ & - \int_D E^{\varphi}_{\Delta}[x, y(x), \pi'(x), p(x)]dx \\ & + \int_D \Delta[x, \bar{y}(x), \pi'(x), \bar{p}(x)]dx \\ & - \int_D \Delta[x, y(x), \pi'(x), p(x)]dx \end{aligned} \tag{3.c.4}$$

where

$$E^{\varphi}_{\Delta}[x, y, p, \bar{p}] \equiv \varphi[x, y, \bar{p}] - \Delta[x, y, p, \bar{p}] \tag{3.c.5}$$

The first integral on the right side of (3.c.4) may not be negative because Y_V is *PQR*; since π is a variety of *class* 2, we may show in the same way as we did for proving Theorem 2.1, that there exists a $0 < \rho_1 < 1$, such that, if $P(V, \bar{V}) < \rho_1$, then the difference between the last two integrals on the right side of (3.c.4) is less than ε.

Let us consider the expression

$$\int_D | E_\Delta^\varphi[x, y(x), \pi'(x), p(x)] | \, dx \tag{3.c.6}$$

by (3.c.5), (3.b.7) and (3.c.3), recalling the defininition of $\Delta[x, y, p, \bar{p}]$, i.e., the definition of $L[x, y, p, \bar{p}]$, since $\pi'(x_i)$ $(i = 1, 2, \cdots, m)$ is bounded, there exists a number k, which depends upon m, n, the variety V and the diameters of the sets D_i $(i = 1, 2, \cdots, m)$, but which depends neither of π nor of ε, such that the expression (3.c.6) is less than $\varepsilon \cdot k$ ([59, vol. 1, § 11, ♯ 142]; [60, § 3, ♯ 9]; [30, § 3, c]). Consequently the absolute value of the integral on the right side of (3.c.4) is also less than $\varepsilon \cdot k$; hence

$$Y_{\bar{V}} > Y_V - \varepsilon(1 + k), \text{ whenever } \rho(V, \bar{V}) < \rho_1 .$$

Thus the theorem is proved.

(d) DEFINITION 3.4. Then integral I_V is said to be *asymptotically evaluable* (abbreviation: *AE*), if it is *PQR* and if there exists a function $L[x, y, p, \bar{p}]$ such that I_V is *LPQR* and if, for every positive R, there exists a function $\varphi^R[x, y, p]$ (as described in § 3.b).

REMARK 3.5. Tonelli [59, vol. 1, page 398–9] gave a procedure by which $\varphi^R[x, y, p]$ may be constructed starting from any *simple* integral $(m = n = 1)$, which is *PQR*: he thus proved that, if a *simple* integral is *PQR*, it is necessarily *AE*. Some criteria of *asymptotic evaluability* are exhibited in [30, § 2, page 140]; although it appears intuitively that every *PQR* integral is also *AE*, this fact was never proved, except in the case $(m = n = 1)$; therefore the statement of any theorem of semicontinuity in whose proof the function $\varphi_R[x, y, p]$ is used, has to contain the hypothesis that this function can be constructed, i.e., that the integral considered is *AE*.

THEOREM 3.6. *If the integral I_V is PQR and AE, it is lower semicontinuous on every ordinary variety.*

Proof. Let us first point out that existence and lowers emicontinuity on any variety of I_V, and those of the integral $\bar{I}_V$ defined by (3.b.2), are equivalent, since the integral

$$\int_D \{\Phi[x, y(x), p(x)] - \bar{\Phi}[x, y(x), p(x)]\}dx \equiv \int_D L[x, y(x), \Omega, p(x)]dx$$

exists and is continuous at every variety, by the Continuity Theorem 1.2.

Let $V = y(x) \equiv y_i(x_i)$ be an ordinary variety, and let $\varepsilon > 0$ be given.

Since $\bar{\Phi}[x, y, p]$ is never negative, it is possible to find a positive number R, such that, if D_i' $(i = 1, 2, \cdots, m)$ is the subset of D_i consisting of the points x_i such that, for each j $(j = 1, 2, \cdots, n)$, the partial derivative $\partial y_i(x_i)/\partial x_i^j$ exists and its absolute value does not exceed R, the inequality

$$\bar{I}_V - \int_{\overset{m}{\underset{1}{\Pi}}{}_i D_i'} \bar{\Phi}[x, y(x), p(x)]dx < \varepsilon/2 \tag{3.d.1}$$

holds.

The integral Y_V, that we associate with I_V and R (see §3.b) is lower semicontinuous on V by Lemma 3.3; i.e., there exists a positive number $\bar{\rho}$ such that, for each variety $\bar{V}$

$$Y_{\bar{V}} > Y_V - \varepsilon/2, \text{ whenever } \rho(V, \bar{V}) > \bar{\rho} . \tag{3.d.2}$$

From (3.b.5) and (3.d.1) we have

$$\bar{I}_V - Y_V < \varepsilon/2$$

whence, by (3.d.2)

$$\bar{I}_{\bar{V}} > \bar{I}_V - \varepsilon, \text{ whenever } \rho(V, \bar{V}) < \bar{\rho}$$

i.e., $\bar{I}_V$ is lower semicontinuous at any ordinary variety, and so is I_V.

DEFINITION 3.7. We shall say that the integral I_V is *lower semicontinuous* at a variety V, such that $I_V = +\infty$, if there exists a positive function $\rho(\varepsilon)$, defined for each positive ε, such that, if $\bar{V}$ is any ordinary variety, then

$$I_{\bar{V}} > \varepsilon, \text{ whenever } \rho(V, \bar{V}) < \rho(\varepsilon) .$$

THEOREM 3.8. *An integral I_V, which is PQR and AE, is lower semicontinuous at every variety V.*

In the case in which V is ordinary, Theorem 3.6 states the lower semicontinuity of I_V on V. If V is not ordinary, the value of I_V on V is $+\infty$ (see Remark 1.7).

Let us again consider $\bar{I}_V$ instead of I_V. Let ε be a given number, and let R be another positive number, such that, if $\bar{\bar{D}}_i$ $(i = 1, 2, \cdots, m)$ denotes the subset of D_i consisting of the points x_i where all the partial derivatives of $y_i(x_i)$ exist and are less than R in absolute value, then

$$\int_{\Pi_1^m \bar{D}_i} \bar{\Phi}[x, y(x), p(x)]dx > \varepsilon + 1 \,. \tag{3.e.1}$$

Like in the proof of Theorem 3.6, we consider again $\varphi^R[x, y, p]$. Y_V exists finite and is lower semicontinuous at V: hence we may find a positive number $\bar{\rho}$, such that, if V is any variety such that

$$\rho(V, \bar{V}) < \bar{\rho} \,, \tag{3.e.2}$$

then

$$Y_{\bar{V}} > Y_V - 1 \,.$$

By (3.b.5) and (3.b.6) ,

$$\begin{aligned} Y_V &\geqq \int_{\Pi_1^m \bar{D}_i} \varphi^R[x, y(x), p(x)]dx \\ &= \int_{\Pi_1^m \bar{D}_i} \bar{\Phi}[x, y(x), p(x)]dx \,, \end{aligned}$$

hence, considering (3.e.1), if (3.e.2) is satisfied,

$$\bar{I}_{\bar{V}} > \varepsilon \,.$$

Therefore $\bar{I}_V$ is semicontinuous at V, and so is I_V. The theorem is thus completely proved.

Conclusion. Let us list four problems which are still open in the area of the study of the semicontinuity of the integrals of the Calculus of Variations in non-parametric form:

Problem 1. No example of any lower semicontinuous integral which is not PQR is known: it appears worth while to investigate whether or not *positive quasi-regularity* is also necessary for lower semicontinuity.

Problem 2. For proving Theorems 3.6, 3.8, we used the construction of the function $\varphi^R[x, y, p]$, and we had to assume that this construction could be made for every R (see § 3.b). It would be interesting to prove Theorem 3.8 without using this construction, i.e., dropping the hypothesis that I_V is AE.

REMARK C.1. The semicontinuity at any variety V of *class* 1, or even just such that all the functions $y_i(x_i)$ are Lipschitzian, can easily be proved for any I_V, which is PQR, without any hypothesis of *asymp-*

totic evaluability, by generalizing the procedure followed in [30, § 3, *First Theorem of Semicontinuity*].

Problem 3. No example of any integral I_V, which is *PQR* without being *AE*, is known. It would be useful to devise a general method by which it would be possible to construct $\varphi^R[x, y, p]$ from R and $\phi[x, y, p]$: thus proving that if I_V is *PQR*, it is necessarily *AE*.

Problem 4. Only varieties which are absolutely continuous in the sense of Tonelli [63] and the *m-uniform* metric were considered in this paper; however, it appears that *positively quasi-regular* integrals are lower semicontinuous even with respect to weaker metrics, on more general classes of varieties. Generalization of the results contained in this paper may be considered.

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