

Pacific Journal of Mathematics

DECOMPOSITION OF HOLOMORPHS

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Let G be a group, and let H be its holomorph. There are two situations in which H is known to be decomposable into the direct product of two proper subgroups. If G is the direct product of two of its proper characteristic subgroups, say G_1 and G_2 , then H is the direct product of the holomorphs of G_1 and G_2 . If G is a complete group, then H is the direct product of G and G^* , where G^* is the centralizer of G in H . In this paper we will show that if G is not the direct product of two proper characteristic subgroups, and if G is not complete, then H is indecomposable. Thus we have a complete characterization of those groups whose holomorphs are indecomposable.

A decomposition of H into the direct product of indecomposable factors is known for the case where G is a finite abelian group [1]. Our present results enable us to generalize this and give a decomposition of H into the direct product of indecomposable factors, whenever G is the direct product of a finite number of characteristically indecomposable characteristic subgroups. In particular this gives a complete decomposition of H whenever G is a finite group.

Peremans [2] has shown that a necessary and sufficient condition for G to be a direct factor of H is that G be either complete or the direct product of a group of order two and a complete group that has no subgroups of index two. This result is related to the present paper. In fact Peremans' result can be deduced from Lemma 1*.

1. **Preliminaries.** Let G be a group, and let A be the group of all automorphisms of G . Let e and I denote the identities of G and A respectively. The holomorph H of G can be regarded as the semi-direct product of G and A , i.e., the set of all pairs (g, σ) , $g \in G$, $\sigma \in A$, with multiplication defined by

$$(g, \sigma)(h, \tau) = (g(\sigma h), \sigma\tau).$$

We identify g in G with (g, I) in H . Then H is a group that contains G as an invariant subgroup, and every automorphism of G can be extended to an inner automorphism of H .

For all a in G we let λ_a denote the inner automorphism of G corresponding to the element a . Thus $\lambda_a g = aga^{-1}$.

All the results of this paper depend on the following lemma:

LEMMA 1. *Let $H = H_1 \times H_2$. Then $G \cap H_1$ and $G \cap H_2$ are characteristic subgroups of G and*

$$G = (G \cap H_1) \times (G \cap H_2) .$$

Proof. We note first that $G \cap H_1$ and $G \cap H_2$ are normal subgroups of H , and hence they are characteristic subgroups of G .

For $i = 1$ or 2 , let ε_i denote the projection of H onto H_i corresponding to the decomposition $H = H_1 \times H_2$. Thus if $\alpha \in H_1$ and $\beta \in H_2$, then $\varepsilon_1(\alpha\beta) = \alpha$ and $\varepsilon_2(\alpha\beta) = \beta$. Put $J_i = \varepsilon_i G$. Clearly $J_i \subseteq H_i$ and J_i is a normal subgroup of H . Let F_i and S_i denote the set of all first and second components respectively of elements of J_i . Thus $F_i \subseteq G$ and $S_i \subseteq A$.

Let (a, σ) be an element of J_1 . Then for some g in G we have $\varepsilon_1 g = (a, \sigma)$. Put $\varepsilon_2 g = (b, \tau)$. Then $g = (a, \sigma)(b, \tau)$. Therefore $\tau = \sigma^{-1}$ and $(\sigma b^{-1}, \sigma) = (b, \tau)^{-1} \in J_2$. Hence $\sigma \in S_2$. It follows that $S_1 \subseteq S_2$. By symmetry $S_2 \subseteq S_1$, and hence $S_1 = S_2$.

Let σ be an element of S_1 and let ξ be an element of A . Put $\varepsilon_i(e, \xi) = (g_i, \xi_i)$, $i = 1, 2$. For some a and c in G we have $(a, \sigma) \in J_1$ and $(c, \sigma) \in J_2$. Now

$$(a, \sigma)(g_2, \xi_2) = (g_2, \xi_2)(a, \sigma)$$

and

$$(c, \sigma)(g_1, \xi_1) = (g_1, \xi_1)(c, \sigma) .$$

Comparing second components we see that σ commutes with both ξ_1 and ξ_2 . Since $\xi = \xi_1 \xi_2$, we have $\sigma \xi = \xi \sigma$. It follows that S_1 is contained in the center of A .

Let (a, σ) be an element of J_1 and let (d, μ) be an element of J_2 . Since σ is contained in the center of A and since $(a, \sigma)^{-1} = (\sigma^{-1} a^{-1}, \sigma^{-1})$, it follows that

$$d(a, \sigma)d^{-1}(e, \lambda_{\sigma a})(a, \sigma)^{-1}(e, \lambda_{\sigma a})^{-1} = d(\sigma d)^{-1} .$$

Therefore $d(\sigma d)^{-1} \in H_1$. Moreover

$$d(\sigma d)^{-1} = (d, \mu)(e, \sigma)(d, \mu)^{-1}(e, \sigma)^{-1} \in H_2 .$$

Hence $d(\sigma d)^{-1} \in H_1 \cap H_2$. This gives us $d(\sigma d)^{-1} = e$ and $\sigma d = d$. Thus σ leaves every element of F_2 fixed. By symmetry, since $\sigma \in S_1 = S_2$, it follows that σ leaves every element of F_1 fixed. Now let g be an arbitrary element of G . Then $g = (f, \nu)(h, \zeta)$ with $(f, \nu) \in J_1$ and $(h, \zeta) \in J_2$. Since $g = f(\nu h)$, $\sigma f = f$, and $\sigma \nu h = \nu \sigma h = \nu h$, it follows that $\sigma g = g$. Hence $\sigma = I$. Therefore S_1 and S_2 consist of the identity alone. It follows that $J_1 \subseteq G \cap H_1$, $J_2 \subseteq G \cap H_2$, and

$$G \subseteq J_1 \times J_2 \subseteq (G \cap H_1) \times (G \cap H_2) \subseteq G .$$

Therefore $G = (G \cap H_1) \times (G \cap H_2)$ and the proof is complete.

2. **Some known results.** Suppose $G = G_1 \times G_2 \times \dots \times G_n$, where the G_i are characteristic subgroups of G . Let A_i denote the group of all automorphisms of G_i . We identify σ_i in A_i with the element σ'_i in A such that

$$\sigma'_i g = \begin{cases} g & \text{if } g \in G_j, j \neq i, \\ \sigma_i g & \text{if } g \in G_i. \end{cases}$$

Then $A = A_1 \times A_2 \times \dots \times A_n$. Moreover H_i , the holomorph of G_i , becomes a subgroup of H , and $H = H_1 \times H_2 \times \dots \times H_n$.

The centralizer of a group in its holomorph is called its conjoint. The conjoint G^* of G consists of the elements (g^{-1}, λ_g) , $g \in G$. The mapping η defined by

$$\eta(g, \sigma) = (g^{-1}, \lambda_g \sigma)$$

is an automorphism of H that maps G onto G^* and maps G^* onto G . Therefore G and G^* are isomorphic, and G is the centralizer of G^* in H . Furthermore Lemma 1 is equivalent to the following:

LEMMA 1*. *Let $H = H_1 \times H_2$. Then $G^* \cap H_1$ and $G^* \cap H_2$ are characteristic subgroups of G^* and*

$$G^* = (G^* \cap H_1) \times (G^* \cap H_2).$$

If G is complete, i.e., if G is a centerless group with only inner automorphisms, then $H = G \times G^*$.

3. **Decomposable and indecomposable holomorphs.** If G is the direct product of two proper characteristic subgroups, then G is said to be characteristically decomposable. If not, then G is said to be characteristically indecomposable.

THEOREM 1. *Let G be a group, and let H be its holomorph. If G is either characteristically decomposable or complete, then H is decomposable. If G is characteristically indecomposable and not complete, then H is indecomposable.*

Proof. We have seen in § 2 that H is decomposable if G is either characteristically decomposable or complete. Suppose that G is characteristically indecomposable and that $H = H_1 \times H_2$. It follows from Lemma 1 that either $G \cap H_1 = G$ or $G \cap H_2 = G$. Thus either $G \subseteq H_1$ or $G \subseteq H_2$. Similarly it follows from Lemma 1* that either $G^* \subseteq H_1$ or $G^* \subseteq H_2$. Without loss of generality suppose that $G \subseteq H_1$. Then H_2 is contained in the centralizer of G , that is $H_2 \subseteq G^*$. If $G^* \subseteq H_1$ we have $H_2 \subseteq H_1$ and $H = H_1$. Thus we need only consider the case $G^* \subseteq H_2$.

Here $G^* = H_2$ and H_1 is contained in the centralizer of G^* . Thus $H_1 \subseteq G$, and hence $H_1 = G$. Now $G \cap G^*$ is the center of G , and $G \cap G^* = H_1 \cap H_2$. Hence G is centerless. Since $H = H_1 \times H_2 = G \times G^*$, it follows that G has only inner automorphisms. Therefore G is complete. This completes the proof of the theorem.

4. Decomposition of the holomorph into indecomposable subgroups.
To complete our discussion we need the following result:

LEMMA 2. *If a group is complete and characteristically indecomposable, then it is indecomposable.*

Proof. Let G be a complete group and suppose $G = G_1 \times G_2$. Since every automorphism of G is inner, it follows that every automorphism of G maps G_1 and G_2 onto themselves. Hence G_1 and G_2 are characteristic subgroups of G . This establishes the lemma.

THEOREM 2. *Suppose G is the direct product of a finite number of characteristically indecomposable characteristic subgroups: $G = G_1 \times G_2 \times \cdots \times G_n$. Suppose that G_i is complete for $1 \leq i \leq r$, and that G_j is not complete for $r + 1 \leq j \leq n$. Then a decomposition of H into indecomposable subgroups is given by*

$$(1) \quad H = \prod_{i=1}^r G_i \times \prod_{i=1}^r G_i^* \times \prod_{i=r+1}^n H_i,$$

where G_i^* and H_i are the conjoint and holomorph respectively of G_i , and where Π denotes a direct product.

Proof. It follows from § 2 that (1) is a decomposition of H . By Lemma 2 the groups G_i and G_i^* are indecomposable for $1 \leq i \leq r$, and by Theorem 1 the groups H_i are indecomposable for $r + 1 \leq i \leq n$.

Since a characteristic subgroup of a characteristic subgroup of G is itself a characteristic subgroup of G it follows that G satisfies the condition of Theorem 2 whenever the characteristic subgroups of G satisfy the descending chain condition. In particular Theorem 2 gives us a decomposition of H into indecomposable subgroups whenever G is a finite group.

If G is the direct product of an infinite number of characteristic subgroups, then H is not the direct product of their holomorphs. Thus Theorem 2 does not hold in this case.

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50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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