DECOMPOSITION OF HOLOMORPHS

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Let $G$ be a group, and let $H$ be its holomorph. There are two situations in which $H$ is known to be decomposable into the direct product of two proper subgroups. If $G$ is the direct product of two of its proper characteristic subgroups, say $G_1$ and $G_2$, then $H$ is the direct product of the holomorphs of $G_1$ and $G_2$. If $G$ is a complete group, then $H$ is the direct product of $G$ and $G^*$, where $G^*$ is the centralizer of $G$ in $H$. In this paper we will show that if $G$ is not the direct product of two proper characteristic subgroups, and if $G$ is not complete, then $H$ is indecomposable. Thus we have a complete characterization of those groups whose holomorphs are indecomposable.

A decomposition of $H$ into the direct product of indecomposable factors is known for the case where $G$ is a finite abelian group [1]. Our present results enable us to generalize this and give a decomposition of $H$ into the direct product of indecomposable factors, whenever $G$ is the direct product of a finite number of characteristically indecomposable characteristic subgroups. In particular this gives a complete decomposition of $H$ whenever $G$ is a finite group.

Peremans [2] has shown that a necessary and sufficient condition for $G$ to be a direct factor of $H$ is that $G$ be either complete or the direct product of a group of order two and a complete group that has no subgroups of index two. This result is related to the present paper. In fact Peremans’ result can be deduced from Lemma 1.

1. Preliminaries. Let $G$ be a group, and let $A$ be the group of all automorphisms of $G$. Let $e$ and $I$ denote the identities of $G$ and $A$ respectively. The holomorph $H$ of $G$ can be regarded as the semi-direct product of $G$ and $A$, i.e., the set of all pairs $(g, \sigma), g \in G, \sigma \in A$, with multiplication defined by

$$(g, \sigma)(h, \tau) = (g(\sigma h), \sigma \tau).$$

We identify $g$ in $G$ with $(g, I)$ in $H$. Then $H$ is a group that contains $G$ as an invariant subgroup, and every automorphism of $G$ can be extended to an inner automorphism of $H$.

For all $a$ in $G$ we let $\lambda_a$ denote the inner automorphism of $G$ corresponding to the element $a$. Thus $\lambda_ag = aga^{-1}$.

All the results of this paper depend on the following lemma:

**Lemma 1.** Let $H = H_1 \times H_2$. Then $G \cap H_1$ and $G \cap H_2$ are characteristic subgroups of $G$ and

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Proof. We note first that $G \cap H_1$ and $G \cap H_2$ are normal subgroups of $H$, and hence they are characteristic subgroups of $G$.

For $i = 1$ or 2, let $\varepsilon_i$ denote the projection of $H$ onto $H_i$ corresponding to the decomposition $H = H_1 \times H_2$. Thus if $\alpha \in H_1$ and $\beta \in H_2$, then $\varepsilon_i(\alpha \beta) = \alpha$ and $\varepsilon_2(\alpha \beta) = \beta$. Put $J_i = \varepsilon_i G$. Clearly $J_i \subseteq H_i$ and $J_i$ is a normal subgroup of $H$. Let $F_i$ and $S_i$ denote the set of all first and second components respectively of elements of $J_i$. Thus $F_i \subseteq G$ and $S_i \subseteq A$.

Let $(a, \sigma)$ be an element of $J_i$. Then for some $g$ in $G$ we have $\varepsilon_i g = (a, \sigma)$. Put $\varepsilon_i g = (b, \tau)$. Then $g = (a, \sigma)(b, \tau)$. Therefore $\tau = \sigma^{-1}$ and $(\sigma b^{-1}, \sigma) = (b, \tau)^{-1} \in J_2$. Hence $\sigma \in S_2$. It follows that $S_1 \subseteq S_2$. By symmetry $S_2 \subseteq S_1$, and hence $S_1 = S_2$.

Let $\sigma$ be an element of $S_1$ and let $\xi$ be an element of $A$. Put $\varepsilon_i(e, \xi) = (g_i, \xi_i)$, $i = 1, 2$. For some $a$ and $c$ in $G$ we have $(a, \sigma) \in J_1$ and $(c, \sigma) \in J_2$. Now

$$(a, \sigma)(g_2, \xi_2) = (g_2, \xi_2)(a, \sigma)$$

and

$$(c, \sigma)(g_1, \xi_1) = (g_1, \xi_1)(c, \sigma).$$

Comparing second components we see that $\sigma$ commutes with both $\xi_1$ and $\xi_2$. Since $\xi = \xi_1 \xi_2$, we have $\sigma \xi = \xi \sigma$. It follows that $S_1$ is contained in the center of $A$.

Let $(a, \sigma)$ be an element of $J_1$ and let $(d, \mu)$ be an element of $J_2$. Since $\sigma$ is contained in the center of $A$ and since $(a, \sigma)^{-1} = (\sigma^{-1} a^{-1}, \sigma^{-1})$, it follows that

$$d(a, \sigma)d^{-1}(e, \lambda_{\sigma a})(a, \sigma)^{-1}(e, \lambda_{\sigma d})^{-1} = d(\sigma d)^{-1}.$$  

Therefore $d(\sigma d)^{-1} \in H_1$. Moreover

$$d(\sigma d)^{-1} = (d, \mu)(e, \sigma)(d, \mu)^{-1}(e, \sigma)^{-1} \in H_2.$$  

Hence $d(\sigma d)^{-1} \in H_1 \cap H_2$. This gives us $d(\sigma d)^{-1} = e$ and $\sigma d = d$. Thus $\sigma$ leaves every element of $F_2$ fixed. By symmetry, since $\sigma \in S_1 = S_2$, it follows that $\sigma$ leaves every element of $F_1$ fixed. Now let $g$ be an arbitrary element of $G$. Then $g = (f, \nu)(h, \xi)$ with $(f, \nu) \in J_1$ and $(h, \xi) \in J_2$. Since $g = f(\nu h)$, $\sigma f = f$, and $\sigma \nu h = \nu \sigma h = \nu h$, it follows that $\sigma g = g$. Hence $\sigma = I$. Therefore $S_1$ and $S_2$ consist of the identity alone. It follows that $J_1 \subseteq G \cap H_1$, $J_2 \subseteq G \cap H_2$, and

$$G \subseteq J_1 \times J_2 \subseteq (G \cap H_1) \times (G \cap H_2) \subseteq G.$$  

Therefore $G = (G \cap H_1) \times (G \cap H_2)$ and the proof is complete.
2. Some known results. Suppose \( G = G_1 \times G_2 \times \cdots \times G_n \), where the \( G_i \) are characteristic subgroups of \( G \). Let \( A_i \) denote the group of all automorphisms of \( G_i \). We identify \( \sigma_i \) in \( A_i \) with the element \( \sigma'_i \) in \( A \) such that
\[
\sigma'_i g = \begin{cases} 
 g & \text{if } g \in G_j, j \neq i , \\
\sigma_i g & \text{if } g \in G_i . 
\end{cases}
\]
Then \( A = A_1 \times A_2 \times \cdots \times A_n \). Moreover \( H_i \), the holomorph of \( G_i \), becomes a subgroup of \( H \), and \( H = H_1 \times H_2 \times \cdots \times H_n \).

The centralizer of a group in its holomorph is called its conjoint. The conjoint \( G^* \) of \( G \) consists of the elements \( (g^{-1}, \lambda_g), g \in G \). The mapping \( \eta \) defined by
\[
\eta(g, \sigma) = (g^{-1}, \lambda_g \sigma)
\]
is an automorphism of \( H \) that maps \( G \) onto \( G^* \) and maps \( G^* \) onto \( G \). Therefore \( G \) and \( G^* \) are isomorphic, and \( G \) is the centralizer of \( G^* \) in \( H \). Furthermore Lemma 1 is equivalent to the following:

**Lemma 1*. Let \( H = H_1 \times H_2 \). Then \( G^* \cap H_1 \) and \( G^* \cap H_2 \) are characteristic subgroups of \( G^* \) and
\[
G^* = (G^* \cap H_i) \times (G^* \cap H_2).
\]

If \( G \) is complete, i.e., if \( G \) is a centerless group with only inner automorphisms, then \( H = G \times G^* \).

3. Decomposable and indecomposable holomorphs. If \( G \) is the direct product of two proper characteristic subgroups, then \( G \) is said to be characteristically decomposable. If not, then \( G \) is said to be characteristically indecomposable.

**Theorem 1*. Let \( G \) be a group, and let \( H \) be its holomorph. If \( G \) is either characteristically decomposable or complete, then \( H \) is decomposable. If \( G \) is characteristically indecomposable and not complete, then \( H \) is indecomposable.

**Proof.** We have seen in § 2 that \( H \) is decomposable if \( G \) is either characteristically decomposable or complete. Suppose that \( G \) is characteristically indecomposable and that \( H = H_1 \times H_2 \). It follows from Lemma 1 that either \( G \cap H_1 = G \) or \( G \cap H_2 = G \). Thus either \( G \subseteq H_1 \) or \( G \subseteq H_2 \). Similarly it follows from Lemma 1* that either \( G^* \subseteq H_1 \) or \( G^* \subseteq H_2 \). Without loss of generality suppose that \( G \subseteq H_1 \). Then \( H_2 \) is contained in the centralizer of \( G \), that is \( H_2 \subseteq G^* \). If \( G^* \subseteq H_1 \) we have \( H_2 \subseteq H_1 \) and \( H = H_1 \). Thus we need only consider the case \( G^* \subseteq H_2 \).
Here $G^* = H_2$ and $H_x$ is contained in the centralizer of $G^*$. Thus $H_i \subseteq G$, and hence $H_i = G$. Now $G \cap G^*$ is the center of $G$, and $G \cap G^* = H_i \cap H_2$. Hence $G$ is centerless. Since $H = H_i \times H_2 = G \times G^*$, it follows that $G$ has only inner automorphisms. Therefore $G$ is complete. This completes the proof of the theorem.

4. Decomposition of the holomorph into indecomposable subgroups. To complete our discussion we need the following result:

**Lemma 2.** If a group is complete and characteristically indecomposable, then it is indecomposable.

**Proof.** Let $G$ be a complete group and suppose $G = G_1 \times G_2$. Since every automorphism of $G$ is inner, it follows that every automorphism of $G$ maps $G_1$ and $G_2$ onto themselves. Hence $G_1$ and $G_2$ are characteristic subgroups of $G$. This establishes the lemma.

**Theorem 2.** Suppose $G$ is the direct product of a finite number of characteristically indecomposable characteristic subgroups: $G = G_1 \times G_2 \times \cdots \times G_n$. Suppose that $G_i$ is complete for $1 \leq i \leq r$, and that $G_j$ is not complete for $r + 1 \leq j \leq n$. Then a decomposition of $H$ into indecomposable subgroups is given by

$$
(1) \quad H = \prod_{i=1}^{r} G_i \times \prod_{i=1}^{r} G_i^* \times \prod_{i=r+1}^{n} H_i,
$$

where $G_i^*$ and $H_i$ are the conjoint and holomorph respectively of $G_i$, and where $\Pi$ denotes a direct product.

**Proof.** It follows from § 2 that (1) is a decomposition of $H$. By Lemma 2 the groups $G_i$ and $G_i^*$ are indecomposable for $1 \leq i \leq r$, and by Theorem 1 the groups $H_i$ are indecomposable for $r + 1 \leq i \leq n$.

Since a characteristic subgroup of a characteristic subgroup of $G$ is itself a characteristic subgroup of $G$ it follows that $G$ satisfies the condition of Theorem 2 whenever the characteristic subgroups of $G$ satisfy the descending chain condition. In particular Theorem 2 gives us a decomposition of $H$ into indecomposable subgroups whenever $G$ is a finite group.

If $G$ is the direct product of an infinite number of characteristic subgroups, then $H$ is not the direct product of their holomorphs. Thus Theorem 2 does not hold in this case.

**Bibliography**


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