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Let G be a group, and let H be its holomorph. There are two situations in which H is known to be decomposable into the direct product of two proper subgroups. If G is the direct product of two of its proper characteristic subgroups, say G_1 and G_2, then H is the direct product of the holomorphs of G_1 and G_2. If G is a complete group, then H is the direct product of G and G^*, where G^* is the centralizer of G in H. In this paper we will show that if G is not the direct product of two proper characteristic subgroups, and if G is not complete, then H is indecomposable. Thus we have a complete characterization of those groups whose holomorphs are indecomposable.

A decomposition of H into the direct product of indecomposable factors is known for the case where G is a finite abelian group [1]. Our present results enable us to generalize this and give a decomposition of H into the direct product of indecomposable factors, whenever G is the direct product of a finite number of characteristically indecomposable characteristic subgroups. In particular this gives a complete decomposition of H whenever G is a finite group.

Peremans [2] has shown that a necessary and sufficient condition for G to be a direct factor of H is that G be either complete or the direct product of a group of order two and a complete group that has no subgroups of index two. This result is related to the present paper. In fact Peremans' result can be deduced from Lemma 1*.

1. Preliminaries. Let G be a group, and let A be the group of all automorphisms of G. Let e and I denote the identities of G and A respectively. The holomorph H of G can be regarded as the semi-direct product of G and A, i.e., the set of all pairs (g, σ), $g \in G, \sigma \in A$, with multiplication defined by

$$(g, \sigma)(h, \tau) = (g(\sigma h), \sigma\tau) .$$

We identify g in G with (g, I) in H. Then H is a group that contains G as an invariant subgroup, and every automorphism of G can be extended to an inner automorphism of H.

For all a in G we let λ_a denote the inner automorphism of G corresponding to the element a. Thus $\lambda_a g = aga^{-1}$.

All the results of this paper depend on the following lemma:

LEMMA 1. *Let $H = H_1 \times H_2$. Then $G \cap H_1$ and $G \cap H_2$ are characteristic subgroups of G and*

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$$G = (G \cap H_1) \times (G \cap H_2) .$$

Proof. We note first that $G \cap H_1$ and $G \cap H_2$ are normal subgroups of H, and hence they are characteristic subgroups of G.

For $i = 1$ or 2, let ε_i denote the projection of H onto H_i corresponding to the decomposition $H = H_1 \times H_2$. Thus if $\alpha \in H_1$ and $\beta \in H_2$, then $\varepsilon_1(\alpha\beta) = \alpha$ and $\varepsilon_2(\alpha\beta) = \beta$. Put $J_i = \varepsilon_i G$. Clearly $J_i \subseteqq H_i$ and J_i is a normal subgroup of H. Let F_i and S_i denote the set of all first and second components respectively of elements of J_i. Thus $F_i \subseteqq G$ and $S_i \subseteqq A$.

Let (a, σ) be an element of J_1. Then for some g in G we have $\varepsilon_1 g = (a, \sigma)$. Put $\varepsilon_2 g = (b, \tau)$. Then $g = (a, \sigma)(b, \tau)$. Therefore $\tau = \sigma^{-1}$ and $(\sigma b^{-1}, \sigma) = (b, \tau)^{-1} \in J_2$. Hence $\sigma \in S_2$. It follows that $S_1 \subseteqq S_2$. By symmetry $S_2 \subseteqq S_1$, and hence $S_1 = S_2$.

Let σ be an element of S_1 and let ξ be an element of A. Put $\varepsilon_i(e, \xi) = (g_i, \xi_i)$, $i = 1, 2$. For some a and c in G we have $(a, \sigma) \in J_1$ and $(c, \sigma) \in J_2$. Now

$$(a, \sigma)(g_2, \xi_2) = (g_2, \xi_2)(a, \sigma)$$

and

$$(c, \sigma)(g_1, \xi_1) = (g_1, \xi_1)(c, \sigma) .$$

Comparing second components we see that σ commutes with both ξ_1 and ξ_2. Since $\xi = \xi_1\xi_2$, we have $\sigma\xi = \xi\sigma$. It follows that S_1 is contained in the center of A.

Let (a, σ) be an element of J_1 and let (d, μ) be an element of J_2. Since σ is contained in the center of A and since $(a, \sigma)^{-1} = (\sigma^{-1}a^{-1}, \sigma^{-1})$, it follows that

$$d(a, \sigma)d^{-1}(e, \lambda_{\sigma a})(a, \sigma)^{-1}(e, \lambda_{\sigma a})^{-1} = d(\sigma d)^{-1} .$$

Therefore $d(\sigma d)^{-1} \in H_1$. Moreover

$$d(\sigma d)^{-1} = (d, \mu)(e, \sigma)(d, \mu)^{-1}(e, \sigma)^{-1} \in H_2 .$$

Hence $d(\sigma d)^{-1} \in H_1 \cap H_2$. This gives us $d(\sigma d)^{-1} = e$ and $\sigma d = d$. Thus σ leaves every element of F_2 fixed. By symmetry, since $\sigma \in S_1 = S_2$, it follows that σ leaves every element of F_1 fixed. Now let g be an arbitrary element of G. Then $g = (f, \nu)(h, \zeta)$ with $(f, \nu) \in J_1$ and $(h, \zeta) \in J_2$. Since $g = f(\nu h)$, $\sigma f = f$, and $\sigma\nu h = \nu\sigma h = \nu h$, it follows that $\sigma g = g$. Hence $\sigma = I$. Therefore S_1 and S_2 consist of the identity alone. It follows that $J_1 \subseteqq G \cap H_1$, $J_2 \subseteqq G \cap H_2$, and

$$G \subseteqq J_1 \times J_2 \subseteqq (G \cap H_1) \times (G \cap H_2) \subseteqq G .$$

Therefore $G = (G \cap H_1) \times (G \cap H_2)$ and the proof is complete.

2. Some known results. Suppose $G = G_1 \times G_2 \times \cdots \times G_n$, where the G_i are characteristic subgroups of G. Let A_i denote the group of all automorphisms of G_i. We identify σ_i in A_i with the element σ'_i in A such that

$$\sigma'_i g = \begin{cases} g & \text{if } g \in G_j, j \neq i , \\ \sigma_i g & \text{if } g \in G_i . \end{cases}$$

Then $A = A_1 \times A_2 \times \cdots \times A_n$. Moreover H_i, the holomorph of G_i, becomes a subgroup of H, and $H = H_1 \times H_2 \times \cdots \times H_n$.

The centralizer of a group in its holomorph is called its conjoint. The conjoint G^* of G consists of the elements (g^{-1}, λ_g), $g \in G$. The mapping η defined by

$$\eta(g, \sigma) = (g^{-1}, \lambda_g \sigma)$$

is an automorphism of H that maps G onto G^* and maps G^* onto G. Therefore G and G^* are isomorphic, and G is the centralizer of G^* in H. Furthermore Lemma 1 is equivalent to the following:

LEMMA 1*. *Let $H = H_1 \times H_2$. Then $G^* \cap H_1$ and $G^* \cap H_2$ are characteristic subgroups of G^* and*

$$G^* = (G^* \cap H_1) \times (G^* \cap H_2) .$$

If G is complete, i.e., if G is a centerless group with only inner automorphisms, then $H = G \times G^*$.

3. Decomposable and indecomposable holomorphs. If G is the direct product of two proper characteristic subgroups, then G is said to be characteristically decomposable. If not, then G is said to be characteristically indecomposable.

THEOREM 1. *Let G be a group, and let H be its holomorph. If G is either characteristically decomposable or complete, then H is decomposable. If G is characteristically indecomposable and not complete, then H is indecomposable.*

Proof. We have seen in § 2 that H is decomposable if G is either characteristically decomposable or complete. Suppose that G is characteristically indecomposable and that $H = H_1 \times H_2$. It follows from Lemma 1 that either $G \cap H_1 = G$ or $G \cap H_2 = G$. Thus either $G \subseteqq H_1$ or $G \subseteqq H_2$. Similarly it follows from Lemma 1* that either $G^* \subseteqq H_1$ or $G^* \subseteqq H_2$. Without loss of generality suppose that $G \subseteqq H_1$. Then H_2 is contained in the centralizer of G, that is $H_2 \subseteqq G^*$. If $G^* \subseteqq H_1$ we have $H_2 \subseteqq H_1$ and $H = H_1$. Thus we need only consider the case $G^* \subseteqq H_2$.

Here $G^* = H_2$ and H_1 is contained in the centralizer of G^*. Thus $H_1 \leqq G$, and hence $H_1 = G$. Now $G \cap G^*$ is the center of G, and $G \cap G^* = H_1 \cap H_2$. Hence G is centerless. Since $H = H_1 \times H_2 = G \times G^*$, it follows that G has only inner automorphisms. Therefore G is complete. This completes the proof of the theorem.

4. Decomposition of the holomorph into indecomposable subgroups. To complete our discussion we need the following result:

LEMMA 2. *If a group is complete and characteristically indecomposable, then it is indecomposable.*

Proof. Let G be a complete group and suppose $G = G_1 \times G_2$. Since every automorphism of G is inner, it follows that every automorphism of G maps G_1 and G_2 onto themselves. Hence G_1 and G_2 are characteristic subgroups of G. This establishes the lemma.

THEOREM 2. *Suppose G is the direct product of a finite number of characteristically indecomposable characteristic subgroups: $G = G_1 \times G_2 \times \cdots \times G_n$. Suppose that G_i is complete for $1 \leqq i \leqq r$, and that G_j is not complete for $r + 1 \leqq j \leqq n$. Then a decomposition of H into indecomposable subgroups is given by*

$$H = \prod_{i=1}^{r} G_i \times \prod_{i=1}^{r} G_i^* \times \prod_{i=r+1}^{n} H_i \,, \tag{1}$$

where G_i^* and H_i are the conjoint and holomorph respectively of G_i, and where Π denotes a direct product.

Proof. It follows from § 2 that (1) is a decomposition of H. By Lemma 2 the groups G_i and G_i^* are indecomposable for $1 \leqq i \leqq r$, and by Theorem 1 the groups H_i are indecomposable for $r + 1 \leqq i \leqq n$.

Since a characteristic subgroup of a characteristic subgroup of G is itself a characteristic subgroup of G it follows that G satisfies the condition of Theorem 2 whenever the characteristic subgroups of G satisfy the descending chain condition. In particular Theorem 2 gives us a decomposition of H into indecomposable subgroups whenever G is a finite group.

If G is the direct product of an infinite number of characteristic subgroups, then H is not the direct product of their holomorphs. Thus Theorem 2 does not hold in this case.

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