A THEOREM ON REGULAR MATRICES

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In this paper it will be proved that if any nonnegative, square matrix $P$ of order $r$ is such that $P^m > 0$ for some positive integer $m$, then $P^{r^2 - 2r + 2} > 0$. This result has already appeared in the literature, [2], but the following is a complete and elementary proof given in detail except for one theorem of I. Schur in [1] which is stated without proof. The term regular is taken from Markov chain theory in which a regular chain is one whose transition matrix has the above property.

A graph $G_P$ associated with any nonnegative, square matrix $P$ of order $r$ is a collection of $r$ distinct points $S = \{s_1, s_2, \ldots, s_r\}$, some or all of which are connected by directed lines. There is a directed line (indicated pictorially by an arrow) from $s_i$ to $s_j$ in the graph $G_P$ if and only if $p_{ij} > 0$ in the matrix $P = (p_{ij})$. A path sequence or path in $G_P$ is any finite sequence of points of $S$ (not necessarily distinct) such that there is a directed line in $G_P$ from every point in the sequence to its immediate successor. The length of a path is one less than the number of occurrences of points in its sequence. A cycle is any path that begins and ends with the same point and a simple cycle is a cycle in which no point occurs twice except, of course, for the first (and last). Two cycles are distinct if their sequences are not cyclic permutations of each other. A nonnegative, square matrix $P$ is regular if $P^m > 0$ for some positive integer $m$. Likewise, a graph $G_P$ associated with a nonnegative, square matrix $P$ is regular if there exists a positive integer $m$ such that an infinite set of paths $A_0, A_1, \ldots, A_n, \ldots$ can be found, the length of each path being $L_n = m + n$, $n = 0, 1, 2, \ldots$. The usual notation $p_{ij}^{(m)}$ is used to denote the $ij$th entry of the matrix $P^m$. In all that follows we shall consider only regular matrices $P$ and their associated graphs $G_P$.

Some immediate consequences of these definitions and the definition of matrix multiplication are the following:

1. There is a path $s_{k_1} \cdots s_{k_{m+1}}$ in $G_P$ if and only if $p_{k_i k_{i+1}}^{(m)} > 0$ in $P^m$.
2. $P$ is regular if and only if $G_P$ is regular.
3. There exists some path from any point in $G_P$ to any point in $G_P$.
4. For any given $i$ and $j$ there exists some $m$ such that $p_{ij}^{(m)} > 0$.
5. If $P^m > 0$ then $P^{m+n} > 0$, $n = 0, 1, 2, \ldots$.

Let $C = \{C_1, C_2, \ldots, C_t\}$ be all the distinct simple cycles of $G_P$ and $\{c_1, c_2, \ldots, c_t\}$ be the corresponding lengths.

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1 This is as treated by Kemeny and Snell in [3].
**Lemma 1.** The length of any cycle $C^*$ is always of the form $c^* = \sum_{i=1}^{t} a_i c_i$, where $a_i$ is some nonnegative integer.

**Proof.** Let any cycle $C^* = s_{k_1}, s_{k_2}, \ldots, s_{k_m}$ be given ($k_i = k_m$). Let $C^* = C^*_i$ and form $C^*_i + 1$ in the following manner from $C^*_i$: Wherever simple cycle $C_i$ occurs in cycle $C^*_i$ delete it except for its last point, thus forming the new cycle $C^*_i + 1$. It is clear that after the $t$th step there will remain only a single point of the original $C^*$, which has of course zero length. If we let $a_i$ be the number of times simple cycle $C_i$ occurred in cycle $C^*_i$ then the lemma follows.

**Theorem 1.** If $G_p$ is any regular graph then it must contain a set of simple cycles whose lengths are relatively prime.

**Proof.** By the regularity assumption and (1) there exists a positive integer $m$ such that cycles of lengths $L_n = m + n$, $n = 0, 1, 2, \ldots$ can be found in $G_p$. Also, from Lemma 1, $L_n = \sum_{i=1}^{t} a_i c_i$ for $n = 0, 1, 2, \ldots$, and suitable $a_i$. Let $d$ be the common factor of the simple cycle lengths $c_i$. Then

$$\sum_{i=1}^{t} a_i c_i = d \sum_{i=1}^{t} a_i c'_i$$

which could never equal $m + n$, $n = 0, 1, 2, \ldots$ unless $d = 1$.

We would like to find a least integer $M$ such that for arbitrary points $s_i$ and $s_j$ there are paths beginning at $s_i$ and ending at $s_j$ and whose lengths are $L_n = M + n$, $n = 0, 1, 2, \ldots$. If we can do this, then, by (1), we shall have also found a least integer $M$ such that $P^* > 0$ where $P$ is the regular matrix associated with $G_p$.

Let us say that a path touches a given set of points if there is some point belonging to both the path and the set. Then we have

**Lemma 2.** Let $G_p$ be a regular graph with $r$ points, let $S$ be a subset containing $r_0$ distinct points of the graph, and let $g$ be any point of $G_p$. Then there always exists a path from $g$ which touches $S$ whose length is less than or equal to $r - r_0$.

**Proof.** If $g \in S$ then the lemma is trivial. Suppose $g \notin S$. By (3) there is at least one path which starts at $g$ and touches the set $S$. Let $p = g_0, g_1, \ldots, s$ be such a path of shortest length. Obviously no point of $S$ can precede the final point $s$ in this path sequence $p$. Furthermore, there can be no repeated points in $p$, for the deletion of any cycle (except for its last point) would produce a path from $g$ to $S$ shorter than path $p$, contrary to the choice of $p$. Therefore, $p$ can have at most $r - r_k$ points.
We shall say that a minimal set of relatively prime integers is a set of relatively prime integers such that if one of the integers is deleted the remaining integers are no longer relatively prime. A step along a path in $G_P$ is a pair of consecutive points of the path sequence.

**Theorem 2.** If $R = \{R_1, R_2, \ldots, R_k\}$ is a set of simple cycles of graph $G_P$ whose lengths $\{r_1, r_2, \ldots, r_k\}$ form a minimal set of relatively prime integers and if $s_i$ and $s_j$ are arbitrary points of $G_P$, then there is always a path which starts at $s_i$, ends at $s_j$, touches each cycle of $R$ and whose length $L \leq (k + 1)r - \sum_{i=1}^{k} r_i - 1$.

**Proof.** Note that the set of distinct points belonging to a simple cycle contains a number of points exactly equal to the length of the cycle. Hence, by Lemma 2 there is a path from an arbitrary point $s_i$ which touches a particular cycle $R_p$ and whose length is less than or equal to $r - r_p$. Thus, we have the following:

<table>
<thead>
<tr>
<th>from</th>
<th>to</th>
<th>greatest number of steps needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>arb. pt.</td>
<td>cycle</td>
<td>$r - r_1$</td>
</tr>
<tr>
<td>cycle</td>
<td>$R_1$</td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>cycle</td>
<td>$R_{k-1}$</td>
<td>$r - r_k$</td>
</tr>
<tr>
<td>$R_k$</td>
<td>arb. pt. $s_j$</td>
<td>$r - 1$</td>
</tr>
<tr>
<td></td>
<td>TOTAL</td>
<td>$L \leq (k + 1)r - \sum_{i=1}^{k} r_i - 1$.</td>
</tr>
</tbody>
</table>

We shall now state without proof I. Schur's theorem cited above and use it in our final theorem.

**Theorem 3.** (Schur) If $\{a_1, a_2, \ldots, a_n\}$ is a set of relatively prime integers with $a_1$ the least and $a_n$ the greatest, then $B = \sum_{i=1}^{n} x_i a_i$ has solutions in nonnegative integers $x_i$ for any $B \geq (a_1 - 1)(a_n - 1)$. This is a best bound for $n = 2$.

**Theorem 4.** If $M$ is the least integer such that paths between any two points of $G_P$ can be found whose lengths are $L_n = M + n, n = 0, 1, 2, \ldots$, then $M \leq r^2 - 2r + 2$.

**Proof.** Given any two points $s_i$ and $s_j$ of $G_P$ we know by Theorem 2 that there is a path from $s_i$ to $s_j$ touching each of the cycles $\{R_i, R_2, \ldots, R_k\}$ and whose length is

$$L \leq (k + 1)r - \sum_{i=1}^{k} r_i - 1.$$
We can, then, interject into this path the simple cycles \( \{R_1, R_2, \ldots, R_k\} \) at the touching points, interjecting cycle \( R_i \), say \( x_i \) times. The length \( L \) of the original path has now been increased to \( L + \sum_{i=1}^{k} x_i r_i = L + B \), the second part of which, by Schur's theorem, can be made to take on any integral value \( B \) where \( B \geq (r_s - 1)(r_g - 1) \), and \( r_s = \min(r_1, r_2, \ldots, r_k) \), \( r_g = \max(r_1, r_2, \ldots, r_k) \). Therefore, we have:

\[
M \leq L + B = (k + 1)r - \sum_{i=1}^{k} r_i - r_s + r_s r_g
\]

**Case I.** Suppose \( k = 2 \). Then \( M \leq 3r - (r_s + r_g) - r_s - r_g + r_s r_g = 3r - 2r_s - 2r_g + r_s r_g = 3r + (r_s - 2)(r_s - 2) - 4 \). The right side of this inequality is obviously maximum when \( r_s \) and \( r_g \) are as large as possible. Recall that \( r_g \leq r \) and \( r_s \leq r - 1 \). Therefore we have:

\[
M \leq 3r + (r - 2)(r - 3) - 4 = r^2 - 2r + 2.
\]

**Case II.** Suppose \( k \geq 3 \). The reader may wish to skip the following formidable looking, though straightforward calculations. They result in a proof that the integer \( M \) with the desired property is in fact smaller when the arbitrary graph contains a larger set of these cycles.

Since the lengths of these cycles are a minimal set of relatively prime integers, it is certainly true that

\[
\sum_{i=1}^{k} r_i \geq r_s + [r_s + 2] + [r_s + 4] + \cdots + [r_s + 2(k - 2)] + r_g
= (k - 1)r_s + (k - 1)(k - 2) + r_g.
\]

Thus, with (7) we have:

\[
M \leq (k + 1)r - [(k - 1)r_s + (k - 1)(k - 2) + r_g] - r_s - r_g + r_s r_g
= (k + 1)r - k r_s - 2r_g + r_s r_g - (k - 1)(k - 2)
= (k + 1)r + (r_s - 2)(r_g - k) - 2k - (k - 1)(k - 2).
\]

Since \( r_g \) must be larger than \( k \), the right side again is maximum when \( r_g \) and \( r_s \) are as large as possible. But \( r_g \leq r \) and \( r_s \leq r - k + 2 \). So \( M \leq (k + 1)r + (r - k)(r - k) - k^2 + k - 2 = r^2 + (1 - k)r + k - 2 \).

This is easily seen to be less than \( r^2 - 2r + 2 \) of Case I, if \( r > 1 \). So in any case \( M \leq r^2 - 2r + 2 \).

To see that \( r^2 - 2r + 2 \) is the least value for an arbitrary graph of \( r \) points and thus for an arbitrary matrix of order \( r \), we need only consider the following example in which \( r = 3 \) and \( M = 5 \).
As a matter of fact it can be shown for any regular matrix $P$ of order $r$ whose graph $G_P$ contains only two cycles, one of length $r$ and one of length $r - 1$, that $P^{r^2-2r+1}$ is not positive. We have, therefore, established the claim of the paper as stated in the opening paragraph.

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