

# Pacific Journal of Mathematics

**A THEOREM ON REGULAR MATRICES**

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In this paper it will be proved that if any nonnegative, square matrix  $P$  of order  $r$  is such that  $P^m > 0$  for some positive integer  $m$ , then  $P^{r^2-2r+2} > 0$ . This result has already appeared in the literature, [2], but the following is a complete and elementary proof given in detail except for one theorem of I. Schur in [1] which is stated without proof. The term regular is taken from Markov chain theory<sup>1</sup> in which a regular chain is one whose transition matrix has the above property.

A graph  $G_P$  associated with any nonnegative, square matrix  $P$  of order  $r$  is a collection of  $r$  distinct points  $S = \{s_1, s_2, \dots, s_r\}$ , some or all of which are connected by directed lines. There is a directed line (indicated pictorially by an arrow) from  $s_i$  to  $s_j$  in the graph  $G_P$  if and only if  $p_{ij} > 0$  in the matrix  $P = (p_{ij})$ . A *path sequence* or *path* in  $G_P$  is any finite sequence of points of  $S$  (not necessarily distinct) such that there is a directed line in  $G_P$  from every point in the sequence to its immediate successor. The *length* of a path is one less than the number of occurrences of points in its sequence. A *cycle* is any path that begins and ends with the same point and a *simple cycle* is a cycle in which no point occurs twice except, of course, for the first (and last). Two cycles are *distinct* if their sequences are not cyclic permutations of each other. A nonnegative, square matrix  $P$  is *regular* if  $P^m > 0$  for some positive integer  $m$ . Likewise, a graph  $G_P$  associated with a nonnegative, square matrix  $P$  is *regular* if there exists a positive integer  $m$  such that an infinite set of paths  $A_0, A_1, \dots, A_n, \dots$  can be found, the length of each path being  $L_n = m + n$ ,  $n = 0, 1, 2, \dots$ . The usual notation  $p_{ij}^{(m)}$  is used to denote the  $ij$ th entry of the matrix  $P^m$ . In all that follows we shall consider only regular matrices  $P$  and their associated graphs  $G_P$ .

Some immediate consequences of these definitions and the definition of matrix multiplication are the following:

- (1) There is a path  $s_{k_1} \dots s_{k_{m+1}}$  in  $G_P$  if and only if  $p_{k_1 k_{m+1}}^{(m)} > 0$  in  $P^m$ .
- (2)  $P$  is regular if and only if  $G_P$  is regular.
- (3) There exists some path from any point in  $G_P$  to any point in  $G_P$ .
- (4) For any given  $i$  and  $j$  there exists some  $m$  such that  $p_{ij}^{(m)} > 0$ .
- (5) If  $P^m > 0$  then  $P^{m+n} > 0$ ,  $n = 0, 1, 2, \dots$ .

Let  $C = \{C_1, C_2, \dots, C_t\}$  be all the distinct simple cycles of  $G_P$  and  $\{c_1, c_2, \dots, c_t\}$  be the corresponding lengths.

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<sup>1</sup> This is as treated by Kemeny and Snell in [3].

LEMMA 1. *The length of any cycle  $C^*$  is always of the form  $c^* = \sum_{i=1}^t a_i c_i$ , where  $a_i$  is some nonnegative integer.*

*Proof.* Let any cycle  $C^* = s_{k_1}, s_{k_2}, \dots, s_{k_m}$  be given ( $k_1 = k_m$ ). Let  $C^* = C_1^*$  and form  $C_{i+1}^*$  in the following manner from  $C_i^*$ : Wherever simple cycle  $C_i$  occurs in cycle  $C_i^*$  delete it except for its last point, thus forming the new cycle  $C_{i+1}^*$ . It is clear that after the  $t$ th step there will remain only a single point of the original  $C^*$ , which has of course zero length. If we let  $a_i$  be the number of times simple cycle  $C_i$  occurred in cycle  $C_i^*$  then the lemma follows.

THEOREM 1. *If  $G_P$  is any regular graph then it must contain a set of simple cycles whose lengths are relatively prime.*

*Proof.* By the regularity assumption and (1) there exists a positive integer  $m$  such that cycles of lengths  $L_n = m + n$ ,  $n = 0, 1, 2, \dots$  can be found in  $G_P$ . Also, from Lemma 1,  $L_n = \sum_{i=1}^t a_i c_i$  for  $n = 0, 1, 2, \dots$ , and suitable  $a_i$ . Let  $d$  be the common factor of the simple cycle lengths  $c_i$ . Then

$$\sum_{i=1}^t a_i c_i = d \sum_{i=1}^t a_i c'_i$$

which could never equal  $m + n$ ,  $n = 0, 1, 2, \dots$  unless  $d = 1$ .

We would like to find a *least* integer  $M$  such that for arbitrary points  $s_i$  and  $s_j$  there are paths beginning at  $s_i$  and ending at  $s_j$  and whose lengths are  $L_n = M + n$ ,  $n = 0, 1, 2, \dots$ . If we can do this, then, by (1), we shall have also found a least integer  $M$  such that  $P^M > 0$  where  $P$  is the regular matrix associated with  $G_P$ .

Let us say that a path *touches* a given set of points if there is some point belonging to both the path and the set. Then we have

LEMMA 2. *Let  $G_P$  be a regular graph with  $r$  points, let  $S$  be a subset containing  $r_k$  distinct points of the graph, and let  $g$  be any point of  $G_P$ . Then there always exists a path from  $g$  which touches  $S$  whose length is less than or equal to  $r - r_k$ .*

*Proof.* If  $g \in S$  then the lemma is trivial. Suppose  $g \notin S$ . By (3) there is at least one path which starts at  $g$  and touches the set  $S$ . Let  $p = g_0, g_1, \dots, s$  be such a path of shortest length. Obviously no point of  $S$  can precede the final point  $s$  in this path sequence  $p$ . Furthermore, there can be no repeated points in  $p$ , for the deletion of any cycle (except for its last point) would produce a path from  $g$  to  $S$  shorter than path  $p$ , contrary to the choice of  $p$ . Therefore,  $p$  can have at most  $r - r_k$  points.

We shall say that a *minimal set* of relatively prime integers is a set of relatively prime integers such that if one of the integers is deleted the remaining integers are no longer relatively prime. A *step* along a path in  $G_P$  is a pair of consecutive points of the path sequence.

**THEOREM 2.** *If  $R = \{R_1, R_2, \dots, R_k\}$  is a set of simple cycles of graph  $G_P$  whose lengths  $\{r_1, r_2, \dots, r_k\}$  form a minimal set of relatively prime integers and if  $s_i$  and  $s_j$  are arbitrary points of  $G_P$ , then there is always a path which starts at  $s_i$ , ends at  $s_j$ , touches each cycle of  $R$  and whose length  $L \leq (k + 1)r - \sum_{i=1}^k r_i - 1$ .*

*Proof.* Note that the set of distinct points belonging to a simple cycle contains a number of points exactly equal to the length of the cycle. Hence, by Lemma 2 there is a path from an arbitrary point  $s_i$  which touches a particular cycle  $R_p$  and whose length is less than or equal to  $r - r_p$ . Thus, we have the following:

	<i>from</i>		<i>to</i>		<i>greatest number of steps needed</i>
arb. pt.	$s_i$		cycle	$R_1$	$r - r_1$
cycle	$R_1$		"	$R_2$	$r - r_2$
⋮	⋮		⋮	⋮	⋮
⋮	⋮		⋮	⋮	⋮
cycle	$R_{k-1}$		cycle	$R_k$	$r - r_k$
"	$R_k$		arb. pt.	$s_j$	$r - 1$
<b>TOTAL</b>					$L \leq (k + 1)r - \sum_{i=1}^k r_i - 1.$

We shall now state without proof I. Schur's theorem cited above and use it in our final theorem.

**THEOREM 3.** (Schur) *If  $\{a_1, a_2, \dots, a_n\}$  is a set of relatively prime integers with  $a_1$  the least and  $a_n$  the greatest, then  $B = \sum_{i=1}^n x_i a_i$  has solutions in nonnegative integers  $x_i$  for any  $B \geq (a_1 - 1)(a_n - 1)$ . This is a best bound for  $n = 2$ .*

**THEOREM 4.** *If  $M$  is the least integer such that paths between any two points of  $G_P$  can be found whose lengths are  $L_n = M + n$ ,  $n = 0, 1, 2, \dots$ , then  $M \leq r^2 - 2r + 2$ .*

*Proof.* Given any two points  $s_i$  and  $s_j$  of  $G_P$  we know by Theorem 2 that there is a path from  $s_i$  to  $s_j$  touching each of the cycles  $\{R_1, R_2, \dots, R_k\}$  and whose length is

$$L \leq (k + 1)r - \sum_{i=1}^k r_i - 1.$$

We can, then, interject into this path the simple cycles  $\{R_1, R_2, \dots, R_k\}$  at the touching points, interjecting cycle  $R_i$  say  $x_i$  times. The length  $L$  of the original path has now been increased to  $L + \sum_{i=1}^k x_i r_i = L + B$ , the second part of which, by Schur's theorem, can be made to take on any integral value  $B$  where  $B \geq (r_s - 1)(r_o - 1)$ , and  $r_s = \min(r_1, r_2, \dots, r_k)$ ,  $r_o = \max(r_1, r_2, \dots, r_k)$ . Therefore, we have:

$$(7) \quad M \leq L + B = (k + 1)r - \sum_{i=1}^k r_i - r_s - r_o + r_s r_o$$

*Case I.* Suppose  $k = 2$ . Then  $M \leq 3r - (r_s + r_o) - r_s - r_o + r_s r_o = 3r - 2r_s - 2r_o + r_s r_o = 3r + (r_o - 2)(r_s - 2) - 4$ . The right side of this inequality is obviously maximum when  $r_s$  and  $r_o$  are as large as possible. Recall that  $r_o \leq r$  and  $r_s \leq r - 1$ . Therefore we have:

$$(8) \quad M \leq 3r + (r - 2)(r - 3) - 4 = r^2 - 2r + 2.$$

*Case II.* Suppose  $k \geq 3$ . The reader may wish to skip the following formidable looking, though straightforward calculations. They result in a proof that the integer  $M$  with the desired property is in fact smaller when the arbitrary graph contains a larger set of these cycles.

Since the lengths of these cycles are a minimal set of relatively prime integers, it is certainly true that

$$\begin{aligned} \sum_{i=1}^k r_i &\geq r_s + [r_s + 2] + [r_s + 4] + \dots + [r_s + 2(k - 2)] + r_o \\ &= (k - 1)r_s + (k - 1)(k - 2) + r_o. \end{aligned}$$

Thus, with (7) we have:

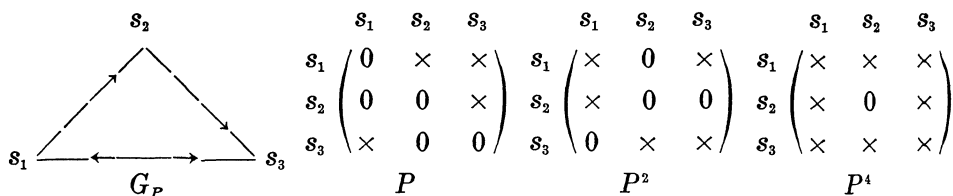
$$\begin{aligned} M &\leq (k + 1)r - [(k - 1)r_s + (k - 1)(k - 2) + r_o] - r_s - r_o + r_s r_o \\ &= (k + 1)r - kr_s - 2r_o + r_s r_o - (k - 1)(k - 2) \\ &= (k + 1)r + (r_s - 2)(r_o - k) - 2k - (k - 1)(k - 2). \end{aligned}$$

Since  $r_o$  must be larger than  $k$ , the right side again is maximum when  $r_o$  and  $r_s$  are as large as possible. But  $r_o \leq r$  and  $r_s \leq r - k + 2$ . So

$$\begin{aligned} M &\leq (k + 1)r + (r - k)(r - k) - k^2 + k - 2 \\ &= r^2 + (1 - k)r + k - 2. \end{aligned}$$

This is easily seen to be less than  $r^2 - 2r + 2$  of Case I, if  $r > 1$ . So in any case  $M \leq r^2 - 2r + 2$ .

To see that  $r^2 - 2r + 2$  is the least value for an arbitrary graph of  $r$  points and thus for an arbitrary matrix of order  $r$ , we need only consider the following example in which  $r = 3$  and  $M = 5$ .



As a matter of fact it can be shown for *any* regular matrix  $P$  of order  $r$  whose graph  $G_P$  contains only two cycles, one of length  $r$  and one of length  $r - 1$ , that  $P^{r^2-2r+1}$  is not positive. We have, therefore, established the claim of the paper as stated in the opening paragraph.

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