A THEOREM ON REGULAR MATRICES

Peter Perkins
A THEOREM ON REGULAR MATRICES

PETER PERKINS

In this paper it will be proved that if any nonnegative, square matrix $P$ of order $r$ is such that $P^m > 0$ for some positive integer $m$, then $P^{r^2-2r+2} > 0$. This result has already appeared in the literature, [2], but the following is a complete and elementary proof given in detail except for one theorem of I. Schur in [1] which is stated without proof. The term regular is taken from Markov chain theory in which a regular chain is one whose transition matrix has the above property.

A graph $G_P$ associated with any nonnegative, square matrix $P$ of order $r$ is a collection of $r$ distinct points $S = \{s_1, s_2, \cdots, s_r\}$, some or all of which are connected by directed lines. There is a directed line (indicated pictorially by an arrow) from $s_i$ to $s_j$ in the graph $G_P$ if and only if $p_{ij} > 0$ in the matrix $P = (p_{ij})$. A path sequence or path in $G_P$ is any finite sequence of points of $S$ (not necessarily distinct) such that there is a directed line in $G_P$ from every point in the sequence to its immediate successor. The length of a path is one less than the number of occurrences of points in its sequence. A cycle is any path that begins and ends with the same point and a simple cycle is a cycle in which no point occurs twice except, of course, for the first (and last). Two cycles are distinct if their sequences are not cyclic permutations of each other. A nonnegative, square matrix $P$ is regular if $P^m > 0$ for some positive integer $m$. Likewise, a graph $G_P$ associated with a nonnegative, square matrix $P$ is regular if there exists a positive integer $m$ such that an infinite set of paths $A_0, A_1, \cdots, A_n, \cdots$ can be found, the length of each path being $L_n = m + n, n = 0, 1, 2, \cdots$. The usual notation $p_{ij}^{(m)}$ is used to denote the $ij$th entry of the matrix $P^m$. In all that follows we shall consider only regular matrices $P$ and their associated graphs $G_P$.

Some immediate consequences of these definitions and the definition of matrix multiplication are the following:

1. There is a path $s_{k_1} \cdots s_{k_{m+1}}$ in $G_P$ if and only if $p_{s_{k_1}}^{(m)} > 0$ in $P^m$.
2. $P$ is regular if and only if $G_P$ is regular.
3. There exists some path from any point in $G_P$ to any point in $G_P$.
4. For any given $i$ and $j$ there exists some $m$ such that $p_{ij}^{(m)} > 0$.
5. If $P^m > 0$ then $P^{m+n} > 0, n = 0, 1, 2, \cdots$.

Let $C = \{C_1, C_2, \cdots, C_l\}$ be all the distinct simple cycles of $G_P$ and $\{c_1, c_2, \cdots, c_l\}$ be the corresponding lengths.

Received November 21, 1960. I wish to thank Professor R. Z. Norman for his suggestions in the writing of this paper.

1 This is as treated by Kemeny and Snell in [3].
**Lemma 1.** The length of any cycle $C^*$ is always of the form $c^* = \sum_{i=1}^{t} a_i c_i$, where $a_i$ is some nonnegative integer.

**Proof.** Let any cycle $C^* = s_{k_1}, s_{k_2}, \ldots, s_{k_m}$ be given ($k_1 = k_m$). Let $C^* = C_1^*$ and form $C_{i+1}^*$ in the following manner from $C_i^*$: Wherever simple cycle $C_i$ occurs in cycle $C_i^*$ delete it except for its last point, thus forming the new cycle $C_{i+1}^*$. It is clear that after the $i$th step there will remain only a single point of the original $C^*$, which has of course zero length. If we let $a_i$ be the number of times simple cycle $C_i$ occurred in cycle $C_i^*$ then the lemma follows.

**Theorem 1.** If $G_P$ is any regular graph then it must contain a set of simple cycles whose lengths are relatively prime.

**Proof.** By the regularity assumption and (1) there exists a positive integer $m$ such that cycles of lengths $L_n = m + n$, $n = 0, 1, 2, \ldots$ can be found in $G_P$. Also, from Lemma 1, $L_n = \sum_{i=1}^{t} a_i c_i$ for $n = 0, 1, 2, \ldots$, and suitable $a_i$. Let $d$ be the common factor of the simple cycle lengths $c_i$. Then

$$\sum_{i=1}^{t} a_i c_i = d \sum_{i=1}^{t} a_i c'_i$$

which could never equal $m + n$, $n = 0, 1, 2, \ldots$ unless $d = 1$.

We would like to find a least integer $M$ such that for arbitrary points $s_i$ and $s_j$ there are paths beginning at $s_i$ and ending at $s_j$ and whose lengths are $L_n = M + n$, $n = 0, 1, 2, \ldots$. If we can do this, then, by (1), we shall have also found a least integer $M$ such that $P^M > 0$ where $P$ is the regular matrix associated with $G_P$.

Let us say that a path touches a given set of points if there is some point belonging to both the path and the set. Then we have

**Lemma 2.** Let $G_P$ be a regular graph with $r$ points, let $S$ be a subset containing $r_k$ distinct points of the graph, and let $g$ be any point of $G_P$. Then there always exists a path from $g$ which touches $S$ whose length is less than or equal to $r - r_k$.

**Proof.** If $g \in S$ then the lemma is trivial. Suppose $g \notin S$. By (3) there is at least one path which starts at $g$ and touches the set $S$. Let $p = g, g, \ldots, s$ be such a path of shortest length. Obviously no point of $S$ can precede the final point $s$ in this path sequence $p$. Furthermore, there can be no repeated points in $p$, for the deletion of any cycle (except for its last point) would produce a path from $g$ to $S$ shorter than path $p$, contrary to the choice of $p$. Therefore, $p$ can have at most $r - r_k$ points.
We shall say that a minimal set of relatively prime integers is a set of relatively prime integers such that if one of the integers is deleted the remaining integers are no longer relatively prime. A step along a path in $G_p$ is a pair of consecutive points of the path sequence.

**Theorem 2.** If $R = \{R_1, R_2, \ldots, R_k\}$ is a set of simple cycles of graph $G_p$ whose lengths $\{r_1, r_2, \ldots, r_k\}$ form a minimal set of relatively prime integers and if $s_i$ and $s_j$ are arbitrary points of $G_p$, then there is always a path which starts at $s_i$, ends at $s_j$, touches each cycle of $R$ and whose length $L \leq (k + 1)r - \sum_{i=1}^{k} r_i - 1$.

**Proof.** Note that the set of distinct points belonging to a simple cycle contains a number of points exactly equal to the length of the cycle. Hence, by Lemma 2 there is a path from an arbitrary point $s_i$ which touches a particular cycle $R_p$ and whose length is less than or equal to $r - r_p$. Thus, we have the following:

<table>
<thead>
<tr>
<th>from</th>
<th>to</th>
<th>greatest number of steps needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>arb. pt. $s_i$</td>
<td>cycle $R_1$</td>
<td>$r - r_1$</td>
</tr>
<tr>
<td>cycle $R_1$</td>
<td>cycle $R_2$</td>
<td>$r - r_2$</td>
</tr>
<tr>
<td>cycle $R_{k-1}$</td>
<td>cycle $R_k$</td>
<td>$r - r_k$</td>
</tr>
<tr>
<td>cycle $R_k$</td>
<td>arb. pt. $s_j$</td>
<td>$r - 1$</td>
</tr>
<tr>
<td></td>
<td>TOTAL</td>
<td>$L \leq (k + 1)r - \sum_{i=1}^{k} r_i - 1$.</td>
</tr>
</tbody>
</table>

We shall now state without proof I. Schur's theorem cited above and use it in our final theorem.

**Theorem 3.** (Schur) If $\{a_1, a_2, \ldots, a_n\}$ is a set of relatively prime integers with $a_1$ the least and $a_n$ the greatest, then $B = \sum_{i=1}^{n} x_i a_i$ has solutions in nonnegative integers $x_i$ for any $B \geq (a_1 - 1)(a_n - 1)$. This is a best bound for $n = 2$.

**Theorem 4.** If $M$ is the least integer such that paths between any two points of $G_p$ can be found whose lengths are $L_n = M + n$, $n = 0, 1, 2, \ldots$, then $M \leq r^2 - 2r + 2$.

**Proof.** Given any two points $s_i$ and $s_j$ of $G_p$ we know by Theorem 2 that there is a path from $s_i$ to $s_j$ touching each of the cycles $\{R_1, R_2, \ldots, R_k\}$ and whose length is

$$L \leq (k + 1)r - \sum_{i=1}^{k} r_i - 1.$$
We can, then, interject into this path the simple cycles \( \{ R_1, R_2, \ldots, R_k \} \) at the touching points, interjecting cycle \( R_i \) say \( x_i \) times. The length \( L \) of the original path has now been increased to \( L + \sum_{i=1}^{k} x_i r_i = L + B \), the second part of which, by Schur's theorem, can be made to take on any integral value \( B \) where \( B \geq (r_s - 1)(r_g - 1) \), and \( r_s = \min(r_1, r_2, \ldots, r_k) \), \( r_g = \max(r_1, r_2, \ldots, r_k) \). Therefore, we have:

\[
M \leq L + B = (k + 1)r - \sum_{i=1}^{k} r_i - r_s - r_g + rsr_g
\]

**Case I.** Suppose \( k = 2 \). Then \( M \leq 3r - (r_s + r_g) - r_s - r_g + r_sr_g = 3r - 2r_s - 2r_g + r_sr_g = 3r + (r_g - 2)(r_s - 2) - 4 \). The right side of this inequality is obviously maximum when \( r_s \) and \( r_g \) are as large as possible. Recall that \( r_g \leq r \) and \( r_s \leq r - 1 \). Therefore we have:

\[
M \leq 3r + (r - 2)(r - 3) - 4 = r^2 - 2r + 2.
\]

**Case II.** Suppose \( k \geq 3 \). The reader may wish to skip the following formidable looking, though straightforward calculations. They result in a proof that the integer \( M \) with the desired property is in fact smaller when the arbitrary graph contains a larger set of these cycles.

Since the lengths of these cycles are a minimal set of relatively prime integers, it is certainly true that

\[
\sum_{i=1}^{k} r_i \geq r_s + [r_s + 2] + [r_s + 4] + \cdots + [r_s + 2(k - 2)] + r_g
\]

\[
= (k - 1)r_s + (k - 1)(k - 2) + r_g.
\]

Thus, with (7) we have:

\[
M \leq (k + 1)r - [(k - 1)r_s + (k - 1)(k - 2) + r_g] - r_s - r_g + r sr_g
\]

\[
= (k + 1)r - kr_s - 2r_g + r_sr_g - (k - 1)(k - 2)
\]

\[
= (k + 1)r + (r_s - 2)(r_g - k) - 2k - (k - 1)(k - 2).
\]

Since \( r_g \) must be larger than \( k \), the right side again is maximum when \( r_g \) and \( r_s \) are as large as possible. But \( r_g \leq k \) and \( r_s \leq r - k + 2 \). So

\[
M \leq (k + 1)r + (r - k)(r - k) - k^2 + k - 2
\]

\[
= r^2 - (1 - k)r + k - 2.
\]

This is easily seen to be less than \( r^2 - 2r + 2 \) of Case I, if \( r > 1 \). So in any case \( M \leq r^2 - 2r + 2 \).

To see that \( r^2 - 2r + 2 \) is the least value for an arbitrary graph of \( r \) points and thus for an arbitrary matrix of order \( r \), we need only consider the following example in which \( r = 3 \) and \( M = 5 \).
As a matter of fact it can be shown for any regular matrix $P$ of order $r$ whose graph $G_P$ contains only two cycles, one of length $r$ and one of length $r - 1$, that $P^{r^2-2r+1}$ is not positive. We have, therefore, established the claim of the paper as stated in the opening paragraph.

**BIBLIOGRAPHY**


Dartmouth College
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Reprinted 1966 in the United States of America
A. V. Balakrishnan, Prediction theory for Markoff processes .................................................. 1171
Dallas O. Banks, Upper bounds for the eigenvalues of some vibrating systems . . . . . 1183
A. Białynicki-Birula, On the field of rational functions of algebraic groups . . . . . 1205
Thomas Andrew Brown, Simple paths on convex polyhedra ............................................. 1211
L. Carlitz, Some congruences for the Bell polynomials ................................................... 1215
Paul Civin, Extensions of homomorphisms ................................................................. 1223
Paul Joseph Cohen and Milton Lees, Asymptotic decay of solutions of differential inequalities ................................................................. 1235
István Fáry, Self-intersection of a sphere on a complex quadric ................................... 1251
Walter Feit and John Griggs Thompson, Groups which have a faithful representation of degree less than \((p - 1/2)\) .................................................. 1257
William James Firey, Mean cross-section measures of harmonic means of convex bodies . ................................................................. 1263
Avner Friedman, The wave equation for differential forms .......................................... 1267
Bernard Russel Gelbaum and Jesus Gil De Lamadrid, Bases of tensor products of Banach spaces ................................................................. 1281
Ronald Kay Getoor, Infinitely divisible probabilities on the hyperbolic plane ................. 1287
Basil Gordon, Sequences in groups with distinct partial products .................................. 1309
Magnus R. Hestenes, Relative self-adjoint operators in Hilbert space ............................ 1315
Fu Cheng Hsiang, On a theorem of Fejér ................................................................. 1359
John McCormick Irwin and Elbert A. Walker, On \(N\)-high subgroups of Abelian groups ................................................................. 1363
John McCormick Irwin, High subgroups of Abelian torsion groups ............................... 1375
R. E. Johnson, Quotient rings of rings with zero singular ideal ...................................... 1385
David G. Kendall and John Leonard Mott, The asymptotic distribution of the time-to-escape for comets strongly bound to the solar system .......................................... 1393
Kurt Kreith, The spectrum of singular self-adjoint elliptic operators ............................... 1401
Lionello Lombardi, The semicontinuity of the most general integral of the calculus of variations in non-parametric form .................................................. 1407
Albert W. Marshall and Ingram Olkin, Game theoretic proof that Chebyshev inequalities are sharp ................................................................. 1421
Wallace Smith Martindale, III, Primitive algebras with involution .................................. 1431
William H. Mills, Decomposition of holomorphs ...................................................... 1443
James Donald Monk, On the representation theory for cylindric algebras .................... 1447
Shu-Teh Chen Moy, A note on generalizations of Shannon-McMillan theorem ................ 1459
Donald Earl Myers, An imbedding space for Schwartz distributions ............................ 1467
John R. Myhill, Category methods in recursion theory .................................................. 1479
Paul Adrian Nickel, On extremal properties for annular radial and circular slit mappings of bordered Riemann surfaces .................................................. 1487
Edward Scott O’Keefe, Primal clusters of two-element algebras .................................... 1505
Nelson Onuchic, Applications of the topological method of Ważewski to certain problems of asymptotic behavior in ordinary differential equations ....................... 1511
Peter Perkins, A theorem on regular matrices ............................................................. 1529
Clinton M. Petty, Centroid surfaces ............................................................................. 1535
Charles Andrew Swanson, Asymptotic estimates for limit circle problems ..................... 1549
Robert James Thompson, On essential absolute continuity ........................................ 1561
Harold H. Johnson, Correction to "Terminating prolongation procedures" ..................... 1571