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A THEOREM ON REGULAR MATRICES

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In this paper it will be proved that if any nonnegative, square matrix P of order r is such that $P^m > 0$ for some positive integer m , then $P^{r^2-2r+2} > 0$. This result has already appeared in the literature, [2], but the following is a complete and elementary proof given in detail except for one theorem of I. Schur in [1] which is stated without proof. The term regular is taken from Markov chain theory¹ in which a regular chain is one whose transition matrix has the above property.

A graph G_P associated with any nonnegative, square matrix P of order r is a collection of r distinct points $S = \{s_1, s_2, \dots, s_r\}$, some or all of which are connected by directed lines. There is a directed line (indicated pictorially by an arrow) from s_i to s_j in the graph G_P if and only if $p_{ij} > 0$ in the matrix $P = (p_{ij})$. A *path sequence* or *path* in G_P is any finite sequence of points of S (not necessarily distinct) such that there is a directed line in G_P from every point in the sequence to its immediate successor. The *length* of a path is one less than the number of occurrences of points in its sequence. A *cycle* is any path that begins and ends with the same point and a *simple cycle* is a cycle in which no point occurs twice except, of course, for the first (and last). Two cycles are *distinct* if their sequences are not cyclic permutations of each other. A nonnegative, square matrix P is *regular* if $P^m > 0$ for some positive integer m . Likewise, a graph G_P associated with a nonnegative, square matrix P is *regular* if there exists a positive integer m such that an infinite set of paths $A_0, A_1, \dots, A_n, \dots$ can be found, the length of each path being $L_n = m + n$, $n = 0, 1, 2, \dots$. The usual notation $p_{ij}^{(m)}$ is used to denote the ij th entry of the matrix P^m . In all that follows we shall consider only regular matrices P and their associated graphs G_P .

Some immediate consequences of these definitions and the definition of matrix multiplication are the following:

- (1) There is a path $s_{k_1} \dots s_{k_{m+1}}$ in G_P if and only if $p_{k_1 k_{m+1}}^{(m)} > 0$ in P^m .
- (2) P is regular if and only if G_P is regular.
- (3) There exists some path from any point in G_P to any point in G_P .
- (4) For any given i and j there exists some m such that $p_{ij}^{(m)} > 0$.
- (5) If $P^m > 0$ then $P^{m+n} > 0$, $n = 0, 1, 2, \dots$.

Let $C = \{C_1, C_2, \dots, C_t\}$ be all the distinct simple cycles of G_P and $\{c_1, c_2, \dots, c_t\}$ be the corresponding lengths.

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¹ This is as treated by Kemeny and Snell in [3].

LEMMA 1. *The length of any cycle C^* is always of the form $c^* = \sum_{i=1}^t a_i c_i$, where a_i is some nonnegative integer.*

Proof. Let any cycle $C^* = s_{k_1}, s_{k_2}, \dots, s_{k_m}$ be given ($k_1 = k_m$). Let $C^* = C_1^*$ and form C_{i+1}^* in the following manner from C_i^* : Wherever simple cycle C_i occurs in cycle C_i^* delete it except for its last point, thus forming the new cycle C_{i+1}^* . It is clear that after the t th step there will remain only a single point of the original C^* , which has of course zero length. If we let a_i be the number of times simple cycle C_i occurred in cycle C_i^* then the lemma follows.

THEOREM 1. *If G_P is any regular graph then it must contain a set of simple cycles whose lengths are relatively prime.*

Proof. By the regularity assumption and (1) there exists a positive integer m such that cycles of lengths $L_n = m + n$, $n = 0, 1, 2, \dots$ can be found in G_P . Also, from Lemma 1, $L_n = \sum_{i=1}^t a_i c_i$ for $n = 0, 1, 2, \dots$, and suitable a_i . Let d be the common factor of the simple cycle lengths c_i . Then

$$\sum_{i=1}^t a_i c_i = d \sum_{i=1}^t a_i c_i'$$

which could never equal $m + n$, $n = 0, 1, 2, \dots$ unless $d = 1$.

We would like to find a *least* integer M such that for arbitrary points s_i and s_j there are paths beginning at s_i and ending at s_j and whose lengths are $L_n = M + n$, $n = 0, 1, 2, \dots$. If we can do this, then, by (1), we shall have also found a least integer M such that $P^M > 0$ where P is the regular matrix associated with G_P .

Let us say that a path *touches* a given set of points if there is some point belonging to both the path and the set. Then we have

LEMMA 2. *Let G_P be a regular graph with r points, let S be a subset containing r_k distinct points of the graph, and let g be any point of G_P . Then there always exists a path from g which touches S whose length is less than or equal to $r - r_k$.*

Proof. If $g \in S$ then the lemma is trivial. Suppose $g \notin S$. By (3) there is at least one path which starts at g and touches the set S . Let $p = g_0, g_1, \dots, s$ be such a path of shortest length. Obviously no point of S can precede the final point s in this path sequence p . Furthermore, there can be no repeated points in p , for the deletion of any cycle (except for its last point) would produce a path from g to S shorter than path p , contrary to the choice of p . Therefore, p can have at most $r - r_k$ points.

We shall say that a *minimal set* of relatively prime integers is a set of relatively prime integers such that if one of the integers is deleted the remaining integers are no longer relatively prime. A *step* along a path in G_P is a pair of consecutive points of the path sequence.

THEOREM 2. *If $R = \{R_1, R_2, \dots, R_k\}$ is a set of simple cycles of graph G_P whose lengths $\{r_1, r_2, \dots, r_k\}$ form a minimal set of relatively prime integers and if s_i and s_j are arbitrary points of G_P , then there is always a path which starts at s_i , ends at s_j , touches each cycle of R and whose length $L \leq (k + 1)r - \sum_{i=1}^k r_i - 1$.*

Proof. Note that the set of distinct points belonging to a simple cycle contains a number of points exactly equal to the length of the cycle. Hence, by Lemma 2 there is a path from an arbitrary point s_i which touches a particular cycle R_p and whose length is less than or equal to $r - r_p$. Thus, we have the following:

<i>from</i>	<i>to</i>	<i>greatest number of steps needed</i>
arb. pt. s_i	cycle R_1	$r - r_1$
cycle R_1	" R_2	$r - r_2$
⋮	⋮	⋮
⋮	⋮	⋮
⋮	⋮	⋮
cycle R_{k-1}	cycle R_k	$r - r_k$
" R_k	arb. pt. s_j	$r - 1$
TOTAL		$L \leq (k + 1)r - \sum_{i=1}^k r_i - 1$.

We shall now state without proof I. Schur's theorem cited above and use it in our final theorem.

THEOREM 3. (Schur) *If $\{a_1, a_2, \dots, a_n\}$ is a set of relatively prime integers with a_1 the least and a_n the greatest, then $B = \sum_{i=1}^n x_i a_i$ has solutions in nonnegative integers x_i for any $B \geq (a_1 - 1)(a_n - 1)$. This is a best bound for $n = 2$.*

THEOREM 4. *If M is the least integer such that paths between any two points of G_P can be found whose lengths are $L_n = M + n$, $n = 0, 1, 2, \dots$, then $M \leq r^2 - 2r + 2$.*

Proof. Given any two points s_i and s_j of G_P we know by Theorem 2 that there is a path from s_i to s_j touching each of the cycles $\{R_1, R_2, \dots, R_k\}$ and whose length is

$$L \leq (k + 1)r - \sum_{i=1}^k r_i - 1 .$$

We can, then, interject into this path the simple cycles $\{R_1, R_2, \dots, R_k\}$ at the touching points, interjecting cycle R_i say x_i times. The length L of the original path has now been increased to $L + \sum_{i=1}^k x_i r_i = L + B$, the second part of which, by Schur's theorem, can be made to take on any integral value B where $B \geq (r_s - 1)(r_o - 1)$, and $r_s = \min(r_1, r_2, \dots, r_k)$, $r_o = \max(r_1, r_2, \dots, r_k)$. Therefore, we have:

$$(7) \quad M \leq L + B = (k + 1)r - \sum_{i=1}^k r_i - r_s - r_o + r_s r_o$$

Case I. Suppose $k = 2$. Then $M \leq 3r - (r_s + r_o) - r_s - r_o + r_s r_o = 3r - 2r_s - 2r_o + r_s r_o = 3r + (r_o - 2)(r_s - 2) - 4$. The right side of this inequality is obviously maximum when r_s and r_o are as large as possible. Recall that $r_o \leq r$ and $r_s \leq r - 1$. Therefore we have:

$$(8) \quad M \leq 3r + (r - 2)(r - 3) - 4 = r^2 - 2r + 2.$$

Case II. Suppose $k \geq 3$. The reader may wish to skip the following formidable looking, though straightforward calculations. They result in a proof that the integer M with the desired property is in fact smaller when the arbitrary graph contains a larger set of these cycles.

Since the lengths of these cycles are a minimal set of relatively prime integers, it is certainly true that

$$\begin{aligned} \sum_{i=1}^k r_i &\geq r_s + [r_s + 2] + [r_s + 4] + \dots + [r_s + 2(k - 2)] + r_o \\ &= (k - 1)r_s + (k - 1)(k - 2) + r_o. \end{aligned}$$

Thus, with (7) we have:

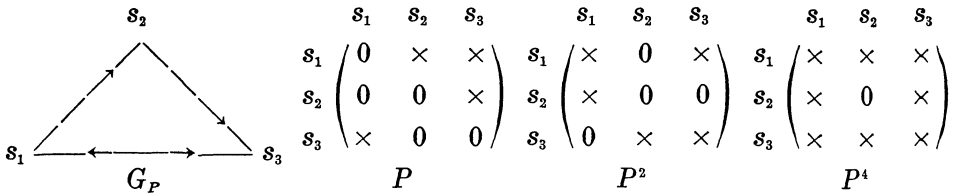
$$\begin{aligned} M &\leq (k + 1)r - [(k - 1)r_s + (k - 1)(k - 2) + r_o] - r_s - r_o + r_s r_o \\ &= (k + 1)r - k r_s - 2r_o + r_s r_o - (k - 1)(k - 2) \\ &= (k + 1)r + (r_s - 2)(r_o - k) - 2k - (k - 1)(k - 2). \end{aligned}$$

Since r_o must be larger than k , the right side again is maximum when r_o and r_s are as large as possible. But $r_o \leq r$ and $r_s \leq r - k + 2$. So

$$\begin{aligned} M &\leq (k + 1)r + (r - k)(r - k) - k^2 + k - 2 \\ &= r^2 + (1 - k)r + k - 2. \end{aligned}$$

This is easily seen to be less than $r^2 - 2r + 2$ of Case I, if $r > 1$. So in any case $M \leq r^2 - 2r + 2$.

To see that $r^2 - 2r + 2$ is the least value for an arbitrary graph of r points and thus for an arbitrary matrix of order r , we need only consider the following example in which $r = 3$ and $M = 5$.



As a matter of fact it can be shown for *any* regular matrix P of order r whose graph G_P contains only two cycles, one of length r and one of length $r - 1$, that P^{r^2-2r+1} is not positive. We have, therefore, established the claim of the paper as stated in the opening paragraph.

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A. V. Balakrishnan, <i>Prediction theory for Markoff processes</i>	1171
Dallas O. Banks, <i>Upper bounds for the eigenvalues of some vibrating systems</i>	1183
A. Białyński-Birula, <i>On the field of rational functions of algebraic groups</i>	1205
Thomas Andrew Brown, <i>Simple paths on convex polyhedra</i>	1211
L. Carlitz, <i>Some congruences for the Bell polynomials</i>	1215
Paul Civin, <i>Extensions of homomorphisms</i>	1223
Paul Joseph Cohen and Milton Lees, <i>Asymptotic decay of solutions of differential inequalities</i>	1235
István Fáry, <i>Self-intersection of a sphere on a complex quadric</i>	1251
Walter Feit and John Griggs Thompson, <i>Groups which have a faithful representation of degree less than $(p - 1/2)$</i>	1257
William James Firey, <i>Mean cross-section measures of harmonic means of convex bodies</i>	1263
Avner Friedman, <i>The wave equation for differential forms</i>	1267
Bernard Russel Gelbaum and Jesus Gil De Lamadrid, <i>Bases of tensor products of Banach spaces</i>	1281
Ronald Kay Getoor, <i>Infinitely divisible probabilities on the hyperbolic plane</i>	1287
Basil Gordon, <i>Sequences in groups with distinct partial products</i>	1309
Magnus R. Hestenes, <i>Relative self-adjoint operators in Hilbert space</i>	1315
Fu Cheng Hsiang, <i>On a theorem of Fejér</i>	1359
John McCormick Irwin and Elbert A. Walker, <i>On N-high subgroups of Abelian groups</i>	1363
John McCormick Irwin, <i>High subgroups of Abelian torsion groups</i>	1375
R. E. Johnson, <i>Quotient rings of rings with zero singular ideal</i>	1385
David G. Kendall and John Leonard Mott, <i>The asymptotic distribution of the time-to-escape for comets strongly bound to the solar system</i>	1393
Kurt Kreith, <i>The spectrum of singular self-adjoint elliptic operators</i>	1401
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Albert W. Marshall and Ingram Olkin, <i>Game theoretic proof that Chebyshev inequalities are sharp</i>	1421
Wallace Smith Martindale, III, <i>Primitive algebras with involution</i>	1431
William H. Mills, <i>Decomposition of holomorphs</i>	1443
James Donald Monk, <i>On the representation theory for cylindric algebras</i>	1447
Shu-Teh Chen Moy, <i>A note on generalizations of Shannon-McMillan theorem</i>	1459
Donald Earl Myers, <i>An imbedding space for Schwartz distributions</i>	1467
John R. Myhill, <i>Category methods in recursion theory</i>	1479
Paul Adrian Nickel, <i>On extremal properties for annular radial and circular slit mappings of bordered Riemann surfaces</i>	1487
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