Pacific Journal of Mathematics

A THEOREM ON REGULAR MATRICES

PETER PERKINS

Vol. 11, No. 4 1961

A THEOREM ON REGULAR MATRICES

PETER PERKINS

In this paper it will be proved that if any nonnegative, square matrix P of order r is such that $P^m > 0$ for some positive integer m, then $P^{r^2-2r+2} > 0$. This result has already appeared in the literature, [2], but the following is a complete and elementary proof given in detail except for one theorem of I. Schur in [1] which is stated without proof. The term regular is taken from Markov chain theory[1] in which a regular chain is one whose transition matrix has the above property.

A *graph* G_P associated with any nonnegative, square matrix P of order r is a collection of r distinct points $S = \{s_1, s_2, \cdots, s_r\}$, some or all of which are connected by directed lines. There is a directed line (indicated pictorially by an arrow) from s_i to s_j in the graph G_P if and only if $p_{ij} > 0$ in the matrix $P = (p_{ij})$. A *path sequence* or *path* in G_P is any finite sequence of points of S (not necessarily distinct) such that there is a directed line in G_P from every point in the sequence to its immediate successor. The *length* of a path is one less than the number of occurrences of points in its sequence. A *cycle* is any path that begins and ends with the same point and a *simple cycle* is a cycle in which no point occurs twice except, of course, for the first (and last). Two cycles are *distinct* if their sequences are not cyclic permutations of each other. A nonnegative, square matrix P is *regular* if $P^m > 0$ for some positive integer m. Likewise, a graph G_P associated with a nonnegative, square matrix P is *regular* if there exists a positive integer m such that an infinite set of paths $A_0, A_1, \cdots, A_n, \cdots$ can be found, the length of each path being $L_n = m + n$, $n = 0, 1, 2, \cdots$. The usual notation $p_{ij}^{(m)}$ is used to denote the ijth entry of the matrix P^m. In all that follows we shall consider only regular matrices P and their associated graphs G_P.

Some immediate consequences of these definitions and the definition of matrix multiplication are the following:

(1) There is a path $s_{k_1} \cdots s_{k_{m+1}}$ in G_P if and only if $p_{k_1 k_{m+1}}^{(m)} > 0$ in P^m.
(2) P is regular if and only if G_P is regular.
(3) There exists some path from any point in G_P to any point in G_P.
(4) For any given i and j there exists some m such that $p_{ij}^{(m)} > 0$.
(5) If $P^m > 0$ then $P^{m+n} > 0$, $n = 0, 1, 2, \cdots$.

Let $C = \{C_1, C_2, \cdots, C_t\}$ be all the distinct simple cycles of G_P and $\{c_1, c_2, \cdots, c_t\}$ be the corresponding lengths.

Received November 21, 1960. I wish to thank Professor R. Z. Norman for his suggestions in the writing of this paper.

[1] This is as treated by Kemeny and Snell in [3].

LEMMA 1. *The length of any cycle* C^* *is always of the form* $c^* = \sum_{i=1}^{t} a_i c_i$, *where* a_i *is some nonnegative integer.*

Proof. Let any cycle $C^* = s_{k_1}, s_{k_2}, \cdots, s_{k_m}$ be given ($k_1 = k_m$). Let $C^* = C_1^*$ and form C_{i+1}^* in the following manner from C_i^*: Wherever simple cycle C_i occurs in cycle C_i^* delete it except for its last point, thus forming the new cycle C_{i+1}^*. It is clear that after the tth step there will remain only a single point of the original C^*, which has of course zero length. If we let a_i be the number of times simple cycle C_i occurred in cycle C_i^* then the lemma follows.

THEOREM 1. *If* G_P *is any regular graph then it must contain a set of simple cycles whose lengths are relatively prime.*

Proof. By the regularity assumption and (1) there exists a positive integer m such that cycles of lengths $L_n = m + n$, $n = 0, 1, 2, \cdots$ can be found in G_P. Also, from Lemma 1, $L_n = \sum_{i=1}^{t} a_i c_i$ for $n = 0, 1, 2, \cdots$, and suitable a_i. Let d be the common factor of the simple cycle lengths c_i. Then

$$\sum_{1=1}^{t} a_i c_i = d \sum_{i=1}^{t} a_i c_i'$$

which could never equal $m + n$, $n = 0, 1, 2, \cdots$ unless $d = 1$.

We would like to find a *least* integer M such that for arbitrary points s_i and s_j there are paths beginning at s_i and ending at s_j and whose lengths are $L_n = M + n$, $n = 0, 1, 2, \cdots$. If we can do this, then, by (1), we shall have also found a least integer M such that $P^M > 0$ where P is the regular matrix associated with G_P.

Let us say that a path *touches* a given set of points if there is some point belonging to both the path and the set. Then we have

LEMMA 2. *Let* G_P *be a regular graph with* r *points, let* S *be a subset containing* r_k *distinct points of the graph, and let* g *be any point of* G_P. *Then there always exists a path from* g *which touches* S *whose length is less than or equal to* $r - r_k$.

Proof. If $g \in S$ then the lemma is trivial. Suppose $g \notin S$. By (3) there is at least one path which starts at g and touches the set S. Let $p = g_0, g_1, \cdots, s$ be such a path of shortest length. Obviously no point of S can precede the final point s in this path sequence p. Furthermore, there can be no repeated points in p, for the deletion of any cycle (except for its last point) would produce a path from g to S shorter than path p, contrary to the choice of p. Therefore, p can have at most $r - r_k$ points.

We shall say that a *minimal set* of relatively prime integers is a set of relatively prime integers such that if one of the integers is deleted the remaining integers are no longer relatively prime. A *step* along a path in G_P is a pair of consecutive points of the path sequence.

THEOREM 2. *If* $R = \{R_1, R_2, \cdots, R_k\}$ *is a set of simple cycles of graph* G_P *whose lengths* $\{r_1, r_2, \cdots, r_k\}$ *form a minimal set of relatively prime integers and if* s_i *and* s_j *are arbitrary points of* G_P, *then there is always a path which starts at* s_i, *ends at* s_j, *touches each cycle of* R *and whose length* $L \leqq (k+1)r - \sum_{i=1}^{k} r_i - 1$.

Proof. Note that the set of distinct points belonging to a simple cycle contains a number of points exactly equal to the length of the cycle. Hence, by Lemma 2 there is a path from an arbitrary point s_i which touches a particular cycle R_p and whose length is less than or equal to $r - r_p$. Thus, we have the following:

from		*to*		*greatest number of steps needed*
arb. pt.	s_i	cycle	R_1	$r - r_1$
cycle	R_1	″	R_2	$r - r_2$
·		·		·
·		·		·
·		·		·
cycle	R_{k-1}	cycle	R_k	$r - r_k$
″	R_k	arb. pt.	s_j	$r - 1$
		TOTAL		$L \leqq (k+1)r - \sum_{i=1}^{k} r_i - 1.$

We shall now state without proof I. Schur's theorem cited above and use it in our final theorem.

THEOREM 3. (Schur) *If* $\{a_1, a_2, \cdots, a_n\}$ *is a set of relatively prime integers with* a_1 *the least and* a_n *the greatest, then* $B = \sum_{i=1}^{n} x_i a_i$ *has solutions in nonnegative integers* x_i *for any* $B \geqq (a_1 - 1)(a_n - 1)$. *This is a best bound for* $n = 2$.

THEOREM 4. *If* M *is the least integer such that paths between any two points of* G_P *can be found whose lengths are* $L_n = M + n$, $n = 0, 1, 2, \cdots$, *then* $M \leqq r^2 - 2r + 2$.

Proof. Given any two points s_i and s_j of G_P we know by Theorem 2 that there is a path from s_i to s_j touching each of the cycles $\{R_1, R_2, \cdots, R_k\}$ and whose length is

$$L \leqq (k+1)r - \sum_{i=1}^{k} r_i - 1 .$$

We can, then, interject into this path the simple cycles $\{R_1, R_2, \cdots, R_k\}$ at the touching points, interjecting cycle R_i say x_i times. The length L of the original path has now been increased to $L + \sum_{i=1}^{k} x_i r_i = L + B$, the second part of which, by Schur's theorem, can be made to take on any integral value B where $B \geqq (r_s - 1)(r_g - 1)$, and $r_s = \min(r_1, r_2, \cdots, r_k)$, $r_g = \max(r_1, r_2, \cdots, r_k)$. Therefore, we have:

$$(7) \qquad M \leqq L + B = (k+1)r - \sum_{i=1}^{k} r_i - r_s - r_g + r_s r_g$$

Case I. Suppose $k = 2$. Then $M \leqq 3r - (r_s + r_g) - r_s - r_g + r_s r_g = 3r - 2r_s - 2r_g + r_s r_g = 3r + (r_g - 2)(r_s - 2) - 4$. The right side of this inequality is obviously maximum when r_s and r_g are as large as possible. Recall that $r_g \leqq r$ and $r_s \leqq r - 1$. Therefore we have:

$$(8) \qquad M \leqq 3r + (r-2)(r-3) - 4 = r^2 - 2r + 2 .$$

Case II. Suppose $k \geqq 3$. The reader may wish to skip the following formidable looking, though straightforward calculations. They result in a proof that the integer M with the desired property is in fact smaller when the arbitrary graph contains a larger set of these cycles.

Since the lengths of these cycles are a minimal set of relatively prime integers, it is certainly true that

$$\begin{aligned}\sum_{i=1}^{k} r_i &\geqq r_s + [r_s + 2] + [r_s + 4] + \cdots + [r_s + 2(k-2)] + r_g \\ &= (k-1)r_s + (k-1)(k-2) + r_g .\end{aligned}$$

Thus, with (7) we have:

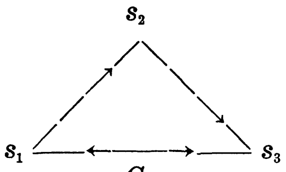
$$\begin{aligned}M &\leqq (k+1)r - [(k-1)r_s + (k-1)(k-2) + r_g] - r_s - r_g + r_s r_g \\ &= (k+1)r - kr_s - 2r_g + r_s r_g - (k-1)(k-2) \\ &= (k+1)r + (r_s - 2)(r_g - k) - 2k - (k-1)(k-2) .\end{aligned}$$

Since r_g must be larger than k, the right side again is maximum when r_g and r_s are as large as possible. But $r_g \leqq$ and $r_s \leqq r - k + 2$. So

$$\begin{aligned}M &\leqq (k+1)r + (r-k)(r-k) - k^2 + k - 2 \\ &= r^2 + (1-k)r + k - 2 .\end{aligned}$$

This is easily seen to be less than $r^2 - 2r + 2$ of Case I, if $r > 1$. So in any case $M \leqq r^2 - 2r + 2$.

To see that $r^2 - 2r + 2$ is the least value for an arbitrary graph of r points and thus for an arbitrary matrix of order r, we need only consider the following example in which $r = 3$ and $M = 5$.



G_P

$$\underset{P}{\begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline s_1 & 0 & \times & \times \\ s_2 & 0 & 0 & \times \\ s_3 & \times & 0 & 0 \end{array}} \qquad \underset{P^2}{\begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline s_1 & \times & 0 & \times \\ s_2 & \times & 0 & 0 \\ s_3 & 0 & \times & \times \end{array}} \qquad \underset{P^4}{\begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline s_1 & \times & \times & \times \\ s_2 & \times & 0 & \times \\ s_3 & \times & \times & \times \end{array}}$$

As a matter of fact it can be shown for *any* regular matrix P of order r whose graph G_P contains only two cycles, one of length r and one of length $r-1$, that P^{r^2-2r+1} is not positive. We have, therefore, established the claim of the paper as stated in the opening paragraph.

BIBLIOGRAPHY

1. Alfred Brauer, *On a problem of partitions*, Amer. J. Math. **LXIV**, (1942), 299-312.

2. J. C. Holladay, and R. S. Varga, *On powers of nonnegative matrices*, Proc. Amer. Math. Soc. Vol. IX, no. 4, (August, 1958), 631.

3. John G. Kemeny and J. Laurie Snell, *Finite Markov Chains*, New York, D. Van Nostrand Company, Incorporated, 1960.

4. David Rosenblatt, *On the Graphs and Asymptotic Forms of Finite Boolean Relation Matrices and Stochastic Matrices*, Naval Research Logistics Quarterly, Vol. IV, no. 2, (June, 1957), 151-167.

DARTMOUTH COLLEGE

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Reprinted 1966 in the United States of America

Pacific Journal of Mathematics

Vol. 11, No. 4 , 1961