

# Pacific Journal of Mathematics

**ASYMPTOTIC ESTIMATES FOR LIMIT CIRCLE PROBLEMS**

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# ASYMPTOTIC ESTIMATES FOR LIMIT CIRCLE PROBLEMS

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**1. Preliminaries.** Characteristic value problems will be considered for the second order, ordinary, linear differential operator  $L$  defined by

$$(1.1) \quad Lx = \frac{1}{k(s)} \left\{ -\frac{d}{ds} \left[ p(s) \frac{dx}{ds} \right] + q(s)x \right\}$$

on the open interval  $\omega_- < s < \omega_+$ , where  $k$ ,  $p$ ,  $q$  are real-valued functions on this interval with the properties that

- (i)  $p$  is differentiable;
- (ii)  $k$  and  $q$  are piecewise continuous; and
- (iii)  $k$  and  $p$  are positive-valued. The points  $\omega_-$  and  $\omega_+$  are in general singularities of  $L$ ; the possibility that they are  $\pm \infty$  is not excluded. It will be convenient to use the notations

$$(1.2) \quad (x, y)_s^t = \int_s^t x(u)\bar{y}(u)k(u)du, \quad \omega_- \leq s < t \leq \omega_+,$$

$$(1.3) \quad [xy](s) = p(s)[x(s)\bar{y}'(s) - x'(s)\bar{y}(s)].$$

Then Green's symmetric formula for  $L$  has the form

$$(1.4) \quad (Lx, y)_s^t - (x, Ly)_s^t = [xy](t) - [xy](s).$$

The symbols  $[xy](\pm)$  will be used as abbreviations for the limits of  $[xy](s)$  as  $s \rightarrow \omega_{\pm}$ , and  $(x, y)$  will be used for the left member of (1.2) when  $s, t$  have been replaced by  $\omega_-, \omega_+$ . Let  $\mathfrak{H}, \mathfrak{H}_{ab}$  denote the Hilbert spaces which are the Lebesgue spaces with respective inner products  $(x, y)$ ,  $(x, y)_a^b$  and norms  $\|x\| = (x, x)^{1/2}$ ,  $\|x\|_a^b = [(x, x)_a^b]^{1/2}$ ,  $\omega_- \leq a < b \leq \omega_+$ .

Let  $a_0$  and  $b_0$  be fixed numbers satisfying  $\omega_- < a_0 < b_0 < \omega_+$  and let  $R_0$  be the rectangle in the  $a - b$ -plane described by the inequalities  $\omega_- < a \leq a_0$ ,  $b_0 \leq b < \omega_+$ . Every closed, bounded subinterval  $[a, b]$  of the basic interval  $(\omega_-, \omega_+)$  can be associated in a one-to-one manner with a point in  $R_0$ . For every such  $[a, b]$  we shall consider the regular Sturm-Liouville problem

$$(1.5) \quad Ly = \mu y, \quad U_a y = U_b y = 0$$

on  $[a, b]$ , where  $U_a, U_b$  are the linear boundary operators

$$(1.6) \quad \begin{aligned} U_a y &= \alpha_0(a)y(a) + \alpha_1(a)y'(a) \\ U_b y &= \beta_0(b)y(b) + \beta_1(b)y'(b), \end{aligned}$$

with  $\alpha_0, \alpha_1$  real-valued functions not both 0 for any value of  $a$  on  $(\omega_-, a_0]$ ,

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Received January 5, 1961.

and with  $\beta_0, \beta_1$  real-valued and not both 0 on  $[b_0, \omega_+]$ . Our problem is to obtain estimates for each characteristic value  $\mu = \mu_{ab}$  of (1.5) for  $a, b$  near  $\omega_-, \omega_+$  under hypotheses that will ensure that the limits of  $\mu_{ab}$  as  $a, b \rightarrow \omega_-, \omega_+$  will exist. Also, we shall obtain estimates for the corresponding characteristic functions  $y = y_{ab} = y_{ab}(s)$  on  $a \leq s \leq b$ . Results like this for differential operators having a singularity at one endpoint were obtained previously by an integral equations approach [8], [9]. The present paper contains extensions of some of these results to operators (1.1) which have singularities at both endpoints. Furthermore, the present approach to the problem will be different; the estimates will now be obtained by means of projection mappings on suitable Hilbert spaces. The method arises from an idea communicated by Professor H. F. Bohnenblust, and affords an elegant and abstract approach to the type of perturbation problem at hand [1]. Also, the present method is powerful enough to handle a variety of domain-perturbed problems that arise in the study of elliptic partial differential equations. Some of these have been considered already [10] and the author has several others in preparation.

Here the method will be illustrated in the case that both of the singularities  $\omega_{\pm}$  of the operator (1.1) are *limit circle* singularities in the well-known classification of H. Weyl [2, p. 225]. In another paper we shall consider the limit point cases (and mixed cases) in which some additional hypotheses are needed on the growth of the coefficient functions in (1.1) as  $s \rightarrow \omega_{\pm}$  to ensure the existence of isolated characteristic values  $\lambda$  of  $L$  on  $(\omega_-, \omega_+)$ ; however, very general boundary operators  $U_a, U_b$  will then permit convergence of  $\mu_{ab}$  to  $\lambda$ . For additional details, see [8]. In the limit circle case herein under consideration, no special assumptions will be imposed on the nature of  $L$  at  $\omega_{\pm}$ , but the generality of the boundary operators must be sacrificed in order to ensure the convergence of  $\mu_{ab}$ . Our purpose here is to obtain asymptotic estimates rather than asymptotic expansions for the characteristic values and functions as  $a, b \rightarrow \omega_-, \omega_+$ . Asymptotic formulae and expansions will be published elsewhere.

**2. Basic and perturbed problems.** Rather than general spectral theory, we are interested in cases that the limits of  $\mu_{ab}$  as  $a, b \rightarrow \omega_-, \omega_+$  exist in the elementary sense. Thus, characteristic values of suitable singular boundary value problems for  $L$  on  $(\omega_-, \omega_+)$  are supposed to exist. These singular problems are described differently according as the points  $\omega_{\pm}$  are in the limit point or limit circle categories. The description is made as follows when both are limit circle singularities [2], [6]: choose a complex number  $l_0$  with  $\text{Im } l_0 \neq 0$ , and let  $L_0$  be the differential operator  $L - l_0$ . A theorem of Weyl [6] states that there exist linearly independent solutions  $\varphi_{\pm} \in \mathfrak{H}$  of  $L_0\varphi = 0$  such that

$$(2.1) \quad [\varphi_- \varphi_-](-) = [\varphi_+ \varphi_+](+) = 0, \quad [\varphi_+ \bar{\varphi}_-](s) = 1.$$

Let  $\mathfrak{D}$  denote the domain consisting of all  $x \in \mathfrak{S}$  which have the following properties:

(a)  $x$  is differentiable on  $(\omega_-, \omega_+)$  and  $x'$  is absolutely continuous on every closed subinterval of this interval:

(b)  $Lx \in \mathfrak{S}$

(c)  $x$  satisfies the end conditions

$$(2.2) \quad [x\varphi_-](-) = [x\varphi_+](+) = 0.$$

Then  $L$  on  $\mathfrak{D}$  is real and essentially self-adjoint [6]. The *basic* characteristic problem

$$(2.3) \quad Lx = \lambda x, \quad x \in \mathfrak{D}$$

is known to have a denumerable set of characteristic values  $\lambda_n$  and corresponding characteristic functions  $x_n$  which are orthonormal and complete in  $\mathfrak{S}$  ( $n = 1, 2, \dots$ ).

Two classes of perturbation problems (1.5) will be considered. The limiting behaviour of *class 1 boundary operators*  $U_a, U_b$  as  $a, b \rightarrow \omega_-, \omega_+$  is rather arbitrary (see §5) while the limiting behaviour of *class 2 operators* (§§2, 3, 4) is restricted as follows:

$$(2.4) \quad \begin{aligned} U_a y &= [y\varphi_-](a)[1 + o(1)] & \text{as } a \rightarrow \omega_- \\ U_b y &= [y\varphi_+](b)[1 + o(1)] & \text{as } b \rightarrow \omega_+ \end{aligned}$$

for every differentiable function  $y$ . A *perturbed domain*  $\mathfrak{D}_{ab}$  is defined for each  $[a, b] \in R_0$  to be the set of all  $y$  in the subspace  $\mathfrak{F}_{ab}$  of  $\mathfrak{S}$  which satisfy the following conditions:

(a)  $y$  is differentiable and  $y'$  is absolutely continuous on  $[a, b]$ ;

(b)  $Ly \in \mathfrak{F}_{ab}$

(c)  $y$  satisfies the homogeneous boundary conditions (1.5) where the boundary operators  $U_a, U_b$  have the limiting behaviour (2.4).

The perturbed characteristic value problem that corresponds to this domain is the regular Sturm-Liouville problem

$$(2.5) \quad Ly = \mu y, \quad y \in \mathfrak{D}_{ab}.$$

In addition, we define a domain  $\mathfrak{D}_a$  for each  $a$  on  $(\omega_-, a_0]$  to be the set of all  $z \in \mathfrak{F}_{a\omega_+}$  which satisfy the following:

(a)  $z$  is differentiable and  $z'$  is absolutely continuous on every closed subinterval of  $[a, \omega_+]$ ;

(b)  $Lz \in \mathfrak{F}_{a\omega_+}$

(c)  $z$  satisfies the conditions

$$(2.6) \quad U_a z = 0, \quad [z\varphi_+](+) = 0.$$

The characteristic value problem

$$(2.7) \quad Lz = \nu z, \quad z \in \mathfrak{D}_a$$

on the half-open interval  $[a, \omega_+)$  may be regarded as intermediate between (2.3) and (2.5), and will be called a semi-perturbed problem.

In order to obtain estimates for the difference between the characteristic values and functions of (2.5) and (2.3), we shall proceed in two steps: (i) the comparison of (2.5) with (2.7), and (ii) the comparison of (2.7) with (2.3). The details of (i) and (ii) are included in §§3 and 4 respectively. Each comparison has independent interest because it is typical for a boundary variational problem when only one endpoint is varied and the unchanged endpoint is (i) an ordinary point; (ii) a singular point of the differential operator. Type (ii) variational problems arise for example in the theory of enclosed quantum mechanical systems[ 4], [5].

**3. Comparison of the  $y$  and  $z$  problems.** The characteristic value problems (2.5) and (2.7) will be compared, with (2.7) regarded as basic and (2.5) regarded as a perturbation on (2.7). In this case, the singular boundary condition  $[z\varphi_+](+) = 0$  is replaced by the regular condition  $U_b z = 0$  at the point  $b$ . We are going to estimate the variation of characteristic values and functions under this perturbation, and show that this variation has the limit 0 as  $b \rightarrow \omega_+$ . The ordinary endpoint  $a$  remains fixed in this section.

Let  $G_{ab}(s, t)$  be the Green's function for the operator  $kL_0$  associated with the boundary conditions (1.5), and let  $G_{ab}$  be the linear transformation on  $\mathfrak{F}_{ab}$  defined by the equation

$$(3.1) \quad G_{ab}y = \int_a^b G_{ab}(s, t)y(t)k(t)dt, \quad y \in \mathfrak{F}_{ab}.$$

Let  $\nu = \nu_a$  be a characteristic value for (2.7) and let  $z_a$  be the corresponding characteristic function. Define a function  $f$  on  $[a, b]$  by<sup>1</sup>

$$(3.2) \quad f = z_a - \gamma_a G_{ab}z_a \quad \text{where } \gamma_a = \nu_a - l_0.$$

It is easily verified because of the linearity of all the operators involved that  $f$  is a solution of the boundary value problem

$$(3.3) \quad L_0 f = 0, \quad U_a f = 0, \quad U_b f = U_b z_a.$$

The solution  $\psi_a$  of  $L_0 y = 0$  that is given by

$$(3.4) \quad \psi_a(s) = \varphi_-(s)U_a\varphi_+ - \varphi_+(s)U_a\varphi_-$$

satisfies the boundary condition  $U_a y = 0$ . Hence the unique solution of (3.3) is

<sup>1</sup> The function on  $[a, b]$  which coincides with  $z_a$  on this interval will also be denoted by  $z_a$ .

$$(3.5) \quad f(s) = (U_b z_a / U_b \psi_a) \psi_a(s), \quad a \leq s \leq b.$$

In fact, if  $g$  is any solution of (3.3), then the function  $h = g - f$  satisfies  $L_0 h = 0$ ,  $U_a h = U_b h = 0$ . This implies that  $h$  is the zero function, or  $g = f$ .

It follows from (2.1) that  $[\varphi_+ \varphi_+](b) \rightarrow 0$  as  $b \rightarrow \omega_+$  and  $[\varphi_- \varphi_-](a) \rightarrow 0$  as  $a \rightarrow \omega_-$ . The identity

$$[\varphi_+ \varphi_+](t) - [\varphi_+ \varphi_+](s) = (l_0 - \bar{l}_0)(\|\varphi_+\|_s^t)^2$$

is a consequence of (1.4), and since  $\varphi_+ \in \mathfrak{H}$ , the limit  $[\varphi_+ \varphi_+](\infty)$  exists. Similarly  $[\varphi_- \varphi_-](\infty)$  exists. From (2.1) and the identity [6]

$$|[\varphi_+ \bar{\varphi}_-](a)|^2 = [\varphi_- \varphi_-](a)[\varphi_+ \varphi_+](a) + |[\varphi_+ \varphi_-](a)|^2$$

we deduce that  $|[\varphi_+ \varphi_-](a)| \rightarrow 1$  as  $a \rightarrow \omega_-$ . Similarly  $|[\varphi_+ \varphi_-](b)| \rightarrow 1$  as  $b \rightarrow \omega_+$ . It has then been established that

$$(3.6) \quad \begin{aligned} U_a \varphi_- &\rightarrow 0, & |U_a \varphi_+| &\rightarrow 1 & \text{as } a \rightarrow \omega_-; \\ U_b \varphi_+ &\rightarrow 0, & |U_b \varphi_-| &\rightarrow 1 & \text{as } b \rightarrow \omega_+. \end{aligned}$$

where (2.4) has been used. Since  $\varphi_{\pm} \in \mathfrak{H}$ , it follows from (3.4) that  $\|\psi_a\|_a^b$  is uniformly bounded for  $[a, b] \in R_0$ . We obtain from (3.4) that

$$U_b \psi_a = U_b \varphi_- U_a \varphi_+ - U_b \varphi_+ U_a \varphi_-$$

and hence there are numbers  $a_0, b_0$  (we may suppose that they coincide with the original choices of  $a_0, b_0$ ) such that  $U_b \psi_a$  is bounded away from zero on  $a \leq a_0, b_0 \leq b$ . These considerations enable us to deduce from (3.2), (3.5) that there exists a constant<sup>2</sup>  $C$  on  $R_0$  such that

$$(3.7) \quad \|\psi_a - \gamma_a G_{ab} \psi_a\|_a^b \leq C \|U_b \psi_a\| \| \psi_a \|_a^b, \quad [a, b] \in R_0.$$

Let  $\mu^i = \mu_{ab}^i$  denote the  $i$ th characteristic value of the regular problem (2.5),  $\mu^1 < \mu^2 < \dots$ , and let  $y^i$  denote the corresponding characteristic function, chosen so that  $\{y^i\}$  is an orthonormal basis in  $\mathfrak{F}_{ab}$ . The following fundamental lemma was obtained by H. F. Bohnenblust in [1] by applying the Parseval completeness relation to the set  $\{y^i\}$ . An outline of the proof is reproduced below.

**LEMMA.** *Let  $P(\delta)$  be the projection mapping from the Hilbert space  $\mathfrak{F}_{ab}$  onto its subspace  $\mathfrak{F}_{ab}(\delta)$  ( $\delta > 0$ ) spanned by all characteristic functions  $y^i$  of (2.5) such that their corresponding  $\mu^i$  satisfy  $|\mu^i - \nu_a| \leq \delta$ . Then for any  $w \in \mathfrak{F}_{ab}$ ,*

$$\|w - P(\delta)w\|_a^b \leq (1 + |\gamma_a|/\delta) \|w - \gamma_a G_{ab} w\|_a^b.$$

<sup>2</sup> The letter  $C$  will be used throughout as a generic notation for the image of a constant function from  $R_0$  into the positive numbers.

*Proof.* The subscripts  $a, b$  will be omitted in this proof. Let  $\alpha_i = (Gw, y^i)$ . It is easily verified that  $(w - \gamma Gw, y^i) = (\mu^i - \nu)\alpha_i$ , and hence by the Parseval identity,

$$\|w - \gamma Gw\|^2 = \sum_i |\mu^i - \nu|^2 |\alpha_i|^2 \geq \delta^2 \sum_i^* |\alpha_i|^2,$$

where the  $*$  denotes summation over only those indices  $i$  such that  $|\mu^i - \nu| > \delta$ . Then

$$\|Gw - P(\delta)Gw\|^2 = \sum^* |\alpha_i|^2 \leq \delta^{-2} \|w - \gamma Gw\|^2,$$

and the conclusion of the lemma follows easily from the Minkowski inequality.

The notation  $\rho_b = C|\gamma_a U_b z_a|$  will be used. It follows from (2.4) and (2.6) that  $\rho_b \rightarrow 0$  as  $b \rightarrow \omega_+$  for each fixed  $a$ . With the choice  $\delta = 2\rho_b$ , we apply the lemma to  $w = z_a$  (see footnote 1) and use (3.7) to obtain

$$\|z_a - P(2\rho_b)z_a\|_a^b \leq (C|U_b z_a| + \frac{1}{2}) \|z_a\|_a^b.$$

We may suppose that  $b_0$  has been selected so that  $C|U_b z_a| \leq 1/4$  on  $b_0 \leq b < \omega_+$ . Hence  $P(2\rho_b)z_a = 0$  implies that  $z_a = 0$  on  $[a, b]$ , and therefore  $\mathfrak{F}_{ab}(2\rho_b)$  has dimension  $\geq 1$ . Hence there exists at least one characteristic value  $\mu = \mu_{ab}$  of (2.5) which satisfies

$$(3.8) \quad |\mu_{ab} - \nu_a| \leq 2\rho_b, \quad [a, b] \in R_0.$$

To prove that there is exactly one, we conclude from the maximum-minimum principle for characteristic values [3], [7] that the absolute value of the  $i$ th characteristic value  $\nu_a^i$  of (2.7) cannot decrease when a boundary condition at  $b$  is adjoined, and hence  $|\nu_a^i| \leq |\mu_{ab}^i|$  ( $i = 1, 2, \dots$ ). Since the numbers  $\nu_a^i$  do not accumulate and since  $\rho_b \rightarrow 0$  as  $b \rightarrow \omega_+$ , there is a constant  $b_0$  such that  $2\rho_b$  is less than the minimum of all the differences  $|\nu_a^j - \nu_a^i|$ , ( $i, j = 1, 2, \dots; i \neq j$ ) whenever  $b \geq b_0$ . If  $0 < \nu_a^1 < \nu_a^2$ , it follows from (3.8) that exactly one characteristic value  $\mu_{ab}$  of (2.5) lies in the interval  $[\nu_a^1, \nu_a^1 + 2\rho_b]$ . A similar statement applies to the case that one or both of  $\nu_a^1, \nu_a^2$  are negative.

In order to prove by induction that there is exactly one  $\mu_{ab}^i$  which satisfies  $|\mu_{ab}^i - \nu_a^i| \leq 2\rho_b$  ( $i = 1, 2, \dots$ ), assume that this is true for each integer  $i \leq n$ . In the case that  $|\nu_a^{n+1}| < |\nu_a^{n+2}|$  there are at most  $n+1$  characteristic values  $\mu_{ab}^i$  which satisfy  $|\mu_{ab}^i| \leq |\nu_a^{n+1}| + 2\rho_b$  since  $|\mu_{ab}^i| \geq |\nu_a^i|$  for each  $i$ . It then follows from the induction assumption that there is at most one characteristic value  $\mu_{ab}^{n+1}$  satisfying  $|\mu_{ab}^{n+1} - \nu_a^{n+1}| \leq 2\rho_b$ , and hence exactly one by (3.8). In the other case  $\nu_a^{n+2} = -\nu_a^{n+1}$ , it follows similarly that there are at most two  $\mu_{ab}^i$  satisfying  $|\nu_a^{n+1}| < |\mu_{ab}^i| \leq |\nu_a^{n+1}| + 2\rho_b$ , and again by (3.8) there is exactly one  $\mu_{ab}^i$  near each of  $\nu_a^{n+1}, \nu_a^{n+2}$ .

**THEOREM 1.** *If the singularity  $\omega_+$  of (1.1) is the limit circle type, then for every characteristic value  $\nu_a$  of (2.7) there exists a rectangle  $R_0$  and a constant  $C$  on  $R_0$  such that<sup>3</sup> a unique characteristic value  $\mu_{ab}$  of the perturbed problem (2.5) lies in the interval  $|\mu_{ab} - \nu_a| \leq C |U_b z_a|$  whenever  $[a, b] \in R_0$ .*

This shows in particular that for each fixed  $a$ , there is a unique  $\mu_{ab}$  of (2.5) such that  $\mu_{ab} \rightarrow \nu_a$  as  $b \rightarrow \omega_+$ . In addition, the estimate of the theorem is valid uniformly on  $\omega_- < a \leq a_0$ . One also finds for the characteristic functions  $y_{ab}$  and  $z_a$  associated with  $\mu_{ab}$  and  $\nu_a$  respectively that the estimate

$$\|y_{ab} - z_a\|_a^b \leq C |U_b z_a|, \quad \|y_{ab}\|_a^b = \|z_a\|_a = 1$$

is valid on  $R_0$ .

**4. Comparison of the  $z$  and  $x$  problems.** The characteristic value problems (2.7) and (2.3) will now be compared, with (2.7) regarded as a perturbation of the basic problem (2.3). The perturbation arises from the singular end condition  $[x\varphi_-](-) = 0$  being replaced by a homogeneous boundary condition at the point  $a$ . The novelty of this section is due to the singular nature of the unchanged endpoint  $\omega_+$ .

Let  $\lambda$  be a characteristic value of (2.3) and let  $x$  be the corresponding normalized characteristic function. Let  $G_a$  be the linear integral operator on  $\mathfrak{F}_{a\omega_+}$  whose kernel is the Green's function for  $kL_0$  associated with the boundary conditions (2.6). This operator is defined similarly to the operator  $G_{ab}$  in (3.1) [6]. Let a function  $g$  on  $[a, \omega_+)$  be defined by<sup>4</sup>

$$(4.1) \quad g = x - \gamma G_a x \quad \text{where } \gamma = \lambda - l_0.$$

The analogue of (3.5) is

$$(4.2) \quad g(s) = (U_a x | U_a \varphi_+) \varphi_+(s), \quad a \leq s < \omega_+.$$

It follows from the postulated boundary conditions (2.2) at  $\omega_-$  that  $[x\varphi_-](a) \rightarrow 0$  as  $a \rightarrow \omega_-$ , and hence by (2.4) that  $U_a x \rightarrow 0$  as  $a \rightarrow \omega_-$ . It was proved above (3.6) that  $|U_a \varphi_+| \rightarrow 1$  as  $a \rightarrow \omega_-$ , and since  $\varphi_+ \in \mathfrak{F}$ , we obtain the inequality

$$(4.3) \quad \|x - \gamma G_a x\|_a \leq C |U_a x| \|x\|_a$$

for some constant  $C$ . The analogue of the lemma in §3 with  $\mathfrak{F}_{ab}$  replaced by  $\mathfrak{F}_{a\omega_+}$  leads to

$$\begin{aligned} \|x - P(\delta)x\|_a &\leq (1 + |\gamma/\delta|) \|x - \gamma G_a x\|_a \\ &\leq (1 + |\gamma/\delta|) C |U_a x| \|x\|_a, \end{aligned}$$

<sup>3</sup> See footnote 2.

<sup>4</sup> The function on  $[a, \omega_+)$  which coincides with  $x$  on this interval will also be denoted by  $x$ .



and the following theorem is obtained.

**THEOREM 2.** *If the singularities  $\omega_{\pm}$  of (1.1) are both of the limit circle type, then for every characteristic value  $\lambda$  of the basic problem (2.3) there exist constants  $a_0$  and  $C$  such that a unique characteristic value  $\nu_a$  of (2.7) lies in the interval  $|\nu_a - \lambda| \leq C|U_a x|$  whenever  $a$  satisfies  $\omega_- < a \leq a_0$ . If  $x, z_a$  are characteristic functions corresponding to  $\lambda, \nu_a$  respectively with norms  $\|x\| = \|z_a\|_a = 1$ , then*

$$(4.4) \quad \|z_a - x\|_a \leq C|U_a x|, \quad \omega_- < a \leq a_0,$$

and in particular  $\|z_a - x\|_a \rightarrow 0$  as  $a \rightarrow \omega_-$ .

We shall next prove the following consequence of (4.4):

$$(4.5) \quad U_b z_a = U_b x + (|U_a x| + |U_b x|)o(1),$$

the order symbol being valid as  $b \rightarrow \omega_+$  uniformly on  $\omega_- < a \leq a_0$ . We use formula (1.4) to obtain

$$\begin{aligned} [z_a \varphi_+](+) - [z_a \varphi_+](b) &= (\nu_a - \bar{l}_0)(z_a, \varphi_+)_b, \\ [x \varphi_+](+) - [x \varphi_+](b) &= (\lambda - \bar{l}_0)(x, \varphi_+)_b. \end{aligned}$$

Since  $[x \varphi_+](+) = [z_a \varphi_+](+) = 0$  by (2.2), (2.6), we deduce from the Schwarz inequality on  $\mathfrak{F}_{b\omega_+}$  that

$$\begin{aligned} |[z_a \varphi_+](b) - [x \varphi_+](b)| &\leq |(\nu_a - \bar{l}_0)(z_a - x, \varphi_+)_b| + |(\nu_a - \lambda)(x, \varphi_+)_b| \\ &\leq |\nu_a - \bar{l}_0| \|z_a - x\|_b \|\varphi_+\|_b \\ &\quad + |\nu_a - \lambda| \|x\| \|\varphi_+\|_b. \end{aligned}$$

The desired conclusion (4.5) then follows from Theorem 2 and (2.4). The following abbreviation will be used:

$$(4.6) \quad \rho_{ab} = |U_a x| + |U_b x|.$$

**THEOREM 3.** *If both singularities  $\omega_{\pm}$  are of the limit circle type, then for every characteristic value  $\lambda$  of (2.3) there exists a rectangle  $R_0$  and a constant  $C$  on  $R_0$  such that exactly one characteristic value  $\mu_{ab}$  of the perturbed problem (2.5) lies in the interval  $|\mu_{ab} - \lambda| \leq C\rho_{ab}$  for every  $[a, b] \in R_0$ . For the characteristic functions  $x, y_{ab}$  associated with  $\lambda, \mu_{ab}$  respectively, normalized by  $\|x\| = \|y_{ab}\|_a^b = 1$ , the estimate  $\|y_{ab} - x\|_a^b \leq C\rho_{ab}$  is valid.*

*Proof.* It follows from Theorems 1 and 2 that

$$\begin{aligned} |\mu_{ab} - \lambda| &\leq |\mu_{ab} - \nu_a| + |\nu_a - \lambda| \\ &\leq C(|U_b z_a| + |U_a x|). \end{aligned}$$

The first statement of the theorem is then a consequence of (4.5) and (4.6). The proof of the second statement is similar and will be omitted.

Finally, we shall obtain uniform estimates for the difference  $y_{ab}(s) - x(s)$  on  $a \leq s \leq b$ . We remark in passing that the asymptotic result  $y_{ab}(s) = x(s)[1 + o(1)]$  as  $a, b \rightarrow \omega_-, \omega_+$  cannot be valid for  $s$  near the boundaries  $a, b$  nor can it be valid near any zeros of  $x(s)$ . Uniform estimates will now be derived by the same technique that proves useful in certain domain-perturbed problems concerning elliptic partial differential equations [1], when  $\varphi_{\pm}(s)$  are bounded on  $(\omega_-, \omega_+)$ .

First it will be shown that  $(\lambda - l_0)G_{ab}x(s)$  gives a uniform estimate for  $y_{ab}(s)$  on  $a \leq s \leq b$ . Let  $\psi_a(s)$  be the function (3.4) and let  $\psi_b(s)$  be defined by

$$\psi_b(s) = \varphi_-(s)U_b\varphi_+ - \varphi_+(s)U_b\varphi_-.$$

Then

$$\begin{aligned} G_{ab}(s, t) &= \sigma^{-1}\psi_a(t)\psi_b(s) \quad \text{if } a \leq t \leq s \leq b, \\ &= \sigma^{-1}\psi_a(s)\psi_b(t) \quad \text{if } a \leq s \leq t \leq b, \end{aligned}$$

where

$$\sigma = U_a\varphi_- U_b\varphi_+ - U_a\varphi_+ U_b\varphi_-.$$

Then  $|\sigma| \rightarrow 1$  as  $a, b \rightarrow \omega_-, \omega_+$ , and the function defined by

$$(\|G_{ab}\|_a^b)^2 = \int_a^b |G_{ab}(s, t)|^2 k(t) dt$$

is a bounded function of  $s, a$ , and  $b$ . Hence the inequality

$$\begin{aligned} |y_{ab}(s) - (\lambda - l_0)G_{ab}x(s)| &= |G_{ab}[(\mu_{ab} - l_0)y_{ab}(s) - (\lambda - l_0)x(s)]| \\ &\leq \|G_{ab}\|_a^b (|\mu_{ab} - l_0| \|y_{ab} - x\|_a^b + |\mu_{ab} - \lambda| \|x\|), \end{aligned}$$

and Theorem 3 show that there exists a constant  $C$  such that

$$(4.6) \quad |y_{ab}(s) - (\lambda - l_0)G_{ab}x(s)| \leq C\rho_{ab} \quad a \leq s \leq b.$$

Let  $h$  be the uniquely determined solution of the boundary value problem

$$L_0 h = 0, \quad U_a h = U_a x, \quad U_b h = U_b x \quad \text{on } a \leq s \leq b.$$

Let the function  $f$  on  $[a, b]$  be defined by

$$f(s) = (\lambda - l_0)G_{ab}x(s) - x(s) + h(s).$$

Since  $f$  satisfies  $L_0 f = 0$ ,  $U_a f = U_b f = 0$ ,  $f$  is identically zero. The following uniform estimate is then a direct consequence of (4.6):

$$(4.7) \quad y_{ab}(s) = x(s) - h(s) + O(\rho_{ab}), \quad a \leq s \leq b.$$

It can be verified without much difficulty that  $h(s) = O(\rho_{ab})$  on a fixed closed subinterval  $I_0$  of  $[a, b]$ , valid for  $[a, b] \in R_0$ . The following uniform result on  $I_0$  is therefore a special case of (4.7):

$$y_{ab}(s) = x(s) + O(\rho_{ab}) \quad [a, b] \in R_0.$$

**5. Class 1 boundary operators.** Instead of the restrictive limiting behaviour (2.4) of the boundary operators  $U_a, U_b$ , the limiting behaviour of class 1 boundary operators is essentially arbitrary. In regard to the perturbation  $a \rightarrow \omega_-$ , a class 1 boundary operator  $U_a$  is defined as follows. Let  $\varphi_+$  be the function defined in §2 and let  $x$  be a characteristic function of the basic problem (2.3) corresponding to the characteristic value  $\lambda$ . Class 1 perturbation problems are possible when the singularity  $\omega_-$  is not an accumulation point of the zeros of  $\varphi_+$  and

$$(5.1) \quad x(s)/\varphi_+(s) = o(1) \quad \text{as } s \rightarrow \omega_-.$$

In this event,  $U_a$  is said to be a *class 1 boundary operator* on  $(\omega_-, a_0]$  whenever the ratio  $\varphi_+(a)U_ax/x(a)U_a\varphi_+$  is bounded on this interval. This rather mild restriction on  $U_a$  implies that

$$(5.2) \quad \varepsilon_a = |U_ax/U_a\varphi_+| = o(1) \quad \text{as } a \rightarrow \omega_-.$$

An example is given in [8, pages 838–840] when  $\omega_- = 0$  is a regular singularity of  $L$ , with  $p(s) = 1$ . In this event, a sufficient condition that the boundedness requirement above (5.2) be satisfied is that the limit  $\sigma = \lim_{a \rightarrow 0} [a\alpha_0(a)/\alpha_1(a)]$  exists (finite or  $\infty$ ) and  $\sigma \neq -\rho$ , where  $\rho$  denotes the smaller of two real, distinct exponents at the singularity 0.

Let  $g$  be defined by (4.1). Then (4.2) is valid but under the assumptions of this section, (4.3) is replaced by

$$(5.3) \quad \|x - \gamma G_a x\|_a \leq C\varepsilon_a \|x\|_a$$

where  $\varepsilon_a$  is defined by (5.2). In the notation of §§2, 3,

$$\|x - P(\delta)x\|_a \leq (1 + |\gamma|/\delta) C\varepsilon_a \|x\|_a.$$

Since  $\varepsilon_a = o(1)$  as  $a \rightarrow \omega_-$ , Theorems 2 and 3 are valid with the replacement  $\varepsilon_a$  instead of  $|U_ax|$ . A similar statement is appropriate in the event that  $U_b$  is a class 1 boundary operator.

In the example of a regular singularity  $\omega_- = 0$  with real exponents  $\rho_1, \rho_2$ , it turns out that  $\varepsilon_a = O(a^{\rho_1 - \rho_2})$  if  $\rho_1 > \rho_2$  and  $\varepsilon_a = O(1/\ln a)$  if  $\rho_1 = \rho_2$  ( $0 < a \leq a_0$ ).

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Reprinted 1966 in the United States of America

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