GROUPS WITH FINITELY MANY AUTOMORPHISMS

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1. Introduction. The connection between the structure of a group and the structure of its endomorphisms or group of automorphisms is a very interesting question but one to which there is as yet a scarcity of answers. We present here two theorems pertaining to this problem. We first show that a finitely generated group has a finite automorphism group if and only if it is a finite and central extension of a cyclic group. The restriction to finitely generated groups is essential. Indeed, there exist indecomposable torsion-free abelian groups of rank the cardinal of the continuum whose automorphism groups are cyclic of order two (see [2], p. 180, 18(b)). The second result which we present is a new and much simpler proof of a theorem to be found in a paper of Baer [1]; namely, a group possessing only finitely many endomorphisms is itself finite.

Before discussing these theorems we shall describe the notation to be used in this paper. Throughout, \( G \) will denote a group with center \( Z \). Let coset representatives \( g_\alpha, g_\beta, \ldots \) of \( Z \) in \( G \) be chosen for all \( \alpha, \beta, \ldots \) elements of \( G/Z \) such that \( g_1 = 1 \). Let \( M = \{m_{\alpha,\beta}\} \) be the corresponding factor set. For the theory of such factor sets, essential to the following, the reader should consult Kurosh [4] or M. Hall [3]. Finally, for any group \( H \), let \( \text{Aut}(H) \) be the automorphism group of \( H \).

2. Preliminary lemmas. The first lemma follows immediately from the definition of equivalence of factor sets.

**Lemma 1.** Let \( N \) and \( R \) be factor sets from the group \( B \) into the abelian group \( A_1 \times A_2 \). Let \( N_i, N_2, R_1, R_2 \) be the factor sets from \( B \) into \( A_1 \) and \( A_2 \) obtained by taking components of \( N \) and \( R \). Then \( N \) and \( R \) will be equivalent if and only if \( N_i \) is equivalent with \( R_i \) and \( N_2 \) with \( R_2 \).

The remaining lemmas are, I believe, entirely or in part scattered throughout the literature. We include proofs for the convenience of the reader.

**Lemma 2.** Let \( \psi \) be an endomorphism of \( Z \) such that the factor sets \( M \) and \( \psi(M) = \{\psi(m_{\alpha,\beta})\} \) of \( G \) are equivalent. Then \( \psi \) may be extended to an endomorphism of \( G \) and if \( \psi \) is an automorphism of \( Z \) then it may be extended to an automorphism of \( G \).

Received February 17, 1961. National Science Foundation Predoctoral Fellow.
Proof. First note that it is clear from the definition of factor set that \( \psi(M) \) is a factor set. Since \( M \) and \( \psi(M) \) are assumed to be equivalent we may choose for each \( \alpha \in G/Z \) an element \( c_\alpha \in Z \) such that
\[
\psi(m_{\alpha, \beta}) = c_\alpha m_{\alpha, \beta} c_\beta
\]
for all \( \alpha, \beta \in G/Z \). Now every element \( g \in G \) can be uniquely expressed in the form \( g = g_\alpha a \) where \( \alpha \in G/Z \) and \( a \in Z \). Thus we may define a mapping \( \varphi \) of \( G \) into itself according to the rule, for all \( g \in G \),
\[
\varphi(g) = g_\alpha c_\alpha \psi(a) .
\]
Then \( \varphi \) is an endomorphism of \( G \) since if \( g = g_\alpha a, h = g_\beta b \) are elements of \( G \), where \( \alpha, \beta \in G/Z, a, b \in Z \) then
\[
\varphi(gh) = \varphi(g_\alpha a g_\beta b) = \varphi(g_\alpha m_{\alpha, \beta} a b) = g_\alpha c_\alpha g_\beta c_\beta \psi(a) \psi(b) = g_\alpha c_\alpha \psi(a) g_\beta c_\beta \psi(b) = \varphi(g) \varphi(h) .
\]
Since \( g_1 = 1 \) implies \( m_{1,1} = g_1^{-1} g_1 g_1 = 1 \) and so \( c_1 = 1 \), we have that \( \varphi \) extends \( \psi \) to \( G \).

Finally, suppose \( \psi \) is an automorphism of \( G \). Let \( g = g_\alpha a \) be an arbitrary element of \( G \), as above. Then \( \varphi(g) = 1 \) implies that \( g_\alpha c_\alpha \psi(a) = 1 \) and hence \( \alpha = 1 \) because \( \alpha \) is the image of \( g_\alpha c_\alpha \psi(a) \) in \( G/Z \). Therefore \( g_1 = c_1 = 1 \) so we have \( \psi(a) = 1 \) and \( g = g_\alpha a = 1 \). \( \varphi \) is consequently a one-to-one mapping of \( G \) into itself. To conclude we prove that \( g \) is the image of some element under \( \varphi \). Choose \( b \in Z \) such that \( \psi(b) = c_\alpha^{-1} a \). Then \( \varphi(g_\alpha b) = g_\alpha c_\alpha \psi(b) = g_\alpha a = g \).

**Lemma 3.** Let \( A \) be a central and characteristic subgroup of the group \( H \). The group \( K \) of those automorphisms of \( H \) which leave \( A \) element-wise fixed and which induce the identity automorphism on \( H/A \) is naturally isomorphic to the group \( \text{Hom}(H/A, A) \) of homomorphisms of \( H/A \) into \( A \).

**Proof.** If \( \sigma \in \text{Hom}(H/A, A) \) define a mapping \( \theta \) of \( H \) into itself by \( \theta(h) = h\sigma(hA) \) for \( h \in H \). \( \theta \) will be an element of \( K \), as can be verified directly, and the correspondence of \( \sigma \) with \( \theta \) is the required isomorphism.

**Corollary.** If \( \text{Aut}(A), \text{Aut}(H/A), \text{and Hom}(H/A, A) \) are finite then \( \text{Aut}(H) \) is finite.
Proof. If $\theta \in \text{Aut}(H)$ let $\theta_1$ be the restriction of $\theta$ to $A$ and $\theta_2$ the automorphism of $H/A$ induced by $\theta$. Then the mapping sending $\theta$ to $(\theta_1, \theta_2) \in \text{Aut}(A) \times \text{Aut}(H/A)$ is a homomorphism with kernel $K$. Hence, the product of the orders of $\text{Aut}(A)$, $\text{Aut}(H/A)$, and $\text{Hom}(H/A, A)$ is a bound for the order of $\text{Aut}(H)$.

3. Finite automorphism groups.

Theorem 1. A finitely generated group $G$ has a finite automorphism group if and only if it has a central cyclic subgroup of finite index.

Proof. First we shall demonstrate the sufficiency of these conditions. In so doing, we may assume $G$ is infinite and therefore a finite and central extension of an infinite cyclic group. The center $Z$ of $G$ is of finite index in $G$ so by Schreier's subgroup theorem (3; p. 97) is finitely generated. Hence $Z$ is the direct sum of a finite abelian group $T$ and an infinite cyclic group. Consequently, $\text{Aut}(G/Z)$ and $\text{Hom}(G/Z, Z) = \text{Hom}(G/Z, T)$ are finite. To conclude, we can apply the above Corollary once we prove that $\text{Aut}(Z)$ if finite. However, this follows from another application of the Corollary with $A = T$ and $H = Z$.

The necessity of the condition is more difficult to prove. If $\text{Aut}(G)$ is finite then $G/Z$ is finite being isomorphic to the group of inner automorphisms of $G$. Again, by Schreier's theorem, $Z$ will be finitely generated. We may assume, in contradiction to the theorem, that $Z$ in a direct decomposition into cyclic groups, contains two or more infinite cyclic factors. In fact let $Z = W \times (a) \times (b)$ where $(a)$ and $(b)$ are the subgroups generated by elements $a$ and $b$ of infinite order and $W$ is a subgroup of $Z$. We need only show that infinitely many automorphisms of $Z$, which are the identity on $W$ and map the group $(a, b)$ generated by $a$ and $b$ onto itself, may be extended to automorphisms of $G$.

For the remainder of this proof we shall use the additive notation for $Z$. Write the factor set $M$ as $m_{a, \beta} = w_{a, \beta} + r_{a, \beta}a + t_{a, \beta}b$ where $w_{a, \beta} \in W$, $s_{a, \beta}$ and $t_{a, \beta}$ are integers. Let the automorphism $\theta$ of $Z$ be defined by $\theta(w) = w$ for $w W$, $\theta(a) = ma + nb$, $\theta(b) = pa + qb$ where $m, n, p,$ and $q$ are integers such that $|mq - np| = 1$. The factor set $\theta(M)$ is then expressible as $\theta(m_{a, \beta}) = w_{a, \beta} + s_{a, \beta}a + t_{a, \beta}b$ where $s_{a, \beta}' = ms_{a, \beta} + pt_{a, \beta}$ and $t_{a, \beta}' = ns_{a, \beta} + qt_{a, \beta}$. Therefore, $S_0 = \{s_{a, \beta}\}$, $T_0 = \{t_{a, \beta}\}$, $S_0' = \{s_{a, \beta}'\}$, and $T_0' = \{t_{a, \beta}'\}$ are factor sets of $G/Z$ with integral values and $S_0' = mS_0 + pT_0$, $T_0' = nS_0 + qT_0$. By Lemmas 1 and 2 the proof is now reduced to the following problem: Find infinitely many quadruplets $(m, n, p, q)$ of integers such that $|mq - np| = 1$ and the factor sets $mS_0 + pT_0$ and $nS_0 + qT_0$ are equivalent with the factor sets $S_0$ and $T_0$ respectively. That is, the factor sets $(m - 1)S_0 + pT_0$ and $nS_0 + (q - 1)T_0$ are both equivalent to the trivial factor sets. However, if $G/Z$ has order
$h$ then $hS_0$ and $hT_0$ both are equivalent to the trivial factor set $(3; p. 223)$. Thus, for any integer $k$ the following values for $m, n, p$ and $q$ suffice:

\begin{align*}
m &= -h^2 - hk + 1 \\
n &= h \\
p &= -h^2k - hk^2 - h \\
q &= hk + 1,
\end{align*}

and the theorem is proved.

4. Groups with finitely many endomorphisms.

**Theorem 2.** A group $G$ which has only finitely many endomorphisms is itself finite.

**Proof.** The group $G$ will then have only finitely many inner automorphisms so $Z/G$ will be finite, say of order $n$. We shall once again use multiplicative notation for $Z$. For any positive integer $k$ the factor sets $M$ and $M^{k+1}$ are equivalent (as above) so that we may, by Lemma 2, extend to endomorphisms of $G$ the endomorphisms $\theta(k)$ of $Z$ which map $z$ to $z^{k+1}$ for $z \in Z$. If $Z$ contains at least one element of infinite order then all the $\theta(k)$ will be distinct. Therefore, we shall assume that $Z$ is a torsion group.

Suppose that we can decompose $Z$ as the direct product $\prod_{\gamma} A_{\gamma}$ of infinitely many nontrivial factors. The elements $m_{a, \beta}$ of the factor set $M$ will have components in only finitely many of the factors $A_{\gamma}$, because there are only finitely many elements $m_{a, \beta}$. If $A_{\delta}$ is any factor containing no component of any element $m_{a, \beta}$ then we can extend the endomorphism $\psi$ of $Z$ to $G$ where $\psi$ is defined by

\[
\psi(a) = a \text{ if } a \in A_{\rho}, \quad \rho \neq \delta \\
\psi(a) = 1 \text{ if } a \in A_{\delta}.
\]

This will give us infinitely many endomorphisms of $G$.

If $Z$ is of bounded order then\footnote{For basic results on abelian groups see [2].} it is the direct sum of cyclic groups so that either $Z$ is finite or the preceding paragraph applies. If $Z$ is of unbounded order write $Z$ as the direct product of its Sylow subgroups. In this case either infinitely many of these subgroups are nontrivial and the above remarks pertain or for some prime $p$ the $p$-Sylow subgroup is of unbounded order. Choose then elements $x_1, x_2, \ldots$ of $Z$ such that $x_i$ has order $p^i$. The endomorphisms $\theta(p^i)$ defined in the first paragraph will all be distinct. For if $n = p\cdot n_1$ where $p$ and $n_1$ are coprime then
$\theta(p^i)(x_i) = x_i$ if and only if $i \leq j + r$. These endomorphisms $\theta(p^i)$ may, as we said, be extended to $G$ so $G$ again has infinitely many endomorphisms. Thus $Z$ is finite so $G$ is also finite.

In closing, the author should like to express his appreciation for the advice and criticism of Dr. G. Baumslag.

REFERENCES
