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GROUPS WITH FINITELY MANY AUTOMORPHISMS

JONATHAN L. ALPERIN

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1. Introduction. The connection between the structure of a group and the structure of its endomorphisms or group of automorphisms is a very interesting question but one to which there is as yet a scarcity of answers. We present here two theorems pertaining to this problem. We first show that a finitely generated group has a finite automorphism group if and only if it is a finite and central extension of a cyclic group. The restriction to finitely generated groups is essential. Indeed, there exist indecomposable torsion-free abelian groups of rank the cardinal of the continuum whose automorphism groups are cyclic of order two (see [2], p. 180, 18(b)). The second result which we present is a new and much simpler proof of a theorem to be found in a paper of Baer [1]; namely, a group possessing only finitely many endomorphisms is itself finite.

Before discussing these theorems we shall describe the notation to be used in this paper. Throughout, G will denote a group with center Z . Let coset representatives g_α, g_β, \dots of Z in G be chosen for all α, β, \dots elements of G/Z such that $g_1 = 1$. Let $M = \{m_{\alpha, \beta}\}$ be the corresponding factor set. For the theory of such factor sets, essential to the following, the reader should consult Kurosh [4] or M. Hall [3]. Finally, for any group H , let $\text{Aut}(H)$ be the automorphism group of H .

2. Preliminary lemmas. The first lemma follows immediately from the definition of equivalence of factor sets.

LEMMA 1. *Let N and R be factor sets from the group B into the abelian group $A_1 \times A_2$. Let N_1, N_2, R_1, R_2 be the factor sets from B into A_1 and A_2 obtained by taking components of N and R . Then N and R will be equivalent if and only if N_1 is equivalent with R_1 and N_2 with R_2 .*

The remaining lemmas are, I believe, entirely or in part scattered throughout the literature. We include proofs for the convenience of the reader.

LEMMA 2. *Let ψ be an endomorphism of Z such that the factor sets M and $\psi(M) = \{\psi(m_{\alpha, \beta})\}$ of G are equivalent. Then ψ may be extended to an endomorphism of G and if ψ is an automorphism of Z then it may be extended to an automorphism of G .*

Proof. First note that it is clear from the definition of factor set that $\psi(M)$ is a factor set. Since M and $\psi(M)$ are assumed to be equivalent we may choose for each $\alpha \in G/Z$ an element $c_\alpha \in Z$ such that

$$\psi(m_{\alpha,\beta}) = c_{\alpha\beta}^{-1} m_{\alpha,\beta} c_\alpha c_\beta$$

for all $\alpha, \beta \in G/Z$. Now every element $g \in G$ can be uniquely expressed in the form $g = g_\alpha a$ where $\alpha \in G/Z$ and $a \in Z$. Thus we may define a mapping φ of G into itself according to the rule, for all $g \in G$,

$$\varphi(g) = g_\alpha c_\alpha \psi(a).$$

Then φ is an endomorphism of G since if $g = g_\alpha a$, $h = g_\beta b$ are elements of G , where $\alpha, \beta \in G/Z$, $a, b \in Z$ then

$$\begin{aligned} \varphi(gh) &= \varphi(g_\alpha a g_\beta b) \\ &= \varphi(g_{\alpha\beta} m_{\alpha,\beta} ab) \\ &= g_{\alpha\beta} c_{\alpha\beta} \psi(m_{\alpha,\beta} ab) \\ &= g_{\alpha\beta} m_{\alpha,\beta} c_\alpha c_\beta \psi(a) \psi(b) \\ &= g_\alpha c_\alpha \psi(a) g_\beta c_\beta \psi(b) \\ &= \varphi(g) \varphi(h). \end{aligned}$$

Since $g_1 = 1$ implies $m_{1,1} = g_1^{-1} g_1 g_1 = 1$ and so $c_1 = 1$, we have that φ extends ψ to G .

Finally, suppose ψ is an automorphism of G . Let $g = g_\alpha a$ be an arbitrary element of G , as above. Then $\varphi(g) = 1$ implies that $g_\alpha c_\alpha \psi(a) = 1$ and hence $\alpha = 1$ because α is the image of $g_\alpha c_\alpha \psi(a)$ in G/Z . Therefore $g_1 = c_1 = 1$ so we have $\psi(a) = 1$ and $g = g_\alpha a = 1$. φ is consequently a one-to-one mapping of G into itself. To conclude we prove that g is the image of some element under φ . Choose $b \in Z$ such that $\psi(b) = c_\alpha^{-1} a$. Then $\varphi(g_\alpha b) = g_\alpha c_\alpha \psi(b) = g_\alpha a = g$.

LEMMA 3. *Let A be a central and characteristic subgroup of the group H . The group K of those automorphisms of H which leave A element-wise fixed and which induce the identity automorphism on H/A is naturally isomorphic to the group $\text{Hom}(H/A, A)$ of homomorphisms of H/A into A .*

Proof. If $\sigma \in \text{Hom}(H/A, A)$ define a mapping θ of H into itself by $\theta(h) = h\sigma(hA)$ for $h \in H$. θ will be an element of K , as can be verified directly, and the correspondence of σ with θ is the required isomorphism.

COROLLARY. *If $\text{Aut}(A)$, $\text{Aut}(H/A)$, and $\text{Hom}(H/A, A)$ are finite then $\text{Aut}(H)$ is finite.*

Proof. If $\theta \in \text{Aut}(H)$ let θ_1 be the restriction of θ to A and θ_2 the automorphism of H/A induced by θ . Then the mapping sending θ to $(\theta_1, \theta_2) \in \text{Aut}(A) \times \text{Aut}(H/A)$ is a homomorphism with kernel K . Hence, the product of the orders of $\text{Aut}(A)$, $\text{Aut}(H/A)$, and $\text{Hom}(H/A, A)$ is a bound for the order of $\text{Aut}(H)$.

3. Finite automorphism groups.

THEOREM 1. *A finitely generated group G has a finite automorphism group if and only if it has a central cyclic subgroup of finite index.*

Proof. First we shall demonstrate the sufficiency of these conditions. In so doing, we may assume G is infinite and therefore a finite and central extension of an infinite cyclic group. The center Z of G is of finite index in G so by Schreier's subgroup theorem (3; p. 97) is finitely generated. Hence Z is the direct sum of a finite abelian group T and an infinite cyclic group. Consequently, $\text{Aut}(G/Z)$ and $\text{Hom}(G/Z, Z) = \text{Hom}(G/Z, T)$ are finite. To conclude, we can apply the above Corollary once we prove that $\text{Aut}(Z)$ is finite. However, this follows from another application of the Corollary with $A = T$ and $H = Z$.

The necessity of the condition is more difficult to prove. If $\text{Aut}(G)$ is finite then G/Z is finite being isomorphic to the group of inner automorphisms of G . Again, by Schreier's theorem, Z will be finitely generated. We may assume, in contradiction to the theorem, that Z in a direct decomposition into cyclic groups, contains two or more infinite cyclic factors. In fact let $Z = W \times (a) \times (b)$ where (a) and (b) are the subgroups generated by elements a and b of infinite order and W is a subgroup of Z . We need only show that infinitely many automorphisms of Z , which are the identity on W and map the group (a, b) generated by a and b onto itself, may be extended to automorphisms of G .

For the remainder of this proof we shall use the additive notation for Z . Write the factor set M as $m_{\alpha, \beta} = w_{\alpha, \beta} + r_{\alpha, \beta}a + t_{\alpha, \beta}b$ where $w_{\alpha, \beta} \in W$, $s_{\alpha, \beta}$ and $t_{\alpha, \beta}$ are integers. Let the automorphism θ of Z be defined by $\theta(w) = w$ for $w \in W$, $\theta(a) = ma + nb$, $\theta(b) = pa + qb$ where m, n, p , and q are integers such that $|mq - np| = 1$. The factor set $\theta(M)$ is then expressible as $\theta(m_{\alpha, \beta}) = w_{\alpha, \beta} + s'_{\alpha, \beta}a + t'_{\alpha, \beta}b$ where $s'_{\alpha, \beta} = ms_{\alpha, \beta} + pt_{\alpha, \beta}$ and $t'_{\alpha, \beta} = ns_{\alpha, \beta} + qt_{\alpha, \beta}$. Therefore, $S_0 = \{s_{\alpha, \beta}\}$, $T_0 = \{t_{\alpha, \beta}\}$, $S'_0 = \{s'_{\alpha, \beta}\}$, and $T'_0 = \{t'_{\alpha, \beta}\}$ are factor sets of G/Z with integral values and $S'_0 = mS_0 + pT_0$, $T'_0 = nS_0 + qT_0$. By Lemmas 1 and 2 the proof is now reduced to the following problem: Find infinitely many quadruplets (m, n, p, q) of integers such that $|mq - np| = 1$ and the factor sets $mS_0 + pT_0$ and $nS_0 + qT_0$ are equivalent with the factor sets S_0 and T_0 respectively. That is, the factor sets $(m-1)S_0 + pT_0$ and $nS_0 + (q-1)T_0$ are both equivalent to the trivial factor sets. However, if G/Z has order

h then hS_0 and hT_0 both are equivalent to the trivial factor set (3; p. 223). Thus, for any integer k the following values for m, n, p and q suffice:

$$\begin{aligned} m &= -h^2 - hk + 1 \\ n &= h \\ p &= -h^2k - hk^2 - h \\ q &= hk + 1, \end{aligned}$$

and the theorem is proved.

4. Groups with finitely many endomorphisms.

THEOREM 2. *A group G which has only finitely many endomorphisms is itself finite.*

Proof. The group G will then have only finitely many inner automorphisms so Z/G will be finite, say of order n . We shall once again use multiplicative notation for Z . For any positive integer k the factor sets M and M^{kn+1} are equivalent (as above) so that we may, by Lemma 2, extend to endomorphisms of G the endomorphisms $\theta(k)$ of Z which map z to z^{kn+1} for $z \in Z$. If Z contains at least one element of infinite order then all the $\theta(k)$ will be distinct. Therefore, we shall assume that Z is a torsion group.

Suppose that we can decompose Z as the direct product $\prod_{\gamma} A_{\gamma}$ of infinitely many nontrivial factors. The elements $m_{\alpha, \beta}$ of the factor set M will have components in only finitely many of the factors A_{γ} , because there are only finitely many elements $m_{\alpha, \beta}$. If A_{δ} is any factor containing no component of any element $m_{\alpha, \beta}$ then we can extend the endomorphism ψ of Z to G where ψ is defined by

$$\begin{aligned} \psi(a) &= a \text{ if } a \in A_{\rho}, \quad \rho \neq \delta \\ \psi(a) &= 1 \text{ if } a \in A_{\delta}. \end{aligned}$$

This will give us infinitely many endomorphisms of G .

If Z is of bounded order then¹ it is the direct sum of cyclic groups so that either Z is finite or the preceding paragraph applies. If Z is of unbounded order write Z as the direct product of its Sylow subgroups. In this case either infinitely many of these subgroups are nontrivial and the above remarks pertain or for some prime p the p -Sylow subgroup is of unbounded order. Choose then elements x_1, x_2, \dots of Z such that x_i has order p^i . The endomorphisms $\theta(p^j)$ defined in the first paragraph will all be distinct. For if $n = p^r n_1$ where p and n_1 are coprime then

¹ For basic results on abelian groups see [2].

$\theta(p^j)(x_i) = x_i$ if and only if $i \leq j + r$. These endomorphisms $\theta(p^j)$ may, as we said, be extended to G so G again has infinitely many endomorphisms. Thus Z is finite so G is also finite.

In closing, the author should like to express his appreciation for the advice and criticism of Dr. G. Baumslag.

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Jonathan L. Alperin, <i>Groups with finitely many automorphisms</i>	1
Martin Arthur Arkowitz, <i>The generalized Whitehead product</i>	7
John D. Baum, <i>Instability and asymptoticity in topological dynamics</i>	25
William Aaron Beyer, <i>Hausdorff dimension of level sets of some Rademacher series</i>	35
Frank Herbert Brownell, III, <i>A note on Cook's wave-matrix theorem</i>	47
Gulbank D. Chakerian, <i>An inequality for closed space curves</i>	53
Inge Futtrup Christensen, <i>Some further extensions of a theorem of Marcinkiewicz</i>	59
Charles Vernon Coffman, <i>Linear differential equations on cones in Banach spaces</i>	69
Eckford Cohen, <i>Arithmetical notes. III. Certain equally distributed sets of integers</i>	77
John Irving Derr and Angus E. Taylor, <i>Operators of meromorphic type with multiple poles of the resolvent</i>	85
Jacob Feldman, <i>On measurability of stochastic processes in products space</i>	113
Robert S. Freeman, <i>Closed extensions of the Laplace operator determined by a general class of boundary conditions, for unbounded regions</i>	121
Robert E. Fullerton, <i>Geometric structure of absolute basis systems in a linear topological space</i>	137
Dieter Gaijer, <i>On conformal mapping of nearly circular regions</i>	149
Andrew Mattei Gleason and Hassler Whitney, <i>The extension of linear functionals defined on H^∞</i>	163
Seymour Goldberg, <i>Closed linear operators and associated continuous linear operators</i>	183
Basil Gordon, Aviezri Siegmund Fraenkel and Ernst Gabor Straus, <i>On the determination of sets by the sets of sums of a certain order</i>	187
Branko Grünbaum, <i>The dimension of intersections of convex sets</i>	197
Paul Daniel Hill, <i>On the number of pure subgroups</i>	203
Robert Peter Holten, <i>Generalized Goursat problem</i>	207
Alfred Horn, <i>Eigenvalues of sums of Hermitian matrices</i>	225
Henry C. Howard, <i>Oscillation and nonoscillation criteria for $y''(x) + f(y(x))p(x) = 0$</i>	243
Taqdir Husain, <i>S-spaces and the open mapping theorem</i>	253
Richard Eugene Isaac, <i>Markov processes and unique stationary probability measures</i>	273
John Rolfe Isbell, <i>Supercomplete spaces</i>	287
John Rolfe Isbell, <i>On finite-dimensional uniform spaces. II</i>	291
N. Jacobson, <i>A note on automorphisms of Lie algebras</i>	303
Antoni A. Kosinski, <i>A theorem on families of acyclic sets and its applications</i>	317
Marvin David Marcus and H. Minc, <i>The invariance of symmetric functions of singular values</i>	327
Ralph David McWilliams, <i>A note on weak sequential convergence</i>	333
John W. Milnor, <i>On axiomatic homology theory</i>	337
Victor Julius Mizel and Malempati Madhusudana Rao, <i>Nonsymmetric projections in Hilbert space</i>	343
Calvin Cooper Moore, <i>On the Frobenius reciprocity theorem for locally compact groups</i>	359
Donald J. Newman, <i>The Gibbs phenomenon for Hausdorff means</i>	367
Jack Segal, <i>Convergence of inverse systems</i>	371
Józef Siciak, <i>On function families with boundary</i>	375
Hyman Joseph Zimmerberg, <i>Two-point boundary conditions linear in a parameter</i>	385