HAUSDORFF DIMENSION OF LEVEL SETS OF SOME RADEMACHER SERIES

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1. Introduction. A special case of a result of Kaczmarz and Steinhaus [4] (Theorem 2 with $a = b$) shows that if \( \{a_i\} \) \((i = 1, 2, \ldots)\) is a sequence of real numbers with \( \sum_{i=1}^{\infty} |a_i| = +\infty \) and \( a_i \to 0 \), then the Rademacher series \( \sum_{i=1}^{\infty} a_i R_i(x) \) assumes every preassigned real value \( c \) (cardinal number of the continuum) times for \( x \) in \((0,1]\). One object of this paper is to refine this result in certain directions. We shall prove

**Theorem 1.** If the sequence \( \{a_i\} \) is in \( l_2 \), but not in \( l_1 \), then \( \sum_{i=1}^{\infty} a_i R_i(x) \) assumes every preassigned real value on a set of Hausdorff dimension 1.

We shall also prove

**Theorem 2.** If \( \{a_i\} \) is a sequence of bounded variation \((\sum_{i=1}^{\infty} |a_i - a_{i-1}| < \infty)\) which is not in \( l_1 \) but \( a_i \to 0 \), then \( \sum_{i=1}^{\infty} a_i R_i(x) \) assumes each preassigned real value on a set of Hausdorff dimension at least \( 1/2 \).

In § 6, we apply the method of proof to a problem on the distribution of digits in decimal expansions of numbers.

In § 7 through 11, we develop a theory of dimension of level sets for series of the type \( \sum_{i=1}^{\infty} r^i R_i(x) \) where \( r \) is a fixed number in the interval \([1/2, 1)\).

2. Preliminary definitions and lemmas.

**Definition 1.** The \( i^{th} \) \((i = 1, 2, \ldots)\) Rademacher function is defined to be \( R_i(x) = 1 - 2\varepsilon_i(x) \) \((0 < x \leq 1)\), where \( \varepsilon_i(x) \) is the \( i^{th} \) digit of the (unique) nonterminating binary expansion of \( x \).

**Definition 2.** Let \( X \) be a subset of Euclidean \( n \)-space. Let \( J_i(X) \) be a finite or countably infinite set of open spheres \( \{J_i\} \) \((i = 1, 2, \ldots)\) with finite diameters \( |J_i| \) whose union covers \( X \) and whose diameters do not exceed \( \varepsilon \) where \( \varepsilon > 0 \). The Hausdorff outer measure of order \( s \),

Received August 26, 1959, and in revised form February 21, 1961.

The work reported in this paper was performed by Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology with the joint support of the U. S. Army, Navy, and Air Force. The author is indebted to I. M. Sheffer and J. R. Kinney for advice.
where $s$ is a positive number, is defined as
\[
\mathcal{A}^s X = \lim_{t \to 0} \inf_{J \in (X)} \sum |J|^s
\]
where the summation is extended over the members of $J_t(X)$ and $\inf_{J \in (X)}$ is with respect to all admissible $J_s(X)$. The Hausdorff dimension of $X$ is defined as $\dim X = \inf_{s \geq 0} \{ s \mid \mathcal{A}^s X = 0 \}$.

**DEFINITION 3.** Suppose $x \in (0, 1)$. $T_n(x)$ shall be the point in Euclidean $n$-space whose $j$th coordinate is given by $T_i(x) = \sum_{e_i = 0}^{\infty} \varepsilon_{i, n+j}(x) 2^{-i} (j = 1, 2, \ldots, n)$.

**LEMMA 1.** If $x$ is a binary irrational (not of the form $p/2^k$ with $p$ and $k$ integers), then $R_{(i-1)n+j}(x) = R_i(T_i(x))$ for $i = 1, 2, \ldots$ and $j = 1, 2, \ldots, n$.

**LEMMA 2.** If $A$ is a subset of $(0, 1]$, then $n \dim A = \dim T_n(A)$.

**Proof.** A binary cube (or binary interval in the case of the line) is defined as a closed cube in $n$-dimensional space whose $2^m$ vertices are of the form
\[
\left( \frac{k_1 + \delta_1}{2^m}, \frac{k_2 + \delta_2}{2^m}, \ldots, \frac{k_n + \delta_n}{2^m} \right),
\]
where the $\delta_i (i = 1, 2, \ldots, n)$ assume independently the values 0 or 1, the $k_i$ are nonnegative integers less than $2^m$, and $m$ is a positive integer. The cube is denoted by $W_{k_1, k_2, \ldots, k_n}$ or $W$. For $n = 1$, $I$ is written in place of $W$. It can be shown that an equivalent definition of dimension is obtained (for a subset of the unit cube $[0 \leq x_i \leq 1 (i = 1, 2, \ldots, n)$] in $n$-space) if one replaces in Definition 2 the spheres by binary cubes and uses the cube edge $(1/2^m)$ in place of the sphere diameter.

Let $k_i/2^m = \sum_{j=1}^{m} e_j 2^{-j}$, where $e_j$ is 0 or 1. With the cube $W_{k_1, k_2, \ldots, k_n}$ we associate the closed interval $I$:
\[
\left[ \sum_{j=1}^{m} \sum_{i=1}^{n} e_j 2^{-[(j-1)n+i]} \right., \sum_{j=1}^{m} \sum_{i=1}^{n} e_j 2^{-[(j-1)n+i]} + 2^{-mn} \right]
\]
and write $I = s(W)$. Let $\{I^n\}$ denote the set of all binary intervals on $[0,1]$ of length of the form $2^{-kn} (k = 0, 1, 2, \ldots)$. $s$ is a one-to-one mapping between $\{I^n\}$ and the set of all binary cubes in $n$-space, and hence has an inverse $s^{-1}$. We note that $l(s(W)) = (e(W))^n$ where $l$ denotes the length of the interval and $e$ denotes length of the cube edge.

We show first that $\dim T_n(A) \leq n \dim A$. It suffices to assume that $A$ contains no points of the form $p/2^k$ ($p$ and $k$ are integers), for if it did, they could be deleted without changing the dimension of $A$ or $T_n(A)$.
Suppose positive $\delta$ and $\varepsilon$ are given arbitrarily. There exists a covering of $A$ by binary intervals $I_i$ ($i = 1, 2, \cdots$) such that $\sum_{i=1}^{\infty} (u(I_i))^{\dim A + \delta} < \varepsilon$. Here we make use of analogue of Theorem 16.1 of [8] for Hausdorff measures defined using binary cube coverings. Replace the covering $I_i$ ($i = 1, 2, \cdots$) by a covering of intervals from $\{I^n\}$ by replacing each interval of $I_i$ of length $2^{-p_i}$ by $2^{p_i-p_i}$ intervals of length $2^{-p_i}$ where $p_i$ is the smallest integer greater than $p_i$ which is a multiple of $n$. Denote the resulting covering intervals by $I_i^*$ ($i = 1, 2, \cdots$). The cubes $s^{-1}(I_i^*)$ will cover $T_n(A)$ and
\[
\sum_{i=1}^{\infty} (e(s^{-1}(I_i^*)))^{n(\dim A + \delta)} = \sum_{i=1}^{\infty} (u(I_i^*))^{\dim A + \delta} < 2^n \varepsilon.
\]
Since this holds for each pair of positive $\varepsilon$ and $\delta$, $\dim T_n(A) \leq n \dim A$.

We now show that $\dim A \leq 1/n \dim T_n(A)$. Suppose that positive $\varepsilon$ and $\delta$ are given arbitrarily. There exists a covering of $T_n(A)$ by binary cubes $W^j$ ($j = 1, 2, \cdots$) such that $\sum_{j=1}^{\infty} (e(W^j))^{\dim T_n(A) + \delta} < \varepsilon$. Let $W^j_k$ ($j = 1, 2, \cdots, k(i)$) ($k(i) \leq 3^n$) be the binary cubes of edge $e(W^j)$ which intersect the cube $W^j$, including $W^j$ itself. The closed binary intervals $s(W^j)$ ($j = 1, 2, \cdots, k(i); i = 1, 2, \cdots$) cover $A$ and
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{k(i)} (s(W^j))^n(\dim T_n(A) + \delta) \leq 3^n \sum_{i=1}^{\infty} (e(W^j))^{\dim T_n(A) + \delta} < 3^n \varepsilon.
\]
Thus $\dim A \leq (1/n) \dim T_n(A)$. This completes the proof of Lemma 2.

We remark that Lemma 2 has an analogue for the well-known Peano curve (see [3], pages 457–8) which maps the unit interval into an $n$-dimensional cube.

**Lemma 3.** If $\{a_i\}$ is in $l_2$, then $\sum_{i=1}^{\infty} a_iR_i(x)$ converges almost everywhere.

This lemma is due to Rademacher. See Theorem 3 in [4].

**Lemma 4.** If $\sum_{i=1}^{\infty} |a_i| = \infty$ and $a_i \to 0$, then given any real number $\alpha$, there exists a binary irrational $x_0 \in (0,1)$ such that $\sum_{i=1}^{\infty} a_iR_i(x_0) = \alpha$.

See Theorem 2 in [4]. The proof of this lemma is similar to that of Riemann’s theorem that any conditionally convergent series of real numbers can be rearranged to converge to any preassigned real number.

**Lemma 5.** If $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ are convergent series of real numbers, then $\sum_{i=1}^{\infty} a_i + b_i$ is convergent with value $\sum a_i + \sum b_i$.

This is Theorem 3, page 78, of [6].
DEFINITION 4. A subset $A$ of $(0,1]$ is of type $G(n, K, M)$ ($n, K, M$ nonnegative integers) if it has the following property. Suppose $\varepsilon'_{n+K}$ ($i = M, M + 1, \ldots$) is an arbitrarily given sequence of 0's and 1's. Then there exists $x \in A$ such that $\varepsilon'_{n+K}(x) = \varepsilon'_{n+K}$ ($i = M, M + 1, \ldots$).

**Lemma 6.** If $A \subset (0,1]$ is of type $G(n, K, M)$, then $\dim A \geq 1/n$.

**Lemma 6A.** If $A \subset (0,1]$ is simultaneously of types $G(2n + 1, 1, 0)$, $G(2n + 1, 3, 0)$, ..., $G(2n + 1, 2n - 1, 0)$ where $n$ is a positive integer, then $\dim A \leq n/(2n + 1)$.

Lemmas 6 and 6A follow from Lemma 2.

3. Proof of Theorem 1. Let $\alpha$ be a real number. Let $n$ be an integer $> 1$. Let

$$E = \left\{ (x_1, x_2, \ldots, x_n) \mid \sum_{j=1}^{n} \sum_{i=1}^{\infty} R_i(x_j)a_{(i-1)n+j} = \alpha \right\},$$

where $0 < x_j \leq 1$. Let $E'$ be the subset of $E$ whose points have only binary irrational coordinates. Let $x' = (x'_1, x'_2, \ldots, x'_n)$ be in $E'$ and let $x$ be the (unique) inverse image of $x'$ under $T_n$; i.e., $x' = T_n(x)$. Observe that $x$ is a binary irrational number. We have

$$\alpha = \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{(i-1)n+j} R_i(x'_j) = \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{(i-1)n+j} R_i(T_n^j(x))$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{(i-1)n+j} R_i((i-1)n+j(x)) = \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{(i-1)n+j} R_i((i-1)n+j(x))$$

$$= \sum_{i=1}^{\infty} a_i R_i(x).$$

Lemmas 1 and 5 justify the third and fourth equalities, respectively. Let $\beta(\alpha, \{a_i\})$ be the set of all $x$ in $(0,1]$ such that $\alpha = \sum_{i=1}^{\infty} a_i R_i(x)$. It follows from (1) that the $x$ defined above is in $\beta(\alpha, \{a_i\})$ and hence that $T_n(\beta) \supset E'$. We now show that $\dim E' \geq n - 1$. For some integer $j(1 \leq j \leq n)$, we must have $\sum_{i=1}^{\infty} |a_{(i-1)n+j}| = \infty$. Without loss of generality, we take $j = n$. Let $A_j (j = 1, 2, \ldots, n - 1)$ be the subset of the interval $(0 < x_j \leq 1]$ where $\sum_{i=1}^{\infty} a_{(i-1)n+j} R_i(x_j)$ converges and $A'_j$ be the subset of $A_j$ whose points are binary irrational. Let $A^* = X_{1 \leq j \leq n-1} A'_j$ be the Cartesian product of the $A'_j$. Suppose $x^* \in A^*$ and $x^* = (x^*_1, x^*_2, \ldots, x^*_{n-1})$. Suppose $\sum_{j=1}^{n-1} \sum_{i=1}^{\infty} R_i(x^*_j)a_{(i-1)n+j} = \alpha$. By Lemma 4, there exists a binary irrational number $x^*_n$ such that $\sum_{i=1}^{\infty} a_{(i-1)n+n} R_i(x^*_n) = \alpha - \alpha_i$. Thus $\sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{(i-1)n+j} R_i(x^*_j) = \alpha + \alpha - \alpha_i = \alpha$. Hence $(x^*_1, x^*_2, \ldots, x^*_{n}) \in E'$. By Lemma 3, the measure of $A_j$ and $A'_j$ is 1. Thus $\dim A^* = n - 1$. But since the projection of $E'$ on the $X_{1 \leq j \leq n-1} x_j$ hyperplane includes $A^*$, $\dim E' \geq n - 1$. 
Using Lemma 2, we have \( \dim \beta(\alpha, \{a_i\}) = (1/n) \dim T_n(\beta(\alpha, \{a_i\})) \geq (1/n) \dim B' \geq (1/n)(n - 1) = 1 - (1/n) \). Since this holds for every integer \( n > 1 \), the theorem follows.

4. Proof of Theorem 2. Let \( n \) be an integer \( > 1 \). We can assume, without loss of generality, that the \( a_i \) are positive. Since \( \{a_i\} \) is not in \( l_1 \), at least one of the \( 2n + 1 \) sequences \( \{a_{(2n+1)i-2n}\}, \{a_{(2n+1)i-(2m-1)}\}, \ldots, \{a_{(2n+1)i}\} \) \( (i = 1, 2, \ldots) \) is not in \( l_1 \). We suppose, without loss of generality, that \( \{a_{(2n+1)i}\} \) is not in \( l_1 \). We take \( s_i = \pm 1 \). Choose \( s_{(2n+1)i-(2n-j)} \) \( (j = 0, 2, 4, \ldots, 2n - 2; i = 1, 2, \ldots) \) as an arbitrary sequence of \( +1 \)'s and \( -1 \)'s except that an infinity are \( -1 \). Put \( s_{(2n+1)i-(2n-j-1)} = -s_{(2n+1)i-(2n-j)} \).

The series \( \sum_{i=1}^{\infty} s_{(2n+1)i-(2n-j)} a_{(2n+1)i-(2n-j)} + s_{(2n+1)i-(2n-j-1)} a_{(2n+1)i-(2n-j-1)} \) converges since \( \sum (a_i - a_{i-1}) \) converges absolutely and hence any subseries \( \sum' (a_i - a_{i-1}) \) converges absolutely. Call its value \( \alpha_{2n-j} \).

Now let \( \alpha \) be the preassigned value and let \( \alpha' = \alpha - \sum_{j=0}^{2n-1} \alpha_{2n-j} \). Choose, by Lemma 4, \( s_{(2n+1)i} = \pm 1 \) so that \( \sum s_{(2n+1)i} a_{(2n+1)i} = \alpha' \). With these choices for \( s_i \), \( \sum_{i=1}^{\infty} s_i a_i = \alpha \). Remembering that \( \varepsilon_i(x) = (1 - R_i(x))/2 \), from Lemma 6A, we have that the set on which \( \sum a_i R_i(x) = \alpha \) has dimension at least \( n/(2n + 1) \). Since \( n \) is an arbitrary integer \( > 1 \), the theorem follows.

5. Remarks.
1. Theorem 1 could be slightly improved as follows. We could consider the sets \( \beta(\gamma, \delta, \{a_i\}) \) of \( x \) where for preassigned numbers \( \gamma \) and \( \delta \) \( (-\infty \leq \gamma \leq \delta \leq +\infty) \), \( \lim_{n \to \infty} \sum_{i=1}^{\infty} a_i R_i(x) = \delta \) and \( \lim_{n \to \infty} \sum_{i=1}^{\infty} a_i R_i(x) = \gamma \). If \( a_i \) is in \( l_2 \) but not in \( l_1 \), then \( \dim \beta(\gamma, \delta, \{a_i\}) = 1 \).
2. It might be interesting to investigate the measure of \( \beta(\alpha, \{a_i\}) \) under the hypothesis of Theorem 1. It might also be interesting to determine, if possible, the dimension function (dimension in sense of [2]) of \( \beta(\alpha, \{a_i\}) \).
3. The conclusion of Theorem 2 is not as precise as that of Theorem 1. However, it may be the best possible conclusion.
4. The function sequence \( \{a_i R_i(x)\} \) is a probablistically independent function sequence. No explicit use of this property is made, but we believe that this property is implicitly used. We hope later to consider extensions to other probablistically independent function sequences; also extensions to certain lacunary trigonometric series should be considered. We note that \( R_k(x) = \text{sign} \{\sin 2^k \pi x\} \).

6. Application. Using the method of proof of Theorem 1, we prove

**Theorem 3.** Let \( \bar{L}(x) = \lim_{n \to \infty} (1/n) \sum_{i=1}^{n} \varepsilon_i(x) \) and \( L(x) = \frac{1}{2^{(x+1)}} \sum_{i=1}^{x} \varepsilon_i(x) \). Then \( L(x) = \bar{L}(x) \) for almost all \( x \) in the interval \( (0, 1) \).
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i(x), \quad \text{where} \quad x \in (0,1). \]

Let \( B = \{ x \mid \overline{L}x > \underline{L}x \} \). Then \( \dim B = 1 \).

**Proof.** We shall show that \( \dim \{ x \mid \overline{L}(x) - \underline{L}(x) \geq (1/k) \} \geq (k - 1)/k \), where \( k \) is an integer \( > 1 \). Let \( B_j = \{ x_j \mid \overline{L}(x_j) = \underline{L}(x_j) = 1/2; \ 0 < x_j < 1 \} \) \((1 \leq j \leq k - 1)\) and \( E^* \) be the Cartesian product of all the \( B_j \). The linear measure of \( B_j \) is 1. Now fix an \((x_1, x_2, \ldots, x_{k-1})\) in \( E^* \). Let \( B^* \) be the subset of \( E^* \) whose points have binary irrational coordinates. Choose \( x_k \) in \((0,1)\) such that \( \overline{L}(x_k) = 1 \) and \( \underline{L}(x_k) = 0 \). For example, \( x_k \) could be the decimal (base 2)

\[ \begin{array}{ccccccc}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6
\end{array} \]

Let \( E \) be the subset of the unit cube \((x_1, x_2, \ldots, x_k) \) \((0 \leq x_j \leq 1, \ 1 \leq j \leq k - 1)\) such that \((x_1, x_2, \ldots, x_{k-1}) \in E^* \) and \( x_k \) as chosen above. Obviously, \( \dim E = k - 1 \). We have, for \( x \in A = \{ x \mid T_n(x) \in E \} \),

\[
\frac{1}{nk} \sum_{i=1}^{nk} \varepsilon_i(x) \\
= \frac{1}{k} \left\{ \frac{k-1}{n} \sum_{j=0}^{n-1} \varepsilon_{j+k}(x) + \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{jk}(x) \right\} \\
= \frac{1}{k} \left\{ \frac{k-1}{n} \sum_{j=1}^{n} \varepsilon_j(x_j) + \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j(x_{k}) \right\}.
\]

Thus,

\[
\lim_{n \to \infty} \frac{1}{nk} \sum_{i=1}^{nk} \varepsilon_i(x) = \frac{1}{k} \left\{ \left( \frac{k-1}{n} \right) \bar{L}(x_j) + \underline{L}(x_k) \right\} \\
= \frac{1}{k} \left\{ \frac{k-1}{2} + 1 \right\} = \frac{1}{k} \left\{ (k - 1) \frac{1}{2} + 1 \right\} = \frac{1}{2} + \frac{1}{2k}.
\]

Similarly,

\[
\lim_{n \to \infty} \frac{1}{nk} \sum_{i=1}^{nk} \varepsilon_i(x) = \frac{1}{2} - \frac{1}{2k}.
\]

Since \( B \supset \{ x \mid \overline{L}(x) - \underline{L}(x) \geq 1/k \} \), we have, using Lemma 2, \( \dim B \geq \dim A = 1/k \dim E = (1/k)(k - 1) = 1 - 1/k \) for every integer \( k > 1 \). The theorem follows.

7. Geometric series. In § 7–11, we investigate the Hausdorff dimension of the set

\[ \beta(\alpha, \{ r^i \}) = \{ x \mid \sum_{i=1}^{\infty} r^i R_0(x) = \alpha; \ 0 < x \leq 1 \} , \]
where \( r \) is a fixed number in \((1/2, 1)\), and \( -\sum_{i=1}^{\infty} r^i < \alpha < \sum_{i=1}^{\infty} r^i \). The
sets \( \beta \) are closed, but not necessarily perfect. Since \( \sum_{i=1}^{\infty} r^i R_i(x) \) converges
absolutely, it is sufficient to consider the sets \( \beta_i(\alpha, r) = \{ x | \sum_{i=1}^{\infty} r^i \varepsilon_i(x) = \alpha; 0 < x \leq 1 \} \) with \( 0 < \alpha < \sum_{i=1}^{\infty} r^i \). Since the dimension of a set is not
changed by adding to it a countable set, we add to \( \beta_i(\alpha, r) \) those binary
rationals \( p/2^k \) (\( p,k \) are nonnegative integers) for which \( \sum_{i=1}^{\infty} r^i \varepsilon_i(p/2^k) = \alpha \) \((0 \leq p/2^k \leq 1)\), where \( \varepsilon_i(p/2^k) \) is the \( i \)th digit of the finite binary
expansion of \( p/2^k \). For the remainder of this paper we take \( \alpha \in (0, \sum_{i=1}^{\infty} r^i) \).
We take \( \log = \log_2 \).

8. Preliminary lemmas.

DEFINITION 5. For \( x \in (0,1] \) and \( 2^{-1/(n-1)} < r \leq 2^{-1/n} \) with \( n \) a fixed
integer \( > 1 \), we define \( T_{n,r}(x) \) as the point in \( n \)-space whose \( j \)th coordinate
is given by

\[
T_{n,r}^j(x) = (1 - r_n) \sum_{i=0}^{\infty} \varepsilon_{i+j}(x) r^{ni} \quad (j = 1, 2, \cdots, n).
\]

If \( x \) is of the form \( p/2^k \) \((p,k \) integers\), \( T_{n,r}^j(x) \) shall be two valued;
one value is given by (2) and the other value is given by (2) with \( \varepsilon_{i+j}(x) \) replaced by \( \varepsilon'_{i+j}(x) \) arising from the finite binary expansion. In
addition, \( T_{n,r}^j(0) = 0 \) \((j = 1, 2, \cdots, n)\).

LEMMA 7. If \( A \) is a subset of \([0,1]\), then \( \dim A = |\log r| \dim T_{n,r}(A) \).
The proof is similar to that of Lemma 2.

DEFINITION 6. Suppose \( r \leq 2^{-1/n} \). \( C_r^n \) is the Cantor set of constant
dissection constructed as follows. Divide the closed interval \([0,1]\) into
three intervals by the points \( r^n, 1 - r^n \) and remove the open middle
interval of length \( 1 - 2r^n \). Repeat this process on the remaining left
and right intervals, removing middle intervals of length \((1 - 2r^n)r^n\).
The process is continued indefinitely. The set remaining is \( C_r^n \). \( C_r^n \) is
the Cartesian product of \( C_r \) with itself \( n \) times.

DEFINITION 7. If \( 2^{-1/(n-1)} < r \leq 2^{-1/n} \), \( l_{r,a} \) is the \( n \)-space hyperplane:
\( \sum_{i=1}^{n} r^i x_i = \alpha(1 - r^n) \).

LEMMA 8. \( T_{n,r}([0,1]) = C_r^n \).

Proof. Suppose \( z \in T_{n,r}([0,1]) \). The coordinates of \( z \) are given by
expressions of type \((1 - r^n) \sum \varepsilon_i r^{ni} \) and hence are in \( C_r^n \) (see [9]). Thus,
\( T_{n,r}([0,1]) \subseteq C_r^n \). Now suppose \( z \in C_r^n \) and \( z = (x_1, x_2, \cdots, x_n) \). There exist
\( \varepsilon_i = 0 \) or \( 1 \) such that \( x_j = (1 - r^n) \sum_{i=1}^{\infty} \varepsilon_i r^{ni} \) \((j = 1, 2, \cdots, n)\). Let \( x = \sum_{i=1}^{\infty} \sum_{j=1}^{n} \varepsilon_i 2^{-(ni+j)} \). At least one of the values of \( T_{n,r}(x) \) is \( z \). Thus,
\( z \in T_{n,r}([0,1]) \) and hence \( C_r^n \subset T_{n,r}([0,1]) \).

**Lemma 9.** Except possibly for a countable set, \( T_{n,\alpha}(\beta_1(\alpha,r)) = l_{r,\alpha} \cap C_r^n \).

**Proof.** Let \( z \in l_{r,\alpha} \cap C_r^n \) and \( z = (x_1, x_2, \ldots, x_n) \).

Since \( z \in l_{r,\alpha} \), \( \sum_{i=1}^{n} r^i x_i = \alpha(1 - r^n) \). Since \( z \in C_r^n \), \( x_i = (1 - r^n) \sum_{j=1}^{\infty} \varepsilon_j r^{n(j-1)+i} \) with \( \varepsilon_j = 0 \) or \( 1 \). Possibly only a finite number of the \( \varepsilon_j \) are different from zero (see [9]). Choose \( x = \sum_{j=1}^{\infty} \sum_{i=1}^{n} \varepsilon_j (1/2)^n(j-1)+i \).

If all the \( \varepsilon \)'s but a finite number are zero, then \( (a) \varepsilon_j = \varepsilon_{n(j-1)+i}(x) \). Otherwise, we have \( (b) \varepsilon_j = \varepsilon_{n(j-1)+i}(x) \). In case \( (b) \),

\[
\sum_{i=1}^{n} \varepsilon_i(x) r^i = \sum_{j=1}^{\infty} \sum_{i=1}^{n} \varepsilon_{n(j-1)+i}(x) r^{n(j-1)+i} = \sum_{j=1}^{\infty} \sum_{i=1}^{n} \varepsilon_j r^{n(j-1)+i}
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{\infty} \varepsilon_j r^{n(j-1)+i} = \sum_{i=1}^{n} r^i \sum_{j=1}^{\infty} \varepsilon_j r^{n(j-1)} = \sum_{i=1}^{n} r^i \frac{x_i}{1 - r^n}
\]

\[
= \frac{1}{1 - r^n} \sum_{i=1}^{n} r^i x_i = \frac{1}{1 - r^n} \alpha(1 - r^n) = \alpha.
\]

A similar computation holds in case \( (a) \) with \( \varepsilon_i(x) \) replaced by \( \varepsilon'_i(x) \). Hence \( z \in \beta_1(\alpha,r) \). Also for this \( x \), one of the values of \( T_{n,\alpha}(x) \) is \( z \). At most a countable number of \( x \) can have two values for \( T_{n,\alpha}(x) \). Hence, except for a countable number, \( z = T_{n,\alpha}(x) \in T_{n,\alpha}(\beta_1(\gamma,r)) \). Therefore, except for a countable set, \( l_{r,\alpha} \cap C_r^n \subset T_{n,\alpha}(\beta_1(\alpha,r)) \). Similar work shows that except for at most a countable number of values of \( T_{n,\alpha}(\beta_1) \), \( T_{n,\alpha}(\beta_1(\alpha,r)) \subset l_{r,\alpha} \cap C_r^n \).

**Lemma 10.** \( \dim C_r^n = 1/| \log r | \).

This follows from Theorem 5 of [2].

9. Case where \( r \) is a root of 2. We consider the case \( r = 2^{-1/n} \) with \( n \) an integer \( > 1 \) and obtain

**Theorem 4.** \( \dim \beta_1(\alpha,2^{-1/n}) = 1 - 1/n \).

**Proof.** Suppose \( n > 1 \). In this case, \( C_r^n \) is the unit cube and \( \dim (l_{2^{-1/n,\alpha} \cap C_r^n}) = n - 1 \). Using Lemmas 7 and 9, we have

\[
\dim \beta_1(\alpha,2^{-1/n}) = | \log 2^{-1/n} | \dim T_{n,\alpha}(\beta_1) = \frac{1}{n} \dim (l_{2^{-1/n,\alpha} \cap C_r^n})
\]

\[
= 1 - 1/n .
\]

If \( n = 1 \), \( \sum \varepsilon_i(x) r^i = \sum \varepsilon_i(x) 2^{-i} \) assumes every value on \((0,1]\) exactly once.
THEOREM 5. For $1 > r \geq 1/2$ and for almost every $\alpha$, $\dim \beta_i(\alpha, r) \leq \log 2r$.
If $2^{-1/n} \geq r \geq 2^{-1/(n-1)}$ for $n > 1$, then $\dim \beta_i(\alpha, r) \leq 1 - 1/n$ for every $\alpha$.

Proof. From Lemma 7, $\dim T_{n,r}([0,1]) = 1/\log r$. Marstrand’s theorem [7], when generalized to $n$ dimensions, states that, for almost every $\alpha$, the hyperplane $l_{r,a}$ intersects the set $T_{n,r}([0,1])$ in a set of dimension $\leq 1/|\log r| - 1$. Thus, from Lemmas 7 and 9, $\dim (\beta_i(\alpha, r)) = |\log r| \dim T_{n,r}(\beta_i(\alpha, r)) = |\log r| \dim (l_{r,a} \cap C_r^n) \leq |\log r| (1/|\log r| - 1) = \log 2r$.

We now prove the second part. We need to show that the dimension of $l_{r,a} \cap C_r^n$ is less than or equal to the dimension of $C_r^{n-1}$. Roughly, we proceed as follows. $C_r^{n-1}$ is a perfect set constructed in Cantor fashion from nested cubes which we call $W_{j-1}^{n-1}$ ($j = 1, 2, \cdots$). These are the cubes of edge $r^{m_j}$ ($m_j$ a positive integer) which are the $(n-1)$-dimensional Cartesian products of the closed non-middle intervals used in constructing $C_r^1$. We denote by $W_{j-1}^n$ the corresponding $n$-dimensional cube whose base is $W_{j-1}^{n-1}$. We show that it requires at most $2^n$ “translates” of each $W_{j-1}^n$ to cover $l_{r,a} \cap C_r^n$.

For arbitrary positive $\varepsilon$ and $\varepsilon'$, there exists a subsequence of the $W_{j-1}^n$ such that

$$\sum' |W_{j-1}^n| \dim c_{r^{n-1}}^{n-1+\varepsilon'} < \delta,$$

where $\sum'$ indicates summation over a subsequence of $j = 1, 2, \cdots$, and $|W_{j-1}^n|$ denotes cube edge. Consider one of the cubes $W_{j-1}^n$ ($j = 1, 2, \cdots$), say $W_{j-1}^n$. Let

$$(k_1^i + \delta_1)r^{m_1}, (k_2^i + \delta_2)r^{m_2}, \cdots, (k_{n-1}^i + \delta_{n-1})r^{m_{n-1}}$$

be the $2^{n-1}$ vertices of $W_{j-1}^n$. Here the $\delta_i$ assume independently the values 0 or 1, and the $k_i^i$ are certain integers.

The $x_n$ coordinates of the intersections of the lines in $n$-space $x_i = (k_i^i + \delta_i)r^{m_j}$ ($i = 1, 2, \cdots, n-1$) with the hyperplane $l_{r,a}$ are given by

$$x_n = \left[\alpha(1 - r^n) - \sum_{i=1}^{n-1} r^i(k_i^i + \delta_i)r^{m_j}\right]/r^n.$$

The extreme values of these intersections are

$$x_n^0 = \left[\alpha(1 - r^n) - \sum_{i=1}^{n-1} r^i k_i^i r^{m_j}\right]/r^n$$

and
We have

\[ x_n^1 = \left[ \alpha(1 - r^n) - \sum_{i=1}^{n} r^i(k_i + 1)r^{n_i} \right]/r^n. \]

We have

\[ x_n^0 - x_n^1 = \frac{1}{r^n} \sum_{i=1}^{n-1} r^i r^{n_i} \leq r^{n_i} / r^n(1 - r) \leq r^{n_i} 2/(1 - 2^{-1/n}). \]

Let \( g(n) \) be three more than the largest integer in \( 2/(1 - 2^{-1/n}) \).

Since \( |W_{i-1}^n| = r^{n_i} \), to each \( W_{i-1}^n \) there corresponds a set of at most \( g(n) \) \( n \)-dimensional cubes of side \( r^{n_i} \), say \( W_{i-1}^n \), \( W_{i-1}^n \), \ldots, \( W_{i-1}^n \) such that \( \bigcup_{i=1}^\infty \bigcup_{\alpha=1}^n W_{i-1}^n \cap C_\alpha \) and

\[ \sum_{i=1}^\infty \sum_{\alpha=1}^n |W_{i-1}^n| \dim C_{i+\tau}^{i+n} \leq g(n) \sum_{i=1}^\infty |W_{i-1}^n| \dim C_{i+\tau}^{i+n} \leq g(n). \]

Hence, \( \dim l_{\alpha,r} \cap C_\alpha \leq \dim C_{i+\tau}^{i+n} = (n - 1)/|\log r^n| \).

Thus, using Lemma 7, we have

\[ \dim \beta_\alpha(\alpha,r) \leq |\log r| \dim(l_{\alpha,r} \cap C_\alpha) \leq 1 - 1/n. \]

We shall show that there are members of the exceptional set of \( \alpha \) in Theorem 5. Take \( r = (\sqrt{5} - 1)/2 \) and \( \alpha = \sum_{i=0}^\infty r^{2i+1} = r/(1 - r^2) \).

Note that for this \( r \), \( r = r^2 + r^3 \). Now let \( A \) be all those \( x \) in \((0,1)\) for which either \( \epsilon_{3i+1}(x) = 0, \epsilon_{3i+2}(x) = \epsilon_{3i+3}(x) = 1 \) or \( \epsilon_{3i+1}(x) = 1, \epsilon_{3i+2}(x) = \epsilon_{3i+3}(x) = 0 \) independently for \( i = 1, 2, \ldots \). Then \( A \) is of type \( G(3,1,0) \) in the sense of Definition 4. For any \( x \in A \), \( \sum_{i=0}^\infty \epsilon_i(x)((\sqrt{5} - 1)/2)^i = \alpha \) and \( \dim A \geq 1/3 \). But \( \log_4(\sqrt{5} - 1) \triangleq 0.31 \). We remark that if \( r = (\sqrt{5} - 1)/2 \) and \( \alpha = \sum_{k=0}^\infty r^{2k+1} \), then it can be shown that \( \dim \beta(\alpha,r) \geq (1/4) \log_4(3 + \sqrt{5})/2 \triangleq 0.35 \).

11. Additional theorem.

**Theorem 6.** Let \( (\sqrt{5} - 1)/2 < r < 1 \). Then \( \dim \beta_\alpha(\alpha,r) \geq 1/n \) where \( n \) is the least integer \( n_0 \) such that \( n_0 > [\log(2r - 1) - \log(r^2 + r - 1)] / (\log r) \).

Note that as \( r \to (\sqrt{5} - 1)/2^+ \), \( n \to \infty \).

To prove the theorem, we need two lemmas.

**Lemma 11.** If a monotone decreasing sequence \( \{a_i\} \) of positive terms has the property that \( a_i \leq \sum_{j=i+1}^\infty a_j < +\infty \) for all \( i \), then every \( \alpha(0 < \alpha \leq \sum_{i=1}^\infty a_i) \) can be expressed as \( \alpha = \sum_{i=1}^{\infty} a_i \), where \( \Sigma' \) indicates some of the terms possibly are omitted from the sum. Further, \( \Sigma' \) can be required to have an infinite number of terms.

The first sentence of the lemma is stated essentially in [5, page 547], except that there the case of \( \Sigma' (\pm) a_i \) is discussed. But one can write
Lemma 12. If \( \sqrt{5} - 1)/2 < r < 1 \) and \( n \) is a positive integer such that \( r^n < (r^2 + r - 1)/(2r - 1) \), then \( \sum_{r_{n+1}^i}^{\infty} r_i \) [where \( \Sigma'' \) indicates that the terms of the form \( r_{n+1}^i \) \((i = 0, 1, 2, \ldots)\) are omitted] has the property that \( r^i < \Sigma''_{j=i+1}^{\infty} r^j \).

Proof. \( \sum_{j=i+1}^{\infty} r^j = \sum_{j=i+1}^{\infty} r^j - \sum_{v}^r r^{n+1} \) where \( \Sigma_v \) is over all \( v \) such that \( nv + 1 \geq i + 1 \). Let \( v \) be the smallest \( v \) allowed. Then

\[
\sum_{j=i+1}^{\infty} r^j = \frac{r^{i+1}}{1 - r} - \frac{r^{n+1}}{1 - r^n} \geq \frac{r^{i+1}}{1 - r} - \frac{r^{i+1}}{1 - r^n}
\]

Therefore \( \sum_{j=i+1}^{\infty} r^j > r^i \) if \( r/(1 - r) - r/(1 - r^n) > 1 \), and hence if \( r/(1 - r^n) < (2r - 1)/(1 - r) \) which reduces to \( r^n < (r^2 + r - 1)/(2r - 1) \).

We now prove Theorem 6. Let

\[
\alpha_1 = r^1 + r^{n+1} + r^{2n+1} + \cdots,
\]
\[
\alpha_2 = r^2 + r^3 + \cdots + r^n + r^{n+2} + \cdots + r^{2n} + r^{2n+2} + \cdots.
\]

Case I. Suppose \( 0 < \alpha < \alpha_2 \). Let

\[
f(x_1) = \sum_{i=1}^{\infty} \varepsilon_i(x_1) r^{(i-1)n+1}.
\]

Then

\[
sup_{0 < x_1 < 1} f(x_1) = f(1) = \alpha_1.
\]

Let \( \omega = \{x_1 | f(x_1) < \alpha\} \). Since \( \lim_{x_1 \to 0^+} f(x_1) = 0 \), \( \omega \) contains an open binary interval \( \omega^* = (0, 1/2^q) \) where \( q \) is an integer. Choose \( x_1^* \in \omega^* \) and let \( \alpha^* = f(x_1^*) \). Note that \( \alpha - \alpha^* < \alpha < \alpha_2 \). By Lemma 12 and then Lemma 11, there exist \( \varepsilon_k^* = 0, 1 \) \((k = 2, 3, \cdots, n, n + 2, \cdots, 2n, 2n + 2, \cdots)\) (for infinitely many \( k, \varepsilon_k^* = 1 \)) such that

\[
\varepsilon_k^* r^2 + \cdots + \varepsilon_k^* r^n + \varepsilon_{k+2}^* r^{n+2} + \cdots + \varepsilon_{2n}^* r^{2n} + \cdots = \alpha - \alpha^*.
\]

Choose \( x = \sum_{i=1}^{\infty} \varepsilon_i^* x^{i-2}, \) where \( \varepsilon_{(i-1)n+1}^* = \varepsilon_i(x_i^*) \) \((i = 1, 2, 3, \cdots)\), and \( \varepsilon_i^* = \varepsilon_i \) \((i = 2, 3, \cdots, n, n + 2, \cdots, 2n, 2n + 2, \cdots)\). Then \( \sum_{i=1}^{\infty} \varepsilon_i^* x^i = \alpha^* + \alpha - \alpha^* = \alpha \). Thus, it is possible to choose \( \varepsilon_{(i-1)n+1}^* = 0 \) \((i > q)\) independently (except that infinitely many are 1) so that \( \sum_{i=1}^{\infty} \varepsilon_i^* x^i = \alpha \). Hence \( \beta_\alpha(r) \) includes a set \( A \) of type \( G(n, 1, q) \) which, by Lemma 6, has dimension \( \geq 1/n \).

Case II. Suppose \( \alpha_2 \leq \alpha \). Let
\[ \omega = \{ x_i | \alpha - \alpha_2 < f(x_i) \} . \]

Since \( \alpha < \alpha_1 + \alpha_2, \alpha - \alpha_2 < \alpha_1. \) Also \( \lim_{x_i \to 0} f(x) = \alpha_1. \) Therefore, \( \omega \)
contains an open binary interval \( \omega^*_i = ((2^q - 1)/2^q, 1). \) Choose \( x_i^* \in \omega^*_i \)
and let \( \alpha^* = f(x_i^*). \) Then \( \alpha - \alpha^* < \alpha - (\alpha - \alpha_2) = \alpha_2. \) The proof is then
completed as in Case I with \( \omega^*_i \) in place in \( \omega^*. \)

We remark that Theorem 6 can be generalized to other absolutely
convergent Rademacher series \( \sum_{i=1}^{\infty} a_i R_i(x); \) namely, those which satisfy
conditions of the form \( 0 < a_i/\sum_{j>1} a_j < (\sqrt{5} - 4/n - 1)/2 \) for a fixed
integer \( n > 1 \) and \( \{a_i\} (i = 1, 2, \ldots) \) a positive monotone sequence.

**Theorem 7.** If \( r \geq 2^{-1/n}, \) then \( \dim_\beta (\alpha, r) \geq 1 - 1/n. \)

The details of the proof will not be given since it is similar to that
of Theorem 6. Since \( r^n \geq 1/2, \) given \( \alpha, \) there exists \( M \) such that \( \varepsilon_{in+j} = 0, 1 \) \( (1 \leq j \leq n - 1, i > M) \) can be chosen independently and then
\( \varepsilon_{in+j} [(1 \leq i \leq M, 1 \leq j \leq n - 1) \) and \( (j = n, i = 1, 2, \ldots)] \) determined so
that \( \alpha = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{in+j} r^{in+j}. \)

*Added in Proof.* A sequel to this paper will appear in Proc. Amer. Math. Soc.

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Pacific Journal of Mathematics
Vol. 12, No. 1 January, 1962

Jonathan L. Alperin, Groups with finitely many automorphisms ................................. 1
Martin Arthur Arkowitz, The generalized Whitehead product ........................................ 7
John D. Baum, Instability and asymptoticity in topological dynamics ............................ 25
William Aaron Beyer, Hausdorff dimension of level sets of some Rademacher series .......... 35
Frank Herbert Brownell, III, A note on Cook's wave-matrix theorem ............................ 47
Gulbank D. Chakerian, An inequality for closed space curves ........................................ 53
Inge Futtrup Christensen, Some further extensions of a theorem of Marcinkiewicz .......... 59
Charles Vernon Coffman, Linear differential equations on cones in Banach spaces .......... 69
Eckford Cohen, Arithmetical notes. III. Certain equally distributed sets of integers ....... 77
John Irving Derr and Angus E. Taylor, Operators of meromorphic type with multiple poles of the resolvent ................................................................. 85
Jacob Feldman, On measurability of stochastic processes in products space ................... 113
Robert S. Freeman, Closed extensions of the Laplace operator determined by a general class of boundary conditions, for unbounded regions ........................................ 121
Robert E. Fullerton, Geometric structure of absolute basis systems in a linear topological space ........................................................................................................... 137
Dieter Gaier, On conformal mapping of nearly circular regions ........................................ 149
Andrew Mattei Gleason and Hassler Whitney, The extension of linear functionals defined on $H^\infty$ ................................................................. 163
Seymour Goldberg, Closed linear operators and associated continuous linear operators ................................................................. 183
Basil Gordon, Aviezri Siegmund Fraenkel and Ernst Gabor Straus, On the determination of sets by the sets of sums of a certain order ..................................................... 187
Branko Grünbaum, The dimension of intersections of convex sets .................................. 197
Paul Daniel Hill, On the number of pure subgroups ......................................................... 203
Robert Peter Holten, Generalized Goursat problem ......................................................... 207
Alfred Horn, Eigenvalues of sums of Hermitian matrices ............................................... 225
Henry C. Howard, Oscillation and nonoscillation criteria for $y''(x) + f(y(x)) p(x) = 0$ .................. 243
Taqdir Husain, S-spaces and the open mapping theorem .................................................. 253
Richard Eugene Isaac, Markov processes and unique stationary probability measures .... 273
John Rolfe Isbell, Supercomplete spaces ........................................................................... 287
John Rolfe Isbell, On finite-dimensional uniform spaces. II ................................................ 291
N. Jacobson, A note on automorphisms of Lie algebras .................................................... 303
Antoni A. Kosinski, A theorem on families of acyclic sets and its applications ................ 317
Marvin David Marcus and H. Minc, The invariance of symmetric functions of singular values ................................................................................................................. 327
Ralph David McWilliams, A note on weak sequential convergence ............................... 333
John W. Milnor, On axiomatic homology theory .............................................................. 337
Victor Julius Mizel and Malempati Madhusudana Rao, Nonsymmetric projections in Hilbert space ................................................................. 343
Calvin Cooper Moore, On the Frobenius reciprocity theorem for locally compact groups ........................................................................................................ 359
Jack Segal, Convergence of inverse systems ..................................................................... 371
Józef Siciak, On function families with boundary ............................................................... 375
Hyman Joseph Zimmerberg, Two-point boundary conditions linear in a parameter ....... 385