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A NOTE ON COOK'S WAVE-MATRIX THEOREM

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F. H. BROWNELL

1. **Introduction.** Consider the linear operator H_0 defined by

$$(1.1) \quad [H_0 u](\vec{x}) = -\nabla^2 u(\vec{x}) + V(\vec{x})u(\vec{x})$$

over all $\vec{x} \in R_n$, n -dimensional Euclidean space, for each $u \in \mathcal{D}_0$. Here ∇^2 is the Laplacian and we take \mathcal{D}_0 as the set of all complex valued functions u over R_n which everywhere possess continuous partials of all orders ≤ 2 and which together with these partials are in absolute value $\leq Q(|\vec{x}|)\exp(-2^{-1}|\vec{x}|^2)$ over R_n for some polynomial Q depending on u . Here V is a fixed, real valued, measurable function over R_n subject to additional assumptions below which will assure that H_0 takes \mathcal{D}_0 into $X = L_2(R_n)$ as a symmetric operator in the Hilbert space X .

Assuming that $V \in L_2(R_n)$ for $n = 3$, Cook [2] has shown that the unique existent (see Theorem I following) self-adjoint extension H of H_0 has the unitary operator

$$(1.2) \quad W(t) = e^{itH} e^{-it\tilde{H}},$$

where \tilde{H} is the similar extension of \tilde{H}_0 and \tilde{H}_0 differs from H_0 only by replacing $V(\vec{x})$ by zero in (1.1), to have existent isometric operators W_{\pm} on X which are the strong limits of $W(t)$ as $t \rightarrow \pm \infty$. Moreover, $W_{\pm} \tilde{H} = H W_{\pm}$, the range spaces $Y_{\pm} = W_{\pm} X$ reduce H , and each H eigenvector is orthogonal to Y_{\pm} . In Theorem II below we give a significant sharpening of these results by weakening the restrictions upon V at ∞ . Thus, with arbitrary $\rho > 0$, any function of the form $C|\vec{x}|^{-1-\rho}$ over $|\vec{x}| \geq b$ will qualify under our assumptions (the Coulomb case $C|\vec{x}|^{-1}$ thus being borderline), while only such of form $C|\vec{x}|^{-3/2-\rho}$ there will do so under Cook's assumptions. In Theorem III we also generalize to dimension $n \geq 3$. Cook's results are used by Ikebe [4] in showing $S = W_+^* W_-$, the "S-matrix", to be unitary with $Y_+ = Y_-$ and in showing the expected connection of the positive part of the spectrum of H with scattering theory under considerably more stringent conditions upon V . Our $n = 3$ existence result II for W_{\pm} also includes that of Jauch & Zinnes ([5], p. 566), who assume $V(\vec{x}) = C|\vec{x}|^{-\beta}$ with $1 < \beta < 3/2$, and that of Hack [3], who replaces $\|V\|_{\gamma} < +\infty$ for some $\gamma \in [2,3)$ by the above noted stronger assumption that $|V(\vec{x})| \leq M|\vec{x}|^{-1-\rho}$ over $|\vec{x}| \geq b$ for some $\rho > 0$.*

2. **Statements.** As notation for our theorems, denote $D_b^{\dagger} = \{\vec{x} \in R_n \mid |\vec{x}| \geq b\}$

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* Note added in proof. See also Kuroda, Nuovo Cim., **12**, (1959), p. 431-454 particularly Theorem 4.1), p. 444.

and $D_b^- = \{\bar{x} \in R_n \mid |\bar{x}| \leq b\}$, $|\bar{x}| = [\sum_{j=1}^n x_j^2]^{1/2}$. Also for real $r \geq 1$ and measurable u over D , let $f_r(u, D) = \left[\int_D |u|^r d\mu_n \right]^{1/r}$ with μ_n n -dimensional Lebesgue measure, and define $\|u\|_r = f_r(u, R_n)$ and ${}_+ \|u\|_r = f_r(u, D_b^+)$ and ${}_-\|u\|_r = f_r(u, D_b^-)$ for specified real $b > 0$. Likewise $f_\infty(u, D) = (\text{ess sup}_{\bar{x} \in D} |u(\bar{x})|)$ for measurable u over D defines $\|u\|_\infty$ and ${}_+ \|u\|_\infty$ similarly. If r is suppressed, this denotes $\gamma = 2$, so that $\|u\|$ and ${}_+ \|u\|$ are the $L_2(R_n)$ and $L_2(D_b^+)$ Hilbert space norms.

We also define on $X = L_2(R_n)$ the unitary Fourier-Plancherel transform operators U and \tilde{U} , having $\tilde{U} = U^* = U^{-1}$, by

$$(2.1) \quad [\tilde{U}w](\bar{y}) = \lim_{r \rightarrow +\infty} (2\pi)^{-n/2} \int_{D_r^-} w(\bar{x}) e^{-i(\bar{x} \cdot \bar{y})} d\mu_n(\bar{x}),$$

$$(2.2) \quad [Uw](\bar{x}) = \lim_{r \rightarrow +\infty} (2\pi)^{-n/2} \int_{D_r^-} w(\bar{y}) e^{i(\bar{x} \cdot \bar{y})} d\mu_n(\bar{y}),$$

for all $w \in X$, the limits being X norm limits. Here $(\bar{x} \cdot \bar{y}) = \sum_{j=1}^n x_j y_j$ is the R_n inner product. We also will need to consider the set G of all functions u of the form

$$(2.3) \quad u = Uw, \quad w(\bar{y}) = \exp(-a^2|\bar{y} - \bar{z}|^2)$$

for some $\bar{z} \in R_n$ and real $a > 0$ depending upon u . With this notation our theorems are as follows.

THEOREM I. *Let real $b > 0$ and let η and γ be extended real satisfying $2 \leq \eta$, $n/2 < \eta$, $\eta \leq +\infty$ and $2 \leq \gamma$, $n/2 < \gamma$, $\gamma \leq +\infty$ for integer $n \geq 1$, the dimension of R_n . Let real valued, measurable V over R_n satisfy both*

- (i) ${}_-\|V\|_\eta < +\infty$,
- (ii) ${}_+\|V\|_\gamma < +\infty$.

Then H_0 in (1.1) takes \mathcal{D}_0 into $X = L_2(R_n)$ as a symmetric operator, and H_0 possesses a unique self-adjoint extension operator H in X .

The special case of I where $\gamma = +\infty$ is our previous Theorem (T.1) of [1], except for the enlargement of the initial domain there to \mathcal{D}_0 here; the modification needed to take care of general γ is very slight. As there define $[Aw](\bar{y}) = |\bar{y}|^\gamma w(\bar{y})$ over $\bar{y} \in R_n$, the domain \mathcal{D}_A of A being all $w \in X$ for which $|\bar{y}|^\gamma w(\bar{y})$ is also finitely square integrable. Then A is easily seen selfadjoint in X , and hence so is $\tilde{H} = UA\tilde{U}$ with domain $\mathcal{D} = U\mathcal{D}_A$; moreover, $\tilde{H}_0 \subseteq \tilde{H}$ is now a consequence of standard Fourier transform theorems (or a simple use of Green's formula). With $\mathcal{D} = U\mathcal{D}_A$, and defining $[Vu](\bar{x}) = V(\bar{x})u(\bar{x})$, we have the following lemma.

LEMMA 2.4. *Let V satisfy the hypotheses of Theorem I. Then*

the function Vu is in X for all $u \in \mathcal{D}$. Moreover, for each real $\alpha > 0$ there exists real $\beta_\alpha > 0$ such that

$$(2.5) \quad \|Vu\| \leq \alpha \|\tilde{H}u\| + \beta_\alpha \|u\|$$

over $u \in \mathcal{D}$.

Since $\tilde{H}_0 \subseteq \tilde{H}$ has $\mathcal{D}_0 \subseteq \mathcal{D}$, from this lemma it follows that H_0 takes \mathcal{D}_0 into X , and Green's formula with the \mathcal{D}_0 exponential bound at ∞ shows that H_0 is symmetric. Also $Hu = \tilde{H}u + Vu$ for $u \in \mathcal{D}$ defines H from \mathcal{D} into X , and $H_0 \subseteq H$ follows from $\tilde{H}_0 \subseteq \tilde{H}$. Also our Lemma 2.4 (replacing Lemma T.2 in [1]) shows H self-adjoint in X without any further change ([1], p.957). Likewise the previous approximation argument ([1], p.958) with Lemma 2.4 shows that H is the closure of $H_1 \subseteq H_0 \subseteq H$ and hence of H_0 , and likewise \tilde{H} is the closure of $\tilde{H}_1 \subseteq \tilde{H}_0 \subseteq \tilde{H}$ and hence of \tilde{H}_0 . Thus H is the unique selfadjoint extension of H_0 and \tilde{H} likewise of \tilde{H}_0 , where H_1 and \tilde{H}_1 are the restrictions of H_0 and \tilde{H}_0 respectively to $\mathcal{D}_1 \subseteq \mathcal{D}_0$, with \mathcal{D}_1 the Hermite functions. Thus Theorem I will be proved as soon as we prove Lemma 2.4 in the next section.

For our main Theorems II and III, we also need the following extension of Cook's [2] Lemma 2.

LEMMA 2.6. *If $u \in G$ (i.e. of form 2.3), then with $0 < K_n < +\infty$ for real $r \geq 1$ and real t*

$$(2.7) \quad \|e^{it\tilde{H}}u\|(\vec{x}) = [4(a^4 + t^2)]^{-n/4} \exp(-a^2[4(a^4 + t^2)]^{-1}|\vec{x} + 2t|\vec{z}^2),$$

$$(2.8) \quad \|e^{it\tilde{H}}u\|_r = [4(a^4 + t^2)]^{-(n/2)(1/2-1/r)} (a^2r)^{-n/2r} (K_n)^{1/r},$$

$$(2.9) \quad \|e^{it\tilde{H}}u\|_\infty = [4(a^4 + t^2)]^{-n/4}.$$

Moreover, for real valued, measurable V satisfying both (i) and (ii) of Theorem I with extended real η and γ , there results for such u both

$$(2.10) \quad \int_{-\infty}^{\infty} \|Ve^{it\tilde{H}}u\| dt < +\infty,$$

$$(2.11) \quad 0 = \lim_{|t| \rightarrow \infty} \|Ve^{it\tilde{H}}u\|,$$

if $2 \leq \eta$ and $2 \leq \gamma < n$.

Since $2 \leq \gamma < n$ in the last part of the lemma, this only applies when dimension $n \geq 3$. From the crucial (2.10) and (2.11) (Corollary 2 and 1 of Cook's Lemma 2), the other arguments of Cook's paper [2] apply without other change and yield all the conclusions of our following Theorems II and III, except for the unstated by Cook orthogonality of each H eigenvector in X to Y_\pm , which is an easy consequence of $W_\pm \tilde{H} = HW_\pm$ and hence $\tilde{H} = W_\pm^* HW_\pm$ and the reduction of H by Y_\pm .

Thus as soon as both Lemmas (2.4) and (2.6) are shown in the next section, all our Theorems I, II, and III will be proved.

THEOREM II. *Let $n = 3$ and for some real $b > 0$ let real valued, measurable V satisfy both (i) and (ii) of Theorem I with $\eta = 2$ and some real γ satisfying $2 \leq \gamma < 3$. Then there exist isometric operators W_+ and W_- on $X = L_2(\mathbb{R}_3)$ such that the unitary operator $W(t)$ in (1.2) has $\lim_{t \rightarrow +\infty} \|W_+u - W(t)u\| = 0 = \lim_{t \rightarrow -\infty} \|W_-u - W(t)u\|$ for every $u \in X$. Moreover, $W_{\pm}\tilde{H} = HW_{\pm}$; $P_{\pm} = W_{\pm}W_{\pm}^*$ are orthogonal projections whose range spaces $Y_{\pm} = P_{\pm}X$ reduce H ; and every $u \in \mathcal{D} = \mathcal{D}_H$ satisfying $Hu = \lambda u$ for some scalar λ is orthogonal to Y_{\pm} .*

This is our new version of Cook's theorem, the special case here $\gamma = 2$ being exactly Cook's statement. Since in most applications the potential V will be bounded at ∞ , and since

$$L_{\infty}(D_b^+) \cap L_2(D_b^+) \subset L_{\infty}(D_b^+) \cap L_{\gamma}(D_b^+)$$

properly for $\gamma > 2$ is easily seen, our version is essentially sharper than Cook's. As pointed out in the introduction it "almost" includes the Coulomb potential, which Cook's does not. (Actually, (2.10) fails for $V(\bar{x}) = C|\bar{x}|^{-1}$, $C \neq 0$.) We also remark that there would be no gain in allowing $2 \leq \eta < 3$ in II instead of specifying $\eta=2$, since $\|V\|_2 \leq \|V\|_{\gamma} [\mu_n(D_b^-)]^{1/2-1/\gamma}$ follows from the Schwarz-Hölder inequality.

THEOREM III. *Let integer $n \geq 4$ and for some real $b > 0$ let real valued, measurable V satisfy both (i) and (ii) of Theorem I with some real η and γ satisfying $n/2 < \eta$ and $n/2 < \gamma < n$. Then the Theorem II conclusions follow.*

As above, the assumptions in III are least restrictive with η as small as possible; and, for $V \in L_{\infty}(D_b^+)$ also holding, are then least restrictive with γ as large as possible.

3. Proof of lemmas. We start by proving Lemma 2.4, considering first the case $1 \leq n \leq 3$. For given $\alpha' > 0$, we see by taking $\omega > 0$ sufficiently small in equation (7) of [1] and by $\sqrt{a^2 + b^2} \leq |a| + |b|$ that

$$(3.1) \quad \|u\|_{\infty} \leq \alpha' \|\tilde{H}u\| + \beta'_{\alpha'} \|u\|$$

over all $u \in \mathcal{D}$ for some real $\beta'_{\alpha'} \geq 1$. Now define real $r \geq 2$ if $\gamma > 2$ in Theorem I (the Lemma (2.4) hypotheses) by requiring $2/\gamma + 2/r = 1$. Then (3.1) with $\beta'_{\alpha'} \geq 1$ yields for $u \in \mathcal{D}$

$$(3.2) \quad \begin{aligned} \|u\|_r &\leq [\|u\|_{\infty}^{r-2} \|u\|^2]^{1/r} = \|u\|^{2/r} (\alpha' \|\tilde{H}u\| + \beta'_{\alpha'} \|u\|)^{1-2/r} \\ &\leq \alpha' \|\tilde{H}u\| + \beta'_{\alpha'} \|u\|. \end{aligned}$$

Thus (3.2), (ii) of I , and the Schwarz-Hölder inequality for the associated powers $r/2$ and $\gamma/2$ yield

$$(3.3) \quad +\|Vu\|^2 \leq +\|V\|_\gamma^2 \|u\|_r^2 \leq +\|V\|_\gamma^2 (\alpha' \|\tilde{H}u\| + \beta'_{\alpha'} \|u\|)^2.$$

Also $-\|V\|_2 \leq [\mu_n(D_b^-)]^{1/2-1/\eta} -\|V\|_\eta < +\infty$, using (i) of I and the Schwarz-Hölder inequality with $\eta \geq 2$, gives from (3.1)

$$(3.4) \quad -\|Vu\|^2 \leq -\|V\|_2^2 \|u\|_\infty^2 \leq -\|V\|_2^2 (\alpha' \|\tilde{H}u\| + \beta'_{\alpha'} \|u\|)^2$$

over $u \in \mathcal{D}$. (3.3) and (3.4) and $\|Vu\|^2 = +\|Vu\|^2 + -\|Vu\|^2$ and $\sqrt{a^2 + b^2} \leq |a| + |b|$ yield (2.5), with $\alpha = M\alpha'$ freely chosen > 0 by choice of α' , and $Vu \in X$ as desired if $\gamma > 2$. If $\gamma = 2$, then $-\|V\|_2 < +\infty$ above with (ii) of I yields $\|V\|_2 < +\infty$; hence (3.1) yields (3.4) with the-script dropped, proving (2.5) and $Vu \in X$. Thus Lemma 2.4 has been shown if $1 \leq n \leq 3$.

Now consider the remaining case $n \geq 4$ of Lemma 2.4. Here $2 \leq n/2 < s \leq +\infty$ for $s = \eta$ and $s = \gamma$, and hence real $\tau \geq 2$ and $\mu \geq 2$ are defined by the requirements $2/\gamma + 2/\tau = 1$ and $2/\eta + 2/\mu = 1$ respectively. Moreover, using $(n + \rho)2^{-1} = \gamma$ or η respectively, we see in [1] at the top of p. 956 that $r' = 4\gamma(2\gamma - 4)^{-1} = 2(1 - 2/\gamma)^{-1} = \tau$ or $r' = 4\eta(2\eta - 4)^{-1} = 2(1 - 2/\eta)^{-1} = \mu$ respectively, and equation (8) there becomes

$$(3.5) \quad \|u\|_\tau \leq \alpha' \|\tilde{H}u\| + \beta'_{\alpha'} \|u\|,$$

$$(3.6) \quad \|u\|_\mu \leq \alpha' \|\tilde{H}u\| + \beta''_{\alpha'} \|u\|$$

respectively over $u \in \mathcal{D}$, with real $\beta'_{\alpha'} > 0$ and $\beta''_{\alpha'} > 0$ existing for each real $\alpha' > 0$. From (3.5) and (3.6) respectively, from (ii) and (i) respectively of I , and from the Schwarz-Hölder inequality we obtain respectively

$$(3.7) \quad +\|Vu\|^2 \leq +\|V\|_\gamma^2 \|u\|_\tau^2 \leq +\|V\|_\gamma^2 (\alpha' \|\tilde{H}u\| + \beta'_{\alpha'} \|u\|)^2,$$

$$(3.8) \quad -\|Vu\|^2 \leq -\|V\|_\eta^2 \|u\|_\mu^2 \leq -\|V\|_\eta^2 (\alpha' \|\tilde{H}u\| + \beta''_{\alpha'} \|u\|)^2$$

over $u \in \mathcal{D}$. Thus (3.7) and (3.8) and $\|Vu\| \leq \sqrt{+\|Vu\|^2 + -\|Vu\|^2} \leq +\|Vu\| + -\|Vu\|$ yields (2.5), with $\alpha = M\alpha' > 0$ freely chosen, and $Vu \in X$ as desired when $n \geq 4$, completing the proof of Lemma 2.4.

Finally we must prove Lemma 2.6. Here from the proof of I (independently of any condition on V), we have $\tilde{H} = UA\tilde{U}$ to be the unique self-adjoint extension of \tilde{H}_0 . Hence $e^{it\tilde{H}} = Ue^{itA}\tilde{U}$ and for u of form (2.3) we compute directly, since the L_1 Fourier transform and the L_2 Fourier-Plancherel transform are well known to coincide almost everywhere for functions in $L_1(\mathbb{R}_n) \cap L_2(\mathbb{R}_n)$,

$$\begin{aligned}
(3.9) \quad [e^{it\tilde{H}}u](\bar{x}) &= (2\pi)^{-n/2} \int_{R_n} \exp(-a^2|\bar{y} - \bar{z}|^2 + it|\bar{y}|^2 + i(\bar{y} \cdot \bar{x})) d\mu_n(\bar{y}) \\
&= \prod_{j=1}^n \left\{ (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-a^2(y - z_j)^2 + ity^2 + iyx_j) dy \right\} \\
&= \exp\left(-a^2|\bar{z}|^2 + 4^{-1}(a^2 - it)^{-1} \sum_{j=1}^n (2a^2z_j + ix_j)^2\right) \\
&\quad \prod_{j=1}^n \left\{ (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-(a^2-it)y^2} dy \right\} \\
&= [2(a^2 - it)]^{-n/2} \exp\left(-a^2|\bar{z}|^2 + 4^{-1}(a^2 - it)^{-1} \sum_{j=1}^n (2a^2z_j + ix_j)^2\right).
\end{aligned}$$

From (3.9) we readily obtain (2.7), from which (2.9) is obvious and (2.8) follows by the direct computation

$$\begin{aligned}
(3.10) \quad \|e^{it\tilde{H}}u\|_r &= [4(a^4 + t^2)]^{-n/4} \left[\int_{R_n} \exp(-a^2r4^{-1}(a^4 + t^2)^{-1}|\bar{y}|^2) d\mu_n(\bar{y}) \right]^{1/r} \\
&= [4(a^4 + t^2)]^{-n/4} [a^{-2}r^{-1}4(a^4 + t^2)]^{n/2r} (K_n)^{1/r}
\end{aligned}$$

with $K_n = \int_{R_n} e^{-|\bar{y}|^2} d\mu_n(\bar{y})$ positive and finite.

Finally to prove last statement of Lemma 2.6 with conclusions (2.10) and (2.11), we here are given V to satisfy (i) and (ii) of I with $2 \leq \gamma < n$ and $2 \leq \eta$. Thus $- \|V\|_2 \leq - \|V\|_\eta [\mu_n(D_{\bar{v}})]^{1/2-1/\eta} < +\infty$, as noted just before III, and by (2.9) for our $u \in G$

$$(3.11) \quad - \|Ve^{it\tilde{H}}u\| \leq - \|V\|_2 [4(a^4 + t^2)]^{-n/4}.$$

Since $n > 2$ here, the right side of (3.11) is in $L_1(-\infty, \infty)$ over t . If $\gamma = 2$, then $+ \|V\|_2 < +\infty$ and (3.11) with the $-$ script replaced by $+$ shows $+ \|Ve^{it\tilde{H}}u\| \in L_1(-\infty, \infty)$ over t . If $\gamma > 2$, then the requirement $2/\gamma + 2/r = 1$ defines real $r \geq 2$, and the Schwarz-Hölder inequality for this r yields from (2.8) and (ii) of I for our $u \in G$

$$(3.12) \quad + \|Ve^{it\tilde{H}}u\| \leq + \|V\|_\gamma M'(\alpha^4 + t^2)^{-(n/2)(1/2-1/r)} = M(\alpha^4 + t^2)^{-n/2\gamma},$$

which is in $L_1(-\infty, \infty)$ by $\gamma < n$. Hence (3.11) and (3.12) and $\|w\| \leq + \|w\| + - \|w\|$ prove (2.10) and (2.11), and the proof of Lemma 2.6 is complete.

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