A NOTE ON COOK’S WAVE-MATRIX THEOREM

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1. Introduction. Consider the linear operator $H_0$ defined by

\[(H_0u)(\vec{x}) = -\nabla^2 u(\vec{x}) + V(\vec{x})u(\vec{x})\]

over all $\vec{x} \in \mathbb{R}_n$, $n$-dimensional Euclidean space, for each $u \in \mathcal{D}_0$. Here $\nabla^2$ is the Laplacian and we take $\mathcal{D}_0$ as the set of all complex valued functions $u$ over $\mathbb{R}_n$ which everywhere possess continuous partials of all orders $\leq 2$ and which together with these partials are in absolute value $\leq Q(|\vec{x}|)\exp(-2^{-1}|\vec{x}|^p)$ over $\mathbb{R}_n$ for some polynomial $Q$ depending on $u$. Here $V$ is a fixed, real valued, measurable function over $\mathbb{R}_n$ subject to additional assumptions below which will assure that $H_0$ takes $\mathcal{D}_0$ into $X = L_2(\mathbb{R}_n)$ as a symmetric operator in the Hilbert space $X$.

Assuming that $V \in L_2(\mathbb{R}_n)$ for $n = 3$, Cook [2] has shown that the unique existent (see Theorem I following) self-adjoint extension $H$ of $H_0$ has the unitary operator

\[(1.2) \quad W(t) = e^{itH}e^{-it\tilde{H}},\]

where $\tilde{H}$ is the similar extension of $\tilde{H}_0$ and $\tilde{H}_0$ differs from $H_0$ only by replacing $V(\vec{x})$ by zero in (1.1), to have existent isometric operators $W_{\pm}$ on $X$ which are the strong limits of $W(t)$ as $t \to \pm \infty$. Moreover, $W_{\pm}\tilde{H} = HW_{\pm}$, the range spaces $Y_{\pm} = W_{\pm}X$ reduce $H$, and each $H$ eigenvector is orthogonal to $Y_{\pm}$. In Theorem II below we give a significant sharpening of these results by weakening the restrictions upon $V$ at $\infty$. Thus, with arbitrary $\rho > 0$, any function of the form $C|\vec{x}|^{-1-\rho}$ over $|\vec{x}| \geq b$ will qualify under our assumptions (the Coulomb case $C|\vec{x}|^{-1}$ thus being borderline), while only such of form $C|\vec{x}|^{-3/2-\rho}$ there will do so under Cook's assumptions. In Theorem III we also generalize to dimension $n \geq 3$. Cook's results are used by Ikebe [4] in showing $S = W^*W_-$, the "$S$-matrix", to be unitary with $Y_+ = Y_-$ and in showing the expected connection of the positive part of the spectrum of $H$ with scattering theory under considerably more stringent conditions upon $V$. Our $n = 3$ existence result II for $W_{\pm}$ also includes that of Jauch & Zinnes ([5], p. 566), who assume $V(\vec{x}) = C|\vec{x}|^{-\beta}$ with $1 < \beta < 3/2$, and that of Hack [3], who replaces $+||V||_\gamma < +\infty$ for some $\gamma \in (2,3)$ by the above noted stronger assumption that $|V(\vec{x})| \leq M|\vec{x}|^{-1-\rho}$ over $|\vec{x}| \geq b$ for some $\rho > 0$.

2. Statements. As notation for our theorems, denote $D_+ = \{\vec{x} \in \mathbb{R}_n | |\vec{x}| \geq b\}$

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and $D_\gamma = \{ \bar{x} \in R_n | |\bar{x}| \leq b \}$, $|\bar{x}| = [\sum_{j=1}^{n} x_j^2]^{1/2}$. Also for real $r \geq 1$ and measurable $u$ over $D$, let $f_\gamma(u, D) = \left[ \int_D |u|^r d\mu_n \right]^{1/r}$ with $\mu_n$ $n$-dimensional Lebesgue measure, and define $||u||_r = f_\gamma(u, R_n)$ and $||u||_\infty = f_\gamma(u, D_\gamma)$ for specified real $b > 0$. Likewise $f_\gamma(u, D) = (\text{ess sup} |u(\bar{x})|)$ for measurable $u$ over $D$ defines $||u||_p$ and $||u||_\infty$ similarly. If $r$ is suppressed, this denotes $\gamma = 2$, so that $||u||$ and $||u||_\infty$ are the $L_2(R_n)$ and $L_\infty(D_\gamma)$ Hilbert space norms.

We also define on $X = L_2(R_n)$ the unitary Fourier-Plancherel transform operators $U$ and $\tilde{U}$, having $\tilde{U} = U^* = U^{-1}$, by

\begin{align}
[\tilde{U}w](\bar{y}) &= \lim_{T \to +\infty} (2\pi)^{-n/2} \int_{D_T} w(\bar{x}) e^{-i\bar{x} \cdot \bar{y}} d\mu_n(\bar{x}) , \\
[Uw](\bar{x}) &= \lim_{T \to +\infty} (2\pi)^{-n/2} \int_{D_T} w(\bar{y}) e^{i\bar{x} \cdot \bar{y}} d\mu_n(\bar{y}) ,
\end{align}

for all $w \in X$, the limits being $X$ norm limits. Here $(\bar{x} \cdot \bar{y}) = \sum_{j=1}^{n} x_j y_j$, is the $R_n$ inner product. We also will need to consider the set $G$ of all functions $u$ of the form

\begin{equation}
\tag{2.3}
u = Uw , \quad w(\bar{y}) = \exp(-a^2|\bar{y}|^2)
\end{equation}

for some $\bar{z} \in R_n$ and real $a > 0$ depending upon $u$. With this notation our theorems are as follows.

**Theorem I.** Let real $b > 0$ and let $\eta$ and $\gamma$ be extended real satisfying $2 \leq \gamma$, $\eta < +\infty$ and $2 \leq \eta$, $\eta < \gamma$, $\gamma < +\infty$ for integer $n \geq 1$, the dimension of $R_n$. Let real valued, measurable $V$ over $R_n$ satisfy both

(i) $||V||_\eta < +\infty$ ,

(ii) $||V||_\gamma < +\infty$ .

Then $H_0$ in (1.1) takes $D_0$ into $X = L_2(R_n)$ as a symmetric operator, and $H_0$ possesses a unique self-adjoint extension operator $\tilde{H}$ in $X$.

The special case of $I$ where $\gamma = +\infty$ is our previous Theorem (T.1) of [1], except for the enlargement of the initial domain there to $D_0$ here; the modification needed to take care of general $\gamma$ is very slight. As there define $[Aw](\bar{y}) = |\bar{y}|^2 w(\bar{y})$ over $\bar{y} \in R_n$, the domain $D_A$ of $A$ being all $w \in X$ for which $|\bar{y}|^2 w(\bar{y})$ is also finitely square integrable. Then $A$ is easily seen selfadjoint in $X$, and hence so is $\tilde{H} = UAU\tilde{U}$ with domain $D = U D_A$; moreover, $\tilde{H}_0 \subseteq \tilde{H}$ is now a consequence of standard Fourier transform theorems (or a simple use of Green's formula). With $D = U D_A$, and defining $[Vw](\bar{x}) = V(\bar{x})u(\bar{x})$, we have the following lemma.

**Lemma 2.4.** Let $V$ satisfy the hypotheses of Theorem I. Then
the function $V\mu$ is in $X$ for all $\mu \in \mathcal{D}$. Moreover, for each real $\alpha > 0$ there exists real $\beta_\alpha > 0$ such that
\begin{equation}
\|V\mu\| \leq \alpha \|\tilde{H}\mu\| + \beta_\alpha \|\mu\|
\end{equation}
over $\mu \in \mathcal{D}$.

Since $\tilde{H}_0 \subseteq \tilde{H}$ has $\mathcal{D}_0 \subseteq \mathcal{D}$, from this lemma it follows that $H_0$ takes $\mathcal{D}_0$ into $X$, and Green’s formula with the $\mathcal{D}_0$ exponential bound at $\infty$ shows that $H_0$ is symmetric. Also $Hu = \tilde{H}u + V\mu$ for $\mu \in \mathcal{D}$ defines $H$ from $\mathcal{D}$ into $X$, and $H_0 \subseteq H$ follows from $\tilde{H}_0 \subseteq \tilde{H}$. Also our Lemma 2.4 (replacing Lemma T.2 in [1]) shows $H$ self-adjoint in $X$ without any further change ([1], p.957). Likewise the previous approximation argument ([1], p.958) with Lemma 2.4 shows that $H$ is the closure of $H_1 \subseteq H_0 \subseteq H$ and hence of $H_0$, and likewise $\tilde{H}$ is the closure of $\tilde{H}_1 \subseteq \tilde{H}_0 \subseteq \tilde{H}$ and hence of $\tilde{H}_0$. Thus $H$ is the unique selfadjoint extension of $H_0$ and $\tilde{H}$ likewise of $\tilde{H}_0$, where $H_1$ and $\tilde{H}_1$ are the restrictions of $H_0$ and $\tilde{H}_0$ respectively to $\mathcal{D}_1 \subseteq \mathcal{D}_0$, with $\mathcal{D}_1$ the Hermite functions. Thus Theorem I will be proved as soon as we prove Lemma 2.4 in the next section.

For our main Theorems II and III, we also need the following extension of Cook’s [2] Lemma 2.

**Lemma 2.6.** If $\mu \in G$ (i.e. of form 2.3), then with $0 < K_\mu < +\infty$ for real $r \geq 1$ and real $t$
\begin{align}
(2.7) \quad & |e^{it\tilde{H}}\mu(x)| = |4(a^4 + t^2)|^{-\frac{n}{4}}\exp(-a^2(4(a^4 + t^2))^{-1}|x| + 2t|x^2|), \\
(2.8) \quad & \|e^{it\tilde{H}}\mu\|_r = |4(a^4 + t^2)|^{-\{n/2(1/2 - 1/r)\}}(a^2r)^{-n/2r}(K_\mu)^{1/r}, \\
(2.9) \quad & \|e^{it\tilde{H}}\mu\|_\infty = |4(a^4 + t^2)|^{-n/4}.
\end{align}
Moreover, for real valued, measurable $V$ satisfying both (i) and (ii) of Theorem I with extended real $\eta$ and $\gamma$, there results for such $\mu$ both
\begin{align}
(2.10) \quad & \int_{-\infty}^{\infty} \|V e^{it\tilde{H}}\mu\|dt < +\infty, \\
(2.11) \quad & 0 = \lim_{|t| \to \infty} \|V e^{it\tilde{H}}\mu\|,
\end{align}
if $2 \leq \eta$ and $2 \leq \gamma < n$.

Since $2 \leq \gamma < n$ in the last part of the lemma, this only applies when dimension $n \geq 3$. From the crucial (2.10) and (2.11) (Corollary 2 and 1 of Cook’s Lemma 2), the other arguments of Cook’s paper [2] apply without other change and yeild all the conclusions of our following Theorems II and III, except for the unstated by Cook orthogonality of each $H$ eigenvector in $X$ to $Y_{\pm}$, which is an easy consequence of $W_{\pm}\tilde{H} = HW_{\pm}$ and hence $\tilde{H} = W_{\pm}^*HW_{\pm}$ and the reduction of $H$ by $Y_{\pm}$. 
Thus as soon as both Lemmas (2.4) and (2.6) are shown in the next section, all our Theorems I, II, and III will be proved.

**Theorem II.** Let \( n = 3 \) and for some real \( b > 0 \) let real valued, measurable \( V \) satisfy both (i) and (ii) of Theorem I with \( \eta = 2 \) and some real \( \gamma \) satisfying \( 2 \leq \gamma < 3 \). Then there exist isometric operators \( W_+ \) and \( W_- \) on \( X = L_2(R_3) \) such that the unitary operator \( W(t) \) in (1.2) has
\[
\lim_{t \to \pm \infty} \| W_+ u - W(t) u \| = 0 = \lim_{t \to \pm \infty} \| W_- u - W(t) u \| \quad \text{for every } u \in X.
\]
Moreover, \( W_+ \bar{H} = H W_+ \); \( P_+ = W_+ W_+^* \) are orthogonal projections whose range spaces \( Y_+ = P_+ X \) reduce \( H \); and every \( u \in \mathcal{D} = \mathcal{D}_u \) satisfying \( H u = \lambda u \) for some scalar \( \lambda \) is orthogonal to \( Y_\pm \).

This is our new version of Cook's theorem, the special case here \( \gamma = 2 \) being exactly Cook's statement. Since in most applications the potential \( V \) will be bounded at \( \infty \), and since
\[
L_\infty(D_\pm) \cap L_\eta(D_\pm) \subset L_\infty(D_\pm) \cap L_\eta(D_\pm)
\]
properly for \( \gamma > 2 \) is easily seen, our version is essentially sharper than Cook's. As pointed out in the introduction it "almost" includes the Coulomb potential, which Cook's does not. (Actually, (2.10) fails for \( V(\bar{x}) = C|\bar{x}|^{-1}, C \neq 0 \).) We also remark that there would be no gain in allowing \( 2 \leq \gamma < 3 \) in II instead of specifying \( \gamma = 2 \), since \( \| F \|_\infty \leq \| V \|_{1,|[\mu_n(D_\pm)]^2} \) follows from the Schwarz-Hölder inequality.

**Theorem III.** Let integer \( n \geq 4 \) and for some real \( b > 0 \) let real valued, measurable \( V \) satisfy both (i) and (ii) of Theorem I with some real \( \eta \) and \( \gamma \) satisfying \( n/2 < \eta \) and \( n/2 < \gamma < n \). Then the Theorem II conclusions follow.

As above, the assumptions in III are least restrictive with \( \eta \) as small as possible; and, for \( V \in L_\infty(D_\pm) \) also holding, are then least restrictive with \( \gamma \) as large as possible.

3. Proof of lemmas. We start by proving Lemma 2.4, considering first the case \( 1 \leq n \leq 3 \). For given \( \alpha' > 0 \), we see by taking \( \omega > 0 \) sufficiently small in equation (7) of [1] and by \( \sqrt{a^2 + b^2} \leq |a| + |b| \) that
\[
\| u \|_\infty \leq \alpha' \| \bar{H} u \| + \beta'_r \| u \|
\]
over all \( u \in \mathcal{D} \) for some real \( \beta'_r \geq 1 \). Now define real \( r \geq 2 \) if \( \gamma > 2 \) in Theorem I (the Lemma (2.4) hypotheses) by requiring \( 2/\gamma + 2/r = 1 \). Then (3.1) with \( \beta'_r \geq 1 \) yields for \( u \in \mathcal{D} \)
\[
\| u \|_r \leq \| u \|_\infty^{(r-1)/r} \| u \|^{1/r} = \| u \|^{1/r}(\alpha' \| \bar{H} u \| + \beta'_r \| u \|)^{1-2/r}
\leq \alpha' \| \bar{H} u \| + \beta'_r \| u \|.
\]
Thus (3.2), (ii) of $I$, and the Schwarz-Hölder inequality for the associated powers $r/2$ and $\gamma/2$ yield

$$
(3.3) \quad \|Vu\|^r \leq \|V\|^{r/2} \|u\|^r \leq \|V\|^{r/2} (\alpha' \|\vec{H}u\| + \beta'_* \|u\|)^r.
$$

Also $-\|V\| \leq \|V\| \leq \|V\| + \|\vec{H}u\| + \beta'_* |u|$ gives from (3.1)

$$
(3.4) \quad -\|Vu\|^2 \leq -\|V\| \|u\|^2 \leq -\|V\| (\alpha' \|\vec{H}u\| + \beta'_* \|u\|)^2.
$$

over $u \in \mathcal{D}$. (3.3) and (3.4) and $\|Vu\|^s = \|Vu\|^s + \|Vu\|^s$ and $\sqrt{a^2 + b^2} \leq |a| + |b|$ yield (2.5), with $\alpha = M\alpha'$ freely chosen $> 0$ by choice of $\alpha'$, and $Vu \in X$ as desired if $\gamma > 2$. If $\gamma = 2$, then $-\|V\| < +\infty$ above with (ii) of $I$ yields $\|V\| < +\infty$; hence (3.1) yields (3.4) with the-script dropped, proving (2.5) and $Vu \in X$. Thus Lemma 2.4 has been shown if $1 \leq n \leq 3$.

Now consider the remaining case $n \geq 4$ of Lemma 2.4. Here $2 \leq n/2 < s \leq +\infty$ for $s = \eta$ and $s = \gamma$, and hence real $\tau \geq 2$ and $\mu \geq 2$ are defined by the requirements $2\gamma + 2/\tau = 1$ and $2/\tau + 2/\mu = 1$ respectively. Moreover, using $(n + \rho)\tau^{-1} = \gamma$ or $\eta$ respectively, we see in [1] at the top of p. 956 that $r' = 4\gamma(2\gamma - 2) = 2(1 - 2/\gamma)^{-1} = \tau$ or $r' = 4\eta(2\eta - 4) = 2(1 - 2/\eta)^{-1} = \mu$ respectively, and equation (8) there becomes

$$
\|u\| \leq \alpha' \|\vec{H}u\| + \beta'_* |u| \tag{3.5}
$$

$$
\|u\| \leq \alpha' \|\vec{H}u\| + \beta'_* |u| \tag{3.6}
$$

respectively over $u \in \mathcal{D}$, with real $\beta'_* > 0$ and $\beta'_* > 0$ existing for each real $\alpha' > 0$. From (3.5) and (3.6) respectively, from (ii) and (i) respectively of $I$, and from the Schwarz-Hölder inequality we obtain respectively

$$
(3.7) \quad \|Vu\|^r \leq \|V\|^{r/2} \|u\|^r \leq \|V\|^{r/2} (\alpha' \|\vec{H}u\| + \beta'_* |u|)^r,
$$

$$
(3.8) \quad -\|Vu\|^2 \leq -\|V\| \|u\|^2 \leq -\|V\| (\alpha' \|\vec{H}u\| + \beta'_* |u|)^2.
$$

over $u \in \mathcal{D}$. Thus (3.7) and (3.8) and $\|Vu\| \leq \sqrt{\|Vu\|^2 + \|\vec{H}u\|^2 \leq \|Vu\| + \|\vec{H}u\|}$ yields (2.5), with $\alpha = M\alpha' > 0$ freely chosen, and $Vu \in X$ as desired when $n \geq 4$, completing the proof of Lemma 2.4.

Finally we must prove Lemma 2.6. Here from the proof of $I$ (independently of any condition on $V$), we have $H = UA\vec{U}$ to be the unique self-adjoint extension of $\vec{H}_s$. Hence $e^{it\vec{H}} = Ue^{itA}\vec{U}$ and for $u$ of form (2.3) we compute directly, since the $L_1$ Fourier transform and the $L_2$ Fourier-Plancherel transform are well known to coincide almost everywhere for functions in $L_1(R_n) \cap L_2(R_n)$,
\[ [e^{it\tilde{H}}u](\tilde{x}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(-a^2|\tilde{y} - \tilde{z}|^2 + it|\tilde{y}|^2 + i(\tilde{y} \cdot \tilde{z}))d\mu_n(\tilde{y}) \]

\[ = \prod_{j=1}^{n} \left( (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-a^2(y - z_j)^2 + ity^2 + iyx_j)dy \right) \]

\[ = \exp(-a^2|\tilde{z}|^2 + 4^{-1}(a^2 - it)\sum_{j=1}^{n} (2a^2z_j + ix_j)^2) \prod_{j=1}^{n} \left( (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-(a^2-1)y^2}dy \right) \]

\[ = [2(a^2 - it)]^{-n/2} \exp(-a^2|\tilde{z}|^2 + 4^{-1}(a^2 - it)\sum_{j=1}^{n} (2a^2z_j + ix_j)^2) \].

From (3.9) we readily obtain (2.7), from which (2.9) is obvious and (2.8) follows by the direct computation

\[ ||e^{it\tilde{H}}u||_r = [4(a^4 + t^2)]^{-n/4} \left[ \int_{\mathbb{R}^n} \exp(-a^2r4^{-1}(a^4 + t^2)^{-1}|\tilde{y}|^2)d\mu_n(\tilde{y}) \right]^{1/r} \]

\[ = [4(a^4 + t^2)]^{-n/4} \left[ 4^{-r/2}r^{-1}4(a^4 + t^2) \right]^{n/2r}(K_n)^{1/r} \]

with \( K_n = \int_{\mathbb{R}^n} e^{-|\tilde{y}|^2}d\mu_n(\tilde{y}) \) positive and finite.

Finally to prove last statement of Lemma 2.6 with conclusions (2.10) and (2.11), we here are given \( V \) to satisfy (i) and (ii) of \( I \) with \( 2 \leq \gamma < n \) and \( 2 \leq \eta \). Thus \( -||V||_2 \leq -||V||_n[\mu_n(D^\gamma)]^{(n/2)\gamma^{-1}} < +\infty \), as noted just before III, and by (2.9) for our \( u \in G \)

\[ (3.11) \quad -||Ve^{it\tilde{H}}u|| \leq -||V||_n[4(a^4 + t^2)]^{-n/4} . \]

Since \( n > 2 \) here, the right side of (3.11) is in \( L_4(-\infty, \infty) \) over \( t \). If \( \gamma = 2 \), then \( +||V||_2 < +\infty \) and (3.11) with the \(-\) script replaced by \(+\) shows \( +||Ve^{it\tilde{H}}u|| \in L_4(-\infty, \infty) \) over \( t \). If \( \gamma > 2 \), then the requirement \( 2/\gamma + 2/r = 1 \) defines real \( r \geq 2 \), and the Schwarz-Hölder inequality for this \( r \) yields from (2.8) and (ii) of \( I \) for our \( u \in G \)

\[ (3.12) \quad +||Ve^{it\tilde{H}}u|| \leq +||V||_nM'(a^4 + t^2)^{-n/2}r^{(1-1/r)} = M(a^4 + t^2)^{-n/2r} , \]

which is in \( L_4(-\infty, \infty) \) by \( \gamma < n \). Hence (3.11) and (3.12) and \( ||w|| \leq +||w|| + -||w|| \) prove (2.10) and (2.11), and the proof of Lemma 2.6 is complete.

REFERENCES

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