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**A NOTE ON COOK'S WAVE-MATRIX THEOREM**

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# A NOTE ON COOK'S WAVE-MATRIX THEOREM

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**1. Introduction.** Consider the linear operator  $H_0$  defined by

$$(1.1) \quad [H_0 u](\vec{x}) = -\nabla^2 u(\vec{x}) + V(\vec{x})u(\vec{x})$$

over all  $\vec{x} \in R_n$ ,  $n$ -dimensional Euclidean space, for each  $u \in \mathcal{D}_0$ . Here  $\nabla^2$  is the Laplacian and we take  $\mathcal{D}_0$  as the set of all complex valued functions  $u$  over  $R_n$  which everywhere possess continuous partials of all orders  $\leq 2$  and which together with these partials are in absolute value  $\leq Q(|\vec{x}|)\exp(-2^{-1}|\vec{x}|^2)$  over  $R_n$  for some polynomial  $Q$  depending on  $u$ . Here  $V$  is a fixed, real valued, measurable function over  $R_n$  subject to additional assumptions below which will assure that  $H_0$  takes  $\mathcal{D}_0$  into  $X = L_2(R_n)$  as a symmetric operator in the Hilbert space  $X$ .

Assuming that  $V \in L_2(R_n)$  for  $n = 3$ , Cook [2] has shown that the unique existent (see Theorem I following) self-adjoint extension  $H$  of  $H_0$  has the unitary operator

$$(1.2) \quad W(t) = e^{itH} e^{-it\tilde{H}},$$

where  $\tilde{H}$  is the similar extension of  $\tilde{H}_0$  and  $\tilde{H}_0$  differs from  $H_0$  only by replacing  $V(\vec{x})$  by zero in (1.1), to have existent isometric operators  $W_{\pm}$  on  $X$  which are the strong limits of  $W(t)$  as  $t \rightarrow \pm\infty$ . Moreover,  $W_{\pm}\tilde{H} = HW_{\pm}$ , the range spaces  $Y_{\pm} = W_{\pm}X$  reduce  $H$ , and each  $H$  eigenvector is orthogonal to  $Y_{\pm}$ . In Theorem II below we give a significant sharpening of these results by weakening the restrictions upon  $V$  at  $\infty$ . Thus, with arbitrary  $\rho > 0$ , any function of the form  $C|\vec{x}|^{-1-\rho}$  over  $|\vec{x}| \geq b$  will qualify under our assumptions (the Coulomb case  $C|\vec{x}|^{-1}$  thus being borderline), while only such of form  $C|\vec{x}|^{-3/2-\rho}$  there will do so under Cook's assumptions. In Theorem III we also generalize to dimension  $n \geq 3$ . Cook's results are used by Ikebe [4] in showing  $S = W_+^* W_-$ , the "S-matrix", to be unitary with  $Y_+ = Y_-$  and in showing the expected connection of the positive part of the spectrum of  $H$  with scattering theory under considerably more stringent conditions upon  $V$ . Our  $n = 3$  existence result II for  $W_{\pm}$  also includes that of Jauch & Zinnes ([5], p. 566), who assume  $V(\vec{x}) = C|\vec{x}|^{-\beta}$  with  $1 < \beta < 3/2$ , and that of Hack [3], who replaces  $+\|V\|_{\gamma} < +\infty$  for some  $\gamma \in [2,3)$  by the above noted stronger assumption that  $|V(\vec{x})| \leq M|\vec{x}|^{-1-\rho}$  over  $|\vec{x}| \geq b$  for some  $\rho > 0$ .\*

**2. Statements.** As notation for our theorems, denote  $D_b^+ = \{\vec{x} \in R_n \mid |\vec{x}| \geq b\}$

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\* *Note added in proof.* See also Kuroda, Nuovo Cim., **12**, (1959), p. 431-454 particularly Theorem 4.1), p. 444.

and  $D_b^- = \{\vec{x} \in R_n \mid |\vec{x}| \leq b\}$ ,  $|\vec{x}| = [\sum_{j=1}^n x_j^2]^{1/2}$ . Also for real  $r \geq 1$  and measurable  $u$  over  $D$ , let  $f_r(u, D) = \left[ \int_D |u|^r d\mu_n \right]^{1/r}$  with  $\mu_n$   $n$ -dimensional Lebesgue measure, and define  $\|u\|_r = f_r(u, R_n)$  and  ${}_+ \|u\|_r = f_r(u, D_b^+)$  and  ${}_-\|u\|_r = f_r(u, D_b^-)$  for specified real  $b > 0$ . Likewise  $f_\infty(u, D) = (\text{ess sup}_{\vec{x} \in D} |u(\vec{x})|)$  for measurable  $u$  over  $D$  defines  $\|u\|_\infty$  and  ${}_+ \|u\|_\infty$  and  ${}_-\|u\|_\infty$  similarly. If  $r$  is suppressed, this denotes  $\gamma = 2$ , so that  $\|u\|$  and  ${}_+ \|u\|$  are the  $L_2(R_n)$  and  $L_2(D_b^\pm)$  Hilbert space norms.

We also define on  $X = L_2(R_n)$  the unitary Fourier-Plancherel transform operators  $U$  and  $\tilde{U}$ , having  $\tilde{U} = U^* = U^{-1}$ , by

$$(2.1) \quad [\tilde{U}w](\vec{y}) = \lim_{T \rightarrow +\infty} (2\pi)^{-n/2} \int_{D_T^-} w(\vec{x}) e^{-i(\vec{x} \cdot \vec{y})} d\mu_n(\vec{x}),$$

$$(2.2) \quad [Uw](\vec{x}) = \lim_{T \rightarrow +\infty} (2\pi)^{-n/2} \int_{D_T^+} w(\vec{y}) e^{i(\vec{x} \cdot \vec{y})} d\mu_n(\vec{y}),$$

for all  $w \in X$ , the limits being  $X$  norm limits. Here  $(\vec{x} \cdot \vec{y}) = \sum_{j=1}^n x_j y_j$  is the  $R_n$  inner product. We also will need to consider the set  $G$  of all functions  $u$  of the form

$$(2.3) \quad u = Uw, \quad w(\vec{y}) = \exp(-a^2|\vec{y} - \vec{z}|^2)$$

for some  $\vec{z} \in R_n$  and real  $a > 0$  depending upon  $u$ . With this notation our theorems are as follows.

**THEOREM I.** *Let real  $b > 0$  and let  $\eta$  and  $\gamma$  be extended real satisfying  $2 \leq \eta$ ,  $n/2 < \eta$ ,  $\eta \leq +\infty$  and  $2 \leq \gamma$ ,  $n/2 < \gamma$ ,  $\gamma \leq +\infty$  for integer  $n \geq 1$ , the dimension of  $R_n$ . Let real valued, measurable  $V$  over  $R_n$  satisfy both*

- (i)  ${}_-\|V\|_\eta < +\infty$ ,
- (ii)  ${}_+ \|V\|_\gamma < +\infty$ .

*Then  $H_0$  in (1.1) takes  $\mathcal{D}_0$  into  $X = L_2(R_n)$  as a symmetric operator, and  $H_0$  possesses a unique self-adjoint extension operator  $H$  in  $X$ .*

The special case of  $I$  where  $\gamma = +\infty$  is our previous Theorem (T.1) of [1], except for the enlargement of the initial domain there to  $\mathcal{D}_0$  here; the modification needed to take care of general  $\gamma$  is very slight. As there define  $[Aw](\vec{y}) = |\vec{y}|^\gamma w(\vec{y})$  over  $\vec{y} \in R_n$ , the domain  $\mathcal{D}_A$  of  $A$  being all  $w \in X$  for which  $|\vec{y}|^\gamma w(\vec{y})$  is also finitely square integrable. Then  $A$  is easily seen selfadjoint in  $X$ , and hence so is  $\tilde{H} = UA\tilde{U}$  with domain  $\mathcal{D} = U\mathcal{D}_A$ ; moreover,  $\tilde{H}_0 \subseteq \tilde{H}$  is now a consequence of standard Fourier transform theorems (or a simple use of Green's formula). With  $\mathcal{D} = U\mathcal{D}_A$ , and defining  $[Vu](\vec{x}) = V(\vec{x})u(\vec{x})$ , we have the following lemma.

**LEMMA 2.4.** *Let  $V$  satisfy the hypotheses of Theorem I. Then*

the function  $Vu$  is in  $X$  for all  $u \in \mathcal{D}$ . Moreover, for each real  $\alpha > 0$  there exists real  $\beta_\alpha > 0$  such that

$$(2.5) \quad \|Vu\| \leq \alpha \|\tilde{H}u\| + \beta_\alpha \|u\|$$

over  $u \in \mathcal{D}$ .

Since  $\tilde{H}_0 \subseteq \tilde{H}$  has  $\mathcal{D}_0 \subseteq \mathcal{D}$ , from this lemma it follows that  $H_0$  takes  $\mathcal{D}_0$  into  $X$ , and Green's formula with the  $\mathcal{D}_0$  exponential bound at  $\infty$  shows that  $H_0$  is symmetric. Also  $Hu = \tilde{H}u + Vu$  for  $u \in \mathcal{D}$  defines  $H$  from  $\mathcal{D}$  into  $X$ , and  $H_0 \subseteq H$  follows from  $\tilde{H}_0 \subseteq \tilde{H}$ . Also our Lemma 2.4 (replacing Lemma T.2 in [1]) shows  $H$  self-adjoint in  $X$  without any further change ([1], p.957). Likewise the previous approximation argument ([1], p.958) with Lemma 2.4 shows that  $H$  is the closure of  $H_1 \subseteq H_0 \subseteq H$  and hence of  $H_0$ , and likewise  $\tilde{H}$  is the closure of  $\tilde{H}_1 \subseteq \tilde{H}_0 \subseteq \tilde{H}$  and hence of  $\tilde{H}_0$ . Thus  $H$  is the unique selfadjoint extension of  $H_0$  and  $\tilde{H}$  likewise of  $\tilde{H}_0$ , where  $H_1$  and  $\tilde{H}_1$  are the restrictions of  $H_0$  and  $\tilde{H}_0$  respectively to  $\mathcal{D}_1 \subseteq \mathcal{D}_0$ , with  $\mathcal{D}_1$  the Hermite functions. Thus Theorem I will be proved as soon as we prove Lemma 2.4 in the next section.

For our main Theorems II and III, we also need the following extension of Cook's [2] Lemma 2.

LEMMA 2.6. *If  $u \in G$  (i.e. of form 2.3), then with  $0 < K_n < +\infty$  for real  $r \geq 1$  and real  $t$*

$$(2.7) \quad \|e^{it\tilde{H}}u\|(\bar{x}) = [4(a^4 + t^2)]^{-n/4} \exp(-a^2[4(a^4 + t^2)]^{-1}|\bar{x} + 2t|\bar{z}^2),$$

$$(2.8) \quad \|e^{it\tilde{H}}u\|_r = [4(a^4 + t^2)]^{-(n/2)(1/2-1/r)} (a^2r)^{-n/2r} (K_n)^{1/r},$$

$$(2.9) \quad \|e^{it\tilde{H}}u\|_\infty = [4(a^4 + t^2)]^{-n/4}.$$

Moreover, for real valued, measurable  $V$  satisfying both (i) and (ii) of Theorem I with extended real  $\eta$  and  $\gamma$ , there results for such  $u$  both

$$(2.10) \quad \int_{-\infty}^{\infty} \|Ve^{it\tilde{H}}u\| dt < +\infty,$$

$$(2.11) \quad 0 = \lim_{|t| \rightarrow \infty} \|Ve^{it\tilde{H}}u\|,$$

if  $2 \leq \eta$  and  $2 \leq \gamma < n$ .

Since  $2 \leq \gamma < n$  in the last part of the lemma, this only applies when dimension  $n \geq 3$ . From the crucial (2.10) and (2.11) (Corollary 2 and 1 of Cook's Lemma 2), the other arguments of Cook's paper [2] apply without other change and yield all the conclusions of our following Theorems II and III, except for the unstated by Cook orthogonality of each  $H$  eigenvector in  $X$  to  $Y_\pm$ , which is an easy consequence of  $W_\pm \tilde{H} = HW_\pm$  and hence  $\tilde{H} = W_\pm^* HW_\pm$  and the reduction of  $H$  by  $Y_\pm$ .

Thus as soon as both Lemmas (2.4) and (2.6) are shown in the next section, all our Theorems I, II, and III will be proved.

**THEOREM II.** *Let  $n = 3$  and for some real  $b > 0$  let real valued, measurable  $V$  satisfy both (i) and (ii) of Theorem I with  $\eta = 2$  and some real  $\gamma$  satisfying  $2 \leq \gamma < 3$ . Then there exist isometric operators  $W_+$  and  $W_-$  on  $X = L_2(\mathbb{R}_3)$  such that the unitary operator  $W(t)$  in (1.2) has  $\lim_{t \rightarrow +\infty} \|W_+u - W(t)u\| = 0 = \lim_{t \rightarrow -\infty} \|W_-u - W(t)u\|$  for every  $u \in X$ . Moreover,  $W_\pm \tilde{H} = HW_\pm$ ;  $P_\pm = W_\pm W_\pm^*$  are orthogonal projections whose range spaces  $Y_\pm = P_\pm X$  reduce  $H$ ; and every  $u \in \mathcal{D} = \mathcal{D}_H$  satisfying  $Hu = \lambda u$  for some scalar  $\lambda$  is orthogonal to  $Y_\pm$ .*

This is our new version of Cook's theorem, the special case here  $\gamma = 2$  being exactly Cook's statement. Since in most applications the potential  $V$  will be bounded at  $\infty$ , and since

$$L_\infty(D_b^+) \cap L_2(D_b^+) \subset L_\infty(D_b^+) \cap L_\gamma(D_b^+)$$

properly for  $\gamma > 2$  is easily seen, our version is essentially sharper than Cook's. As pointed out in the introduction it "almost" includes the Coulomb potential, which Cook's does not. (Actually, (2.10) fails for  $V(\bar{x}) = C|\bar{x}|^{-1}$ ,  $C \neq 0$ .) We also remark that there would be no gain in allowing  $2 \leq \eta < 3$  in II instead of specifying  $\eta = 2$ , since  $- \|V\|_2 \leq - \|V\|_\eta [\mu_n(D_b^-)]^{1/2-1/\eta}$  follows from the Schwarz-Hölder inequality.

**THEOREM III.** *Let integer  $n \geq 4$  and for some real  $b > 0$  let real valued, measurable  $V$  satisfy both (i) and (ii) of Theorem I with some real  $\eta$  and  $\gamma$  satisfying  $n/2 < \eta$  and  $n/2 < \gamma < n$ . Then the Theorem II conclusions follow.*

As above, the assumptions in III are least restrictive with  $\eta$  as small as possible; and, for  $V \in L_\infty(D_b^+)$  also holding, are then least restrictive with  $\gamma$  as large as possible.

**3. Proof of lemmas.** We start by proving Lemma 2.4, considering first the case  $1 \leq n \leq 3$ . For given  $\alpha' > 0$ , we see by taking  $\omega > 0$  sufficiently small in equation (7) of [1] and by  $\sqrt{a^2 + b^2} \leq |a| + |b|$  that

$$(3.1) \quad \|u\|_\infty \leq \alpha' \|\tilde{H}u\| + \beta'_\omega \|u\|$$

over all  $u \in \mathcal{D}$  for some real  $\beta'_\omega \geq 1$ . Now define real  $r \geq 2$  if  $\gamma > 2$  in Theorem I (the Lemma (2.4) hypotheses) by requiring  $2/\gamma + 2/r = 1$ . Then (3.1) with  $\beta'_\omega \geq 1$  yields for  $u \in \mathcal{D}$

$$(3.2) \quad \|u\|_r \leq [\|u\|_\infty^{r-2} \|u\|^2]^{1/r} = \|u\|^{2/r} (\alpha' \|\tilde{H}u\| + \beta'_\omega \|u\|)^{1-2/r} \\ \leq \alpha' \|\tilde{H}u\| + \beta'_\omega \|u\|.$$

Thus (3.2), (ii) of  $I$ , and the Schwarz-Hölder inequality for the associated powers  $r/2$  and  $\gamma/2$  yield

$$(3.3) \quad +\|Vu\|^2 \leq +\|V\|_{\frac{\gamma}{2}}^2 \|u\|_{\frac{r}{2}}^2 \leq +\|V\|_{\frac{\gamma}{2}}^2 (\alpha' \|\tilde{H}u\| + \beta'_{\alpha'} \|u\|)^2.$$

Also  $-\|V\|_2 \leq [\mu_n(D_b^-)]^{1/2-1/\eta} -\|V\|_{\eta} < +\infty$ , using (i) of  $I$  and the Schwarz-Hölder inequality with  $\eta \geq 2$ , gives from (3.1)

$$(3.4) \quad -\|Vu\|^2 \leq -\|V\|_2^2 \|u\|_{\infty}^2 \leq -\|V\|_2^2 (\alpha' \|\tilde{H}u\| + \beta'_{\alpha'} \|u\|)^2$$

over  $u \in \mathcal{D}$ . (3.3) and (3.4) and  $\|Vu\|^2 = +\|Vu\|^2 + -\|Vu\|^2$  and  $\sqrt{a^2 + b^2} \leq |a| + |b|$  yield (2.5), with  $\alpha = M\alpha'$  freely chosen  $> 0$  by choice of  $\alpha'$ , and  $Vu \in X$  as desired if  $\gamma > 2$ . If  $\gamma = 2$ , then  $-\|V\|_2 < +\infty$  above with (ii) of  $I$  yields  $\|V\|_2 < +\infty$ ; hence (3.1) yields (3.4) with the-script dropped, proving (2.5) and  $Vu \in X$ . Thus Lemma 2.4 has been shown if  $1 \leq n \leq 3$ .

Now consider the remaining case  $n \geq 4$  of Lemma 2.4. Here  $2 \leq n/2 < s \leq +\infty$  for  $s = \eta$  and  $s = \gamma$ , and hence real  $\tau \geq 2$  and  $\mu \geq 2$  are defined by the requirements  $2/\gamma + 2/\tau = 1$  and  $2/\eta + 2/\mu = 1$  respectively. Moreover, using  $(n + \rho)2^{-1} = \gamma$  or  $\eta$  respectively, we see in [1] at the top of p. 956 that  $r' = 4\gamma(2\gamma - 4)^{-1} = 2(1 - 2/\gamma)^{-1} = \tau$  or  $r' = 4\eta(2\eta - 4)^{-1} = 2(1 - 2/\eta)^{-1} = \mu$  respectively, and equation (8) there becomes

$$(3.5) \quad \|u\|_{\tau} \leq \alpha' \|\tilde{H}u\| + \beta'_{\alpha'} \|u\|,$$

$$(3.6) \quad \|u\|_{\mu} \leq \alpha' \|\tilde{H}u\| + \beta''_{\alpha'} \|u\|$$

respectively over  $u \in \mathcal{D}$ , with real  $\beta'_{\alpha'} > 0$  and  $\beta''_{\alpha'} > 0$  existing for each real  $\alpha' > 0$ . From (3.5) and (3.6) respectively, from (ii) and (i) respectively of  $I$ , and from the Schwarz-Hölder inequality we obtain respectively

$$(3.7) \quad +\|Vu\|^2 \leq +\|V\|_{\frac{\tau}{2}}^2 \|u\|_{\frac{\tau}{2}}^2 \leq +\|V\|_{\frac{\tau}{2}}^2 (\alpha' \|\tilde{H}u\| + \beta'_{\alpha'} \|u\|)^2,$$

$$(3.8) \quad -\|Vu\|^2 \leq -\|V\|_{\frac{\mu}{2}}^2 \|u\|_{\frac{\mu}{2}}^2 \leq -\|V\|_{\frac{\mu}{2}}^2 (\alpha' \|\tilde{H}u\| + \beta''_{\alpha'} \|u\|)^2$$

over  $u \in \mathcal{D}$ . Thus (3.7) and (3.8) and  $\|Vu\| \leq \sqrt{+\|Vu\|^2 + -\|Vu\|^2} \leq +\|Vu\| + -\|Vu\|$  yields (2.5), with  $\alpha = M\alpha' > 0$  freely chosen, and  $Vu \in X$  as desired when  $n \geq 4$ , completing the proof of Lemma 2.4.

Finally we must prove Lemma 2.6. Here from the proof of  $I$  (independently of any condition on  $V$ ), we have  $\tilde{H} = UA\tilde{U}$  to be the unique self-adjoint extension of  $\tilde{H}_0$ . Hence  $e^{it\tilde{H}} = Ue^{itA}\tilde{U}$  and for  $u$  of form (2.3) we compute directly, since the  $L_1$  Fourier transform and the  $L_2$  Fourier-Plancherel transform are well known to coincide almost everywhere for functions in  $L_1(\mathbb{R}_n) \cap L_2(\mathbb{R}_n)$ ,

$$\begin{aligned}
 [e^{it\tilde{H}}u](\vec{x}) &= (2\pi)^{-n/2} \int_{R_n} \exp(-a^2|\vec{y} - \vec{z}|^2 + it|\vec{y}|^2 + i(\vec{y} \cdot \vec{x})) d\mu_n(\vec{y}) \\
 &= \prod_{j=1}^n \left\{ (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-a^2(y - z_j)^2 + ity^2 + iyx_j) dy \right\} \\
 (3.9) \quad &= \exp\left(-a^2|\vec{z}|^2 + 4^{-1}(a^2 - it)^{-1} \sum_{j=1}^n (2a^2z_j + ix_j)^2\right) \\
 &\quad \prod_{j=1}^n \left\{ (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-(a^2-it)y^2} dy \right\} \\
 &= [2(a^2 - it)]^{-n/2} \exp\left(-a^2|\vec{z}|^2 + 4^{-1}(a^2 - it)^{-1} \sum_{j=1}^n (2a^2z_j + ix_j)^2\right).
 \end{aligned}$$

From (3.9) we readily obtain (2.7), from which (2.9) is obvious and (2.8) follows by the direct computation

$$\begin{aligned}
 (3.10) \quad \|e^{it\tilde{H}}u\|_r &= [4(a^4 + t^2)]^{-n/4} \left[ \int_{R_n} \exp(-a^2r4^{-1}(a^4 + t^2)^{-1}|\vec{y}|^2) d\mu_n(\vec{y}) \right]^{1/r} \\
 &= [4(a^4 + t^2)]^{-n/4} [a^{-2}r^{-1}4(a^4 + t^2)]^{n/2r} (K_n)^{1/r}
 \end{aligned}$$

with  $K_n = \int_{R_n} e^{-|\vec{y}|^2} d\mu_n(\vec{y})$  positive and finite.

Finally to prove last statement of Lemma 2.6 with conclusions (2.10) and (2.11), we here are given  $V$  to satisfy (i) and (ii) of  $I$  with  $2 \leq \gamma < n$  and  $2 \leq \eta$ . Thus  $- \|V\|_2 \leq - \|V\|_\eta [\mu_n(D_{\bar{v}})]^{1/2-1/\eta} < +\infty$ , as noted just before III, and by (2.9) for our  $u \in G$

$$(3.11) \quad - \|Ve^{it\tilde{H}}u\| \leq - \|V\|_2 [4(a^4 + t^2)]^{-n/4}.$$

Since  $n > 2$  here, the right side of (3.11) is in  $L_1(-\infty, \infty)$  over  $t$ . If  $\gamma = 2$ , then  $+ \|V\|_2 < +\infty$  and (3.11) with the  $-$  script replaced by  $+$  shows  $+ \|Ve^{it\tilde{H}}u\| \in L_1(-\infty, \infty)$  over  $t$ . If  $\gamma > 2$ , then the requirement  $2/\gamma + 2/r = 1$  defines real  $r \geq 2$ , and the Schwarz-Hölder inequality for this  $r$  yields from (2.8) and (ii) of  $I$  for our  $u \in G$

$$(3.12) \quad + \|Ve^{it\tilde{H}}u\| \leq + \|V\|_\gamma M'(\alpha^4 + t^2)^{-(n/2)(1/2-1/r)} = M(\alpha^4 + t^2)^{-n/2\gamma},$$

which is in  $L_1(-\infty, \infty)$  by  $\gamma < n$ . Hence (3.11) and (3.12) and  $\|w\| \leq + \|w\| + - \|w\|$  prove (2.10) and (2.11), and the proof of Lemma 2.6 is complete.

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Jonathan L. Alperin, <i>Groups with finitely many automorphisms</i> .....	1
Martin Arthur Arkowitz, <i>The generalized Whitehead product</i> .....	7
John D. Baum, <i>Instability and asymptoticity in topological dynamics</i> .....	25
William Aaron Beyer, <i>Hausdorff dimension of level sets of some Rademacher series</i> .....	35
Frank Herbert Brownell, III, <i>A note on Cook's wave-matrix theorem</i> .....	47
Gulbank D. Chakerian, <i>An inequality for closed space curves</i> .....	53
Inge Futtrup Christensen, <i>Some further extensions of a theorem of Marcinkiewicz</i> .....	59
Charles Vernon Coffman, <i>Linear differential equations on cones in Banach spaces</i> .....	69
Eckford Cohen, <i>Arithmetical notes. III. Certain equally distributed sets of integers</i> .....	77
John Irving Derr and Angus E. Taylor, <i>Operators of meromorphic type with multiple poles of the resolvent</i> .....	85
Jacob Feldman, <i>On measurability of stochastic processes in products space</i> .....	113
Robert S. Freeman, <i>Closed extensions of the Laplace operator determined by a general class of boundary conditions, for unbounded regions</i> .....	121
Robert E. Fullerton, <i>Geometric structure of absolute basis systems in a linear topological space</i> .....	137
Dieter Gaijer, <i>On conformal mapping of nearly circular regions</i> .....	149
Andrew Mattei Gleason and Hassler Whitney, <i>The extension of linear functionals defined on <math>H^\infty</math></i> .....	163
Seymour Goldberg, <i>Closed linear operators and associated continuous linear operators</i> .....	183
Basil Gordon, Aviezri Siegmund Fraenkel and Ernst Gabor Straus, <i>On the determination of sets by the sets of sums of a certain order</i> .....	187
Branko Grünbaum, <i>The dimension of intersections of convex sets</i> .....	197
Paul Daniel Hill, <i>On the number of pure subgroups</i> .....	203
Robert Peter Holten, <i>Generalized Goursat problem</i> .....	207
Alfred Horn, <i>Eigenvalues of sums of Hermitian matrices</i> .....	225
Henry C. Howard, <i>Oscillation and nonoscillation criteria for <math>y''(x) + f(y(x))p(x) = 0</math></i> .....	243
Taqdir Husain, <i>S-spaces and the open mapping theorem</i> .....	253
Richard Eugene Isaac, <i>Markov processes and unique stationary probability measures</i> .....	273
John Rolfe Isbell, <i>Supercomplete spaces</i> .....	287
John Rolfe Isbell, <i>On finite-dimensional uniform spaces. II</i> .....	291
N. Jacobson, <i>A note on automorphisms of Lie algebras</i> .....	303
Antoni A. Kosinski, <i>A theorem on families of acyclic sets and its applications</i> .....	317
Marvin David Marcus and H. Minc, <i>The invariance of symmetric functions of singular values</i> .....	327
Ralph David McWilliams, <i>A note on weak sequential convergence</i> .....	333
John W. Milnor, <i>On axiomatic homology theory</i> .....	337
Victor Julius Mizel and Malempati Madhusudana Rao, <i>Nonsymmetric projections in Hilbert space</i> .....	343
Calvin Cooper Moore, <i>On the Frobenius reciprocity theorem for locally compact groups</i> .....	359
Donald J. Newman, <i>The Gibbs phenomenon for Hausdorff means</i> .....	367
Jack Segal, <i>Convergence of inverse systems</i> .....	371
Józef Siciak, <i>On function families with boundary</i> .....	375
Hyman Joseph Zimmerberg, <i>Two-point boundary conditions linear in a parameter</i> .....	385