AN INEQUALITY FOR CLOSED SPACE CURVES

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1. Among a number of interesting results in a paper of I. Fáry (see [2]) appears the following. Let $C$ be a rectifiable closed curve of length $L(C)$ and total curvature $\kappa(C)$ enclosed by a sphere $S$ of radius $r$ in Euclidean 3-space. Then

\begin{equation}
L(C) \leq \frac{4}{\pi} r \kappa(C) .
\end{equation}

The proof of (1) rests upon the corresponding inequality for plane closed curves, which states that if $C$ is enclosed by a circle of radius $r$, then

\begin{equation}
L(C) \leq r \kappa(C) .
\end{equation}

The latter inequality gives a sharp result, with equality obtained in case $C$ is a circle of radius $r$.

In this paper we sharpen (1) to the following result. Let $C$ be a rectifiable closed curve enclosed by a $k - 1$ dimensional sphere $S$ of radius $r$ in Euclidean $k$-space, $k \geq 2$. Then

\begin{equation}
L(C) \leq r \kappa(C) .
\end{equation}

The proof of (3) again depends on the plane case and is motivated by the following construction. We form the cone $T$ over the curve $C$ with apex at the center of $S$, slit along a longest generator and develop the result in a plane. The resulting plane arc $C'$ is completed to a closed plane curve $C''$ by attaching an arc of a circle. It is noted that the curvature of $C'$ is equal pointwise to the geodesic curvature of $C$ with respect to $T$, which in turn is not greater, pointwise, than the curvature of $C$. The length of $C'$ is the same as that of $C$. The inequality (2) applied to $C''$ now gives (3).

2. In this section we prove some lemmas which lead directly to the main theorem.

**Lemma 1.** Let $C$ be a rectifiable plane arc of length $L$. For any line $G$, let $n(p, \theta)$ be the number of intersections of $G$ with $C$, where $(p, \theta)$, $p \geq 0$, $0 \leq \theta < 2\pi$, are the normal coordinates of $G$. Then

\begin{equation}
L = \frac{1}{2} \int_{0}^{2\pi} \int_{0}^{\infty} n(p, \theta) \, dp \, d\theta .
\end{equation}

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This striking formula of Crofton is proved by Blaschke, [1], page 46.

**Lemma 2.** Let \( C \) be a closed plane curve parametrized by arc length \( s \). Let \( \mathbf{r} = \mathbf{r}(s), \, 0 \leq s \leq L, \) be the tracing vector of, \( C \), and assume \( \mathbf{r}'' \) exists and is continuous except at a finite number of points \( \mathbf{r}(s_0), \ldots, \mathbf{r}(s_m) \), where there are corners with “exterior” angles \( \alpha_1, \ldots, \alpha_m \) respectively. Given any direction \( \theta, \, 0 \leq \theta < 2\pi \), let \( n(\theta) \) be the number of tangents to \( C \) orthogonal to that direction, where a tangent to \( C \) at \( \mathbf{r}(s_i), \, i = 1, 2, \ldots, m, \) means a line through the point but not crossing \( C \) at that point. Then

\[
\frac{1}{2} \int_0^{2\pi} n(\theta) \, d\theta = \int |\mathbf{r}''(s)| \, ds + \sum_{i=1}^m \alpha_i = \text{total curvature of } C,
\]

where the integral on the right is extended over the smooth part of \( C \).

**Proof.** We may write \( n(\theta) = \sum_{i=0}^m n_i(\theta) \), where \( n_0(\theta) \) counts the number of tangents to the smooth part of \( C \) and \( n_i(\theta), \, i \neq 0, \) counts the number of tangents at \( \mathbf{r}(s_i) \). Clearly \( n_i \) takes only the values 0 or 1, for \( i \neq 0 \), and

\[
\frac{1}{2} \int_0^{2\pi} n_i(\theta) \, d\theta = \alpha_i, \; i \neq 0.
\]

Finally, we have that

\[
\frac{1}{2} \int_0^{2\pi} n_0(\theta) \, d\theta = \int |\mathbf{r}''(s)| \, ds,
\]

since the left hand side is just the measure of the spherical image (counting multiplicity) of the smooth part of \( C \).

**Lemma 3.** Let \( \overline{x}_0, \overline{x}_1, \ldots, \overline{x}_n \), be the successive vertices of a plane polygon \( \overline{P} \) enclosed by a circle \( S \) of radius \( r_0 \). Suppose further that the “initial” and “end” points, \( \overline{x}_0 \) and \( \overline{x}_n \) respectively, lie on \( S \). Let \( \alpha_i, \, 0 \leq \alpha_i \leq \pi \), be the angle between \( \overline{x}_{i+1} - \overline{x}_i \) and \( \overline{x}_i - \overline{x}_{i-1}, \, i = 1, \ldots, n-1 \). If \( \overline{x}_0 \neq \overline{x}_n \), let \( \alpha_n, \, 0 \leq \alpha_n \leq \pi \), be the angle between \( \overline{x}_1 - \overline{x}_0 \) and the unit tangent vector to \( S \) (with counterclockwise orientation) at \( \overline{x}_0 \), and let \( \alpha_n, \, 0 \leq \alpha_n \leq \pi \), be the angle between \( \overline{x}_n - \overline{x}_{n-1} \) and the unit tangent vector to \( S \) (with counterclockwise orientation) at \( \overline{x}_n \). If \( \overline{x}_0 = \overline{x}_n \), then simply let \( \alpha_n(= \alpha_i), \, 0 \leq \alpha_n \leq \pi \), be the angle between \( \overline{x}_1 - \overline{x}_0 \) and \( \overline{x}_0 - \overline{x}_{n-1} \). Let \( L(\overline{P}) \) be the length of \( \overline{P} \).

Then if \( \overline{x}_0 \neq \overline{x}_n \), we have that

\[
L(\overline{P}) \leq r_0 \sum_{i=0}^n \alpha_i.
\]
If $\bar{x}_0 = \bar{x}_n$, we have

$$(8') \quad L(\bar{P}) \leq r_0 \sum_{i=0}^{n-1} \alpha_i.$$  

(This lemma is a special case of Fáry’s theorem for the plane. See [2], page 121. The proof we give here is essentially that of Fáry.)

**Proof.** We consider first the case where $\bar{x}_0 \neq \bar{x}_n$. Let $\bar{S}$ be the arc of $S$ traversed in a counterclockwise direction in going along $S$ from $\bar{x}_n$ to $\bar{x}_0$. Let $C = \bar{P} \cup \bar{S}$. Let $\delta$ be the angle subtended at the center of $S$ by $\bar{S}$. Then Lemma 1 gives,

$$L(\bar{P}) + r_0 \delta = L(C) = \frac{1}{2} \int_0^{2\pi} \int_{r_0} n(p, \theta) \, dp \, d\theta.$$  

It is easy to see, however, that $n(p, \theta) \leq n(\theta)$ for $0 \leq \theta < 2\pi$. Hence, by (9) and (5), we have

$$L(\bar{P}) + r_0 \delta \leq \frac{1}{2} r_0 \int_0^{2\pi} n(\theta) \, d\theta = r_0 \left( \sum_{i=0}^{n} \alpha_i + \delta \right).$$  

This gives the assertion for $\bar{x}_0 \neq \bar{x}_n$. The case $\bar{x}_0 = \bar{x}_n$ is now clear.

**Lemma 4.** Let $P$ be a closed polygon enclosed by a $k - 1$ dimensional sphere $S$ of radius $r$ in Euclidean $k$-space. Let $\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_n = \bar{y}_0$, be the successive vertices of $P$. Let $\beta_i, 0 \leq \beta_i \leq \pi$, be the angle between $\bar{y}_{i+1} - \bar{y}_i$ and $\bar{y}_i - \bar{y}_{i-1}, \ i = 0, 1, \ldots, n - 1$, where $\bar{y}_{-1}$ is defined to be $\bar{y}_n$. Define the total curvature, $\kappa(P)$, of $P$, by

$$(11) \quad \kappa(P) = \sum_{i=0}^{n-1} \beta_i, \quad \text{(See Milnor, [3], p. 249.)}$$

Let $L(P)$ be the length of $P$. Then

$$(12) \quad L(P) \leq r \kappa(P).$$

**Proof.** Let $\bar{o}$ be the center of $S$. Assume that the vertices of $P$ are labeled so that $\bar{y}_0$ is no closer to $\bar{o}$ than any other vertex. Let $\beta_i', 0 \leq \beta_i' \leq \pi$, be the angle between $\bar{y}_i - \bar{o}$ and $\bar{y}_i - \bar{y}_{i+1}$; let $\beta_i'', 0 \leq \beta_i'' \leq \pi$, be the angle between $\bar{y}_i - \bar{o}$ and $\bar{y}_i - \bar{y}_{i-1}, \ i = 0, 1, \ldots, n - 1$. The triangle inequality applied to a spherical triangle cut out of a sphere centered at $\bar{y}_i$ shows that

$$\beta_i' + \beta_i'' \geq \pi - \beta_i, \text{ and } (\pi - \beta_i') + (\pi - \beta_i'') \geq \pi - \beta_i.$$

Hence,

$$|\pi - (\beta_i' + \beta_i'')| \leq \beta_i, \quad \text{i = 0, 1, \ldots, n - 1.}$$
We now form the cone over $P$ with apex at $o$, cut along the edge connecting $\tilde{o}$ to $\tilde{y}_0$ and develop the result in a plane as follows. Let $\tilde{p}$ be a fixed point in the plane $R^2$. We map $\tilde{y}_0$ into any point $\tilde{x}_0 \in R^2$ satisfying $|\tilde{x}_0 - \tilde{p}| = |\tilde{y}_0 - \tilde{o}| = r_0$. We next map $\tilde{y}_1$ into a point $\tilde{x}_1 \in R^2$ satisfying $|\tilde{x}_1 - \tilde{p}| = |\tilde{y}_1 - \tilde{o}| = r_1$, and such that the angle $\delta_i$, from $\tilde{x}_0 - \tilde{p}$ to $\tilde{x}_i - \tilde{p}$, measured in a counterclockwise direction, is equal to the angle $\delta_i$, $0 \leq \delta_i \leq \pi$, between $\tilde{y}_0 - \tilde{o}$ and $\tilde{y}_1 - \tilde{o}$. In general we map $\tilde{y}_i$ into $\tilde{x}_i \in R^2$ so that $|\tilde{x}_i - \tilde{p}| = |\tilde{y}_i - \tilde{o}| = r_i$ and the angle $\delta_i$ from $\tilde{x}_{i-1} - \tilde{p}$ to $\tilde{x}_i - \tilde{p}$, measured counterclockwise, is equal to the angle $\delta_i$, $0 \leq \delta_i \leq \pi$, between $\tilde{y}_{i-1} - \tilde{o}$ and $\tilde{y}_i - \tilde{o}$. This construction gives us a polygon $\tilde{P}$ in $R^2$. Construct the circle $S'$ of radius $r_0$ centered at $\tilde{p}$. Then $\tilde{P}$ is enclosed by $S'$, and $\tilde{x}_0$ and $\tilde{x}_n$ (in general $\tilde{x}_0 \neq \tilde{x}_n$) are on $S'$. It is easily seen that the angles $\alpha_i$ and $\alpha_n$ described in Lemma 3 are equal to $(\pi/2) - \beta_i > 0$ and $(\pi/2) - \beta_n > 0$ respectively if $\tilde{x}_0 \neq \tilde{x}_n$ and are both equal to $\pi - (\beta_i + \beta_n) > 0$ if $\tilde{x}_0 = \tilde{x}_n$. Hence if $\tilde{x}_0 \neq \tilde{x}_n$,

\[
\sum_{i=0}^{n} \alpha_i = \frac{\pi}{2} - \beta_i' + \sum_{i=1}^{n-1} |\pi - (\beta_i' + \beta_n')| + \frac{\pi}{2} - \beta_n'
\]

and if $\tilde{x}_0 = \tilde{x}_n$,

\[
\sum_{i=0}^{n} \alpha_i = \sum_{i=0}^{n-1} |\pi - (\beta_i' + \beta_n')|.
\]

Therefore, by (8), (8'), (14), and (14'),

\[
L(P) = L(\tilde{P}) \leq r_0 \sum_{i=0}^{n} |\pi - (\beta_i' + \beta_n')| \leq r_0 \sum_{i=0}^{n-1} \beta_i = r_0 \kappa(P) \leq r \kappa(P).
\]

3. Theorem 1. Let $C$ be a rectifiable closed curve enclosed by a $k - 1$ dimensional sphere $S$ of radius $r$ in Euclidean $k$-space, $k \geq 2$. Let $L(C)$ be the length of $C$ and $\kappa(C)$ be the total curvature of $C$. $(\kappa(C) = 1.\bar{u}b. \kappa(P))$, where $P$ runs over all polygons inscribed in $C$. See Milnor, [3].) Then

\[
L(C) \leq r \kappa(C).
\]

Proof. Given any $\varepsilon > 0$, there is a polygon $P$ inscribed in $C$ such that $L(C) - L(P) \leq \varepsilon$. We have that $\kappa(P) \leq \kappa(C)$. Hence

\[
L(C) - \varepsilon \leq L(P) \leq r \kappa(P) \leq r \kappa(C).
\]

The theorem follows.
COROLLARY. Let $C$ be a closed curve of class $C''$ enclosed by a unit $k - 1$ dimensional sphere in Euclidean $k$-space. Let $\kappa(s) = |\vec{r}''(s)|$ = curvature of $C$ at $\vec{r}(s)$, $0 \leq s \leq L(C)$. Then

\begin{equation}
\max \kappa \geq 1.
\end{equation}

Proof.

\[ L(C) \leq \kappa(C) = \int_0^{L(C)} \kappa(s) \, ds \leq \max \kappa \cdot L(C). \]

Note that we have used the fact that the above integral form for the total curvature coincides with the previous definition. This is proved by Milnor in [3].

REFERENCES
