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**ARITHMETICAL NOTES. III. CERTAIN EQUALLY
DISTRIBUTED SETS OF INTEGERS**

ECKFORD COHEN

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1. Introduction. In this note we shall generalize the following two results in the classical theory of numbers. Let n denote a positive integer with distinct prime divisors p_1, \dots, p_m ,

$$(1.1) \quad n = p_1^{e_1} \cdots p_m^{e_m} \quad (m > 0), \quad n = 1 \quad (m = 0),$$

and place $\Omega(n) = e_1 + \cdots + e_m$, $\Omega(1) = 0$, so that $\Omega(n)$ is the total number of prime divisors of n . For real $x \geq 1$, let $S'(x)$ denote the number of square-free numbers $n \leq x$ such that $\Omega(n)$ is even, and let $S''(x)$ denote the number of square-free $n \leq x$ such that $\Omega(n)$ is odd. It is well-known [6, §161] that

$$(1.2) \quad S'(x) \sim \frac{3x}{\pi^2}, \quad S''(x) \sim \frac{3x}{\pi^2} \quad \text{as } x \rightarrow \infty.$$

Correspondingly, let $T'(x)$ denote the total number of integers $n \leq x$ such that $\Omega(n)$ is even and $T''(x)$ the total number of $n \leq x$ with $\Omega(n)$ odd. Then [6, §167]

$$(1.3) \quad T'(x) \sim \frac{x}{2}, \quad T''(x) \sim \frac{x}{2} \quad \text{as } x \rightarrow \infty.$$

The proof of (1.2) is based upon the deep estimate [6, §155] for the Möbius function $\mu(n)$,

$$(1.4) \quad M(x) \equiv \sum_{n \leq x} \mu(n) = o(x),$$

while the proof of (1.3) is based upon the analogous estimate [6, §167] for Liouville's function $\lambda(n)$,

$$(1.5) \quad L(x) \equiv \sum_{n \leq x} \lambda(n) = o(x).$$

In Theorem 3.3 we prove a generalization of (1.2) and in Theorem 3.4 the corresponding generalization of (1.3). The respective proofs are based upon an estimate (Theorem 3.1) corresponding to (1.4) for an appropriate extension of $\mu(n)$ and an estimate (Theorem 3.2) corresponding to (1.5) for the analogous extension of $\lambda(n)$. The proofs of these estimates are in the manner of Delange's proofs [3, I(b), (c)] of (1.4) and (1.5), both being based upon a classical Tauberian theorem (Lemma 3.2) for the Lambert summability process. We also require some elementary

estimates contained in §2, and a lemma on inversion functions (Lemma 2.1).

2. Preliminary results. For an arbitrary set A of positive integers n , the *characteristic function* $a(n)$ and *inversion function* $b(n)$ of A are defined by

$$\sum_{d|n} b(d) = a(n) \equiv \begin{cases} 1 & (n \in A) \\ 0 & (n \notin A) . \end{cases}$$

The *enumerative function* $A(x)$ of A is the number of $n \leq x$ contained in A , and the *generating function* is the function $f(s) = \sum_{n=1}^{\infty} a(n)/n^s$, $s > 1$.

We shall be concerned with several special sets of integers. Let Z denote the set of positive integers, $k \in Z$. Then P_k will represent the set of k th powers of Z , and Q_k the set of k -free integers of Z . The set of k -full integers, that is, the integers (1.1) with each $e_i \geq k$, will be denoted R_k . We shall use S_k to denote the integers (1.1) in which each e_i has the value 1 or k . Finally, the set of integers (1.1) such that $e_i \equiv 0$ or $1 \pmod{k}$, $i = 1, \dots, m$, will be denoted T_k . The characteristic functions P_k, Q_k, R_k, S_k , and T_k will be denoted respectively $p_k(n), q_k(n), r_k(n), s_k(n)$, and $t_k(n)$; the corresponding enumerative functions will be denoted $P_k(x), Q_k(x), R_k(x), S_k(x), T_k(x)$. Also let $Q = Q_2$, $Q(x) = Q_2(x)$, and $q(n) = q_2(n)$. All of the sets defined are understood to include the integer 1.

REMARK 2.1. It will be observed that $T_1 = Z$, $S_1 = Q_2$, $S_2 = Q_3$.

In addition to the above notation, we shall use $\lambda_k(n)$ to denote the inversion function of P_k and $\mu_k(n)$ the inversion function of R_k or Q_k according as $k > 1$ or $k = 1$. By familiar properties of $\mu(n)$ and $\lambda(n)$, [4, Theorem 263 and 300], it follows that

$$(2.1) \quad \mu_1(n) = \mu(n) , \quad \lambda_2(n) = \lambda(n) .$$

LEMMA 2.1. *The functions $\mu_k(n)$, $\lambda_k(n)$ are multiplicative. If p is a prime and e a positive integer, then for $k \geq 1$,*

$$(2.2) \quad \mu_k(p^e) = \begin{cases} 1 & \text{if } e = k \neq 1 , \\ -1 & \text{if } e = 1 , \\ 0 & \text{otherwise} , \end{cases}$$

while for $k > 1$,

$$(2.3) \quad \lambda_k(p^e) = \begin{cases} 1 & \text{if } e \equiv 0 \pmod{k} , \\ -1 & \text{if } e \equiv 1 \pmod{k} , \\ 0 & \text{otherwise} . \end{cases}$$

REMARK 2.2. The multiplicativity property in connection with (2.2) and (2.3) completely determine $\mu_k(n)$, $k \geq 1$, and $\lambda_k(n)$ for $k \geq 2$.

Proof. By definition, if $k > 1$,

$$(2.4) \quad \sum_{d|n} \mu_k(d) = r_k(n) = \begin{cases} 1 & \text{if } n \in R_k \\ 0 & \text{if } n \notin R_k. \end{cases}$$

Hence, application of the Möbius inversion formula yields

$$(2.5) \quad \mu_k(n) \sum_{d|n} \mu(d) r_k\left(\frac{n}{d}\right), \quad k > 1.$$

Since $\mu(n)$ and $r_k(n)$ are multiplicative, it follows by (2.5) that $\mu_k(n)$ is also multiplicative (cf. [4, Theorem 265]). Also by (2.5), $\mu_k(p^e) = r_k(p^e) - r_k(p^{e-1})$, from which (2.2) results in case $k > 1$. The case $k = 1$ of (2.2) is a consequence of (2.1). The proof of (2.3) is similar and can be omitted.

We recall next some known elementary estimates for $P_k(x)$, $Q_k(x)$, and $R_k(x)$. Let $\zeta(s)$, $s > 1$, denote the Riemann ζ -function.

LEMMA 2.2. *If $k > 1$, then*

$$(2.6) \quad P_k(x) = \sqrt[k]{x} + O(1),$$

$$(2.7) \quad Q_k(x) = \frac{x}{\zeta(k)} + O(\sqrt[k]{x}),$$

$$(2.8) \quad R_k(x) = c_k \sqrt[k]{x} + O\left(\frac{1}{x^{k+1}}\right),$$

where c_k is a certain nonzero constant depending upon k .

The result (2.6) is trivial, (2.7) is the classical estimate of Gegenbauer (cf. [2, §2]), and (2.8) is a well-known result of Erdős and Szekeres (cf. [1]). In particular, we have

LEMMA 2.3. *If $k > 1$, then*

$$(2.9) \quad P_k(x) \sim \sqrt[k]{x}, \quad R_k(x) \sim c_k \sqrt[k]{x} \quad \text{as } x \rightarrow \infty,$$

$$(2.10) \quad Q_k(x) \sim \frac{x}{\zeta(k)}, \quad \left(Q(x) \sim \frac{6x}{\pi^2}\right) \quad \text{as } x \rightarrow \infty.$$

We now deduce, for application in §3, estimates for $S_k(x)$ and $T_k(x)$ corresponding to those in Lemma 2.3 for $P_k(x)$, $Q_k(x)$, and $R_k(x)$.

LEMMA 2.4. *If $k > 1$, then*

$$(2.11) \quad T_k(x) \sim \frac{6\zeta(k)x}{\pi^2} \quad \text{as } x \rightarrow \infty ;$$

if $k \geq 1$, then

$$(2.12) \quad S_k(x) \sim \frac{6\alpha_k x}{\pi^2} \quad \text{as } x \rightarrow \infty ,$$

where

$$(2.13) \quad \alpha_k = \begin{cases} \zeta(k) \prod_p \left(1 - \frac{1}{p^{k-1}} + \frac{1}{p^{k-2}} - \dots - \frac{1}{p^{2k-1}}\right), \\ \zeta(2k) \prod_p \left(1 + \frac{2}{p^k} - \frac{1}{p^{k+1}} + \frac{1}{p^{k+2}} - \frac{1}{p^{k+3}} + \dots + \frac{1}{p^{2k-1}}\right), \\ 1, \end{cases}$$

according as k is even, k is odd and $\neq 1$, or $k = 1$, the products ranging over the primes p .

REMARK 2.3. It will be noted that $\alpha_2 = \zeta(2)/\zeta(3) = \pi^2/6\zeta(3)$.

Proof. The elementary estimate (2.11) was proved in [1, Corollary 2.1]. The result in (2.12), in the cases $k = 1$ and $k = 2$, is a consequence of (2.10) and Remarks 2.1 and 2.3. To complete the proof of (2.12) one may therefore suppose that $k > 2$.

Under this restriction, we consider the generating function $f_k(s)$ of $s_k(n)$. In particular, if $s > 1$, we have (cf. [4, §17.4])

$$(2.14) \quad \begin{aligned} f_k(s) &\equiv \sum_{n=1}^{\infty} \frac{s_k(n)}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{ks}}\right) \\ &= \prod_p \left(1 + \frac{1}{p^s}\right) \left[1 + \frac{1}{p^{ks}} \left(1 + \frac{1}{p^s}\right)^{-1}\right]. \end{aligned}$$

Since

$$\sum_p \frac{1}{p^{ks}} \left(1 + \frac{1}{p^s}\right)^{-1} \leq \sum_p \frac{1}{p^{ks}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{ks}} = \zeta(ks), \quad ks > 1,$$

it follows from (2.14) that

$$(2.15) \quad f_k(s) = \left(\frac{\zeta(s)}{\zeta(2s)}\right) g_k(s), \quad s > 1,$$

where

$$(2.16) \quad g_k(s) \equiv \sum_{n=1}^{\infty} \frac{a_k(n)}{n^s} = \prod_p \left(1 + \frac{1}{p^{sk}} - \frac{1}{p^{s(k+1)}} + \dots\right), \quad s > \frac{1}{k},$$

the product, and hence the series, in (2.16) being absolutely convergent

for $s > 1/k$. By Dirichlet multiplication [4. §17.1] one deduces from (2.15) and (2.16) that

$$s_k(n) = \sum_{d\delta=n} q(d)a_k(\delta),$$

because $\zeta(s)/\zeta(2s)$ is the generating function of $q(n)$, [cf. [4, Theorem 302]]. Applying (2.7) in the case $Q(x) \equiv Q_2(x)$, it follows that

$$S_k(x) \equiv \sum_{m \leq x} s_k(n) = \sum_{d\delta \leq x} q(d)a_k(\delta) = \sum_{n \leq x} a_k(n)Q\left(\frac{x}{n}\right),$$

and hence that

$$S_k(x) = \frac{6x}{\pi^2} \sum_{n \leq x} \frac{a_k(n)}{n} + O\left(x^{1/2} \sum_{n > x} \frac{|a_k(n)|}{n^{1/2}}\right).$$

Recalling that the series in (2.16) converges absolutely for $s > 1/k$, one obtains, since $k > 2$,

$$s_k(x) = \frac{6x}{\pi^2} \sum_{n=1}^{\infty} \frac{a_k(n)}{n} + o\left(x \sum_{n > x} \frac{a_k(n)}{n}\right) + o(x^{1/2}),$$

so that

$$(2.17) \quad S_k(x) = \frac{6\beta_k x}{\pi^2} + o(x), \quad \beta_k = g_k(1).$$

It is readily verified, using (2.16) with $s = 1$, that $\beta_k = \alpha_k$, which completes the proof of (2.12).

3. The principal results. We introduce some further definitions and notation. A divisor d of n will be called *unitary* if $d\delta = n$, $(d, \delta) = 1$. The function $\Omega'(n)$ is defined by $\Omega'(n) = \Omega(g)$ where g is the maximal, unitary, square-free divisor of n . Let S'_k and S''_k , denote, respectively, the subsets of S_k for which $\Omega'(n)$ is even or odd, $n \in S_k$. Analogously, let T'_k and T''_k denote the respective subsets of T_k for which $\Omega(n)$ is even or odd, $n \in T_k$, k even. In addition, we shall use $S'_k(x), S''_k(x), T'_k(x), T''_k(x)$ to denote the enumerative functions of S'_k, S''_k, T'_k, T''_k , respectively.

REMARK 3.1. It will be observed that $S'_1(x) = S'(x)$, $S''_1(x) = S''(x)$, $T'_2(x) = T'(x)$, $T''_2(x) = T''(x)$. In addition, we have, by Lemma 2.1, $\mu_k(n) = (-1)^{\Omega'(n)} s_k(n)$, and in case n is even, $\lambda_k(n) = (-1)^{\Omega(n)} t_k(n)$.

In addition to the lemmas of §2 we shall need the following three known theorems.

LEMMA 3.1 (cf. [5, 259, p. 449]). *For bounded coefficients a_n , the series,*

$$\sum_{n=1}^{\infty} a_n \left(\frac{x^n}{1-x^n} \right)$$

is convergent, provided $|x| < 1$.

LEMMA 3.2 ([3, p, 38]). *If the series*

$$\sum_{n=1}^{\infty} na_n \left(\frac{x^n}{1-x^n} \right) = S,$$

converges for $0 \leq x < 1$, and

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=1}^{\infty} na_n \left(\frac{x^n}{1-x^n} \right) = S,$$

then the series $\sum_{n=1}^{\infty} a_n$ converges with sum S provided $a_n = O(1/n)$.

LEMMA 3.3 ([7, p. 225]). *Suppose that the series $\sum_{n=1}^{\infty} a_n x^n$ converges for $0 \leq x < 1$ and diverges for $x = 1$. If further, $s_n \equiv a_1 + \dots + a_n > 0$ for all n , and $s_n \sim Cn$ (C constant) as $n \rightarrow \infty$, then*

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=1}^{\infty} a_n x^n = C.$$

THEOREM 3.1. *If $k \geq 1$, then*

$$(3.1) \quad M_k(x) \equiv \sum_{n \leq x} \mu_k(n) = o(x).$$

Proof. By Lemmas 2.1 and 3.1, and the definition of $\mu_k(n)$,

$$\begin{aligned} \sum_{n=1}^{\infty} \mu_k(n) \left(\frac{x^n}{1-x^n} \right) &= \sum_{n=1}^{\infty} \mu_k(n) \sum_{m=1}^{\infty} x^{nm} = \sum_{h=1}^{\infty} \left(\sum_{d|h} \mu_k(d) \right) x^h \\ &= \begin{cases} \sum_{h=1}^{\infty} r_k(h) x^h = \sum_{n \in R_k} x^n & \text{if } k > 1, \\ x & \text{if } k = 1. \end{cases} \end{aligned}$$

By (2.9), the set R_k has density 0; hence Lemma 3.3 with $C = 0$ can be applied to the power series so that

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=1}^{\infty} \mu_k(n) \left(\frac{x^n}{1-x^n} \right) = 0, \quad k \geq 1.$$

Since $|\mu_k(n)| \leq 1$, Lemma 3.2 is applicable with $a_n = \mu_k(n)/n$, and one concludes that

$$(3.2) \quad \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n} = o.$$

Put $A_k(x) \equiv \sum_{n \leq x} (\mu_k(n)/n)$; then by partial summation,

$$(3.3) \quad M_k(x) = - \sum_{n \leq x} A_k(n) + A_k(x)([x] + 1).$$

Since $A_k(x) = o(1)$ by (3.2), the theorem results from (3.3).

THEOREM 3.2. *If $k \geq 2$, then*

$$(3.4) \quad L_k(x) \equiv \sum_{n \leq x} \mu_k(n) = o(x).$$

The proof is similar to that of Theorem 3.1 and is therefore omitted. Note that (3.1) reduces to (1.4) in case $k = 1$ and that (3.4) to (1.5) in case $k = 2$.

THEOREM 3.3. *If $k \geq 1$, then*

$$(3.5) \quad S'_k(x) \sim \frac{3\alpha_k x}{\pi^2}, \quad S''_k(x) \sim \frac{3\alpha_k x}{\pi^2} \quad \text{as } x \rightarrow \infty,$$

α_k being defined by (2.13).

Proof. By (2.12), Remark 3.1, and (3.1), one obtains

$$S'_k(x) + S''_k(x) = S_k(x) = \frac{6\alpha_k x}{\pi^2} + o(x),$$

$$S'_k(x) - S''_k(x) = M_k(x) = o(x),$$

and (3.5) results immediately.

Similarly, one may deduce from (2.11), Remark 3.1 and (3.4),

THEOREM 3.4. *If $k > 1$, k even, then*

$$(3.6) \quad T'_k(x) \sim \frac{3\zeta(k)x}{\pi^2}, \quad T''_k(x) \sim \frac{3\zeta(k)x}{\pi^2} \quad \text{as } x \rightarrow \infty.$$

Finally, it will be observed that (3.5) becomes (1.2) in case $k = 1$; while (3.6) becomes (1.3) when $k = 2$.

It is possible to extend (3.6) so as to hold for all $k > 1$. Let g^* denote the largest unitary divisor of $n \in T_k$, such that all prime factors of g^* have multiplicity $e \equiv 1 \pmod{k}$. Place $\Omega^*(n) = \omega(g^*)$, where $\omega(n)$ is the number of distinct prime divisors of n , and let $T_k^*(x)$ and $T_k^{**}(x)$ denote the number of $n \leq x$ contained in T_k according as $\Omega^*(n)$ is even or odd, respectively. Then

THEOREM 3.4'. *If $k > 1$,*

$$(3.7) \quad T_k^*(x) \sim \frac{3\zeta(k)x}{\pi^2}, \quad T_k^{**}(x) \sim \frac{3\zeta(k)x}{\pi^2} \quad \text{as } x \rightarrow \infty.$$

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