ON MEASURABILITY OF STOCHASTIC PROCESSES IN PRODUCTS SPACE

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1. Introduction. Let $\mathcal{E}$ be a $\sigma$-algebra of subsets of $X$, and $T$ a set. Let $\Omega = X^T$, and let $\mathcal{C}$ be the $\sigma$-algebra of subsets of $\Omega$ generated by the finite cylinder sets, i.e., sets of the form $A = \{\omega \in \Omega : \omega(t_i) \in A_i, \ldots, \omega(t_n) \in A_n\}$, $A_i, \ldots, A_n \in \mathcal{E}$. Let $P_0$ be a probability measure on $\mathcal{C}$. Thus the coordinate variables $x_t(\omega) = \omega(t)$, $t \in T$, are the Kolmogorov version [5] of the stochastic process with joint distributions $F_1, \ldots, F_n(A_1, \ldots, A_n) = P_0(A)$. For various purposes, it is appropriate to enlarge this $\sigma$-algebra and extend the measure. In the present paper two methods of doing this will be mentioned, and one of the methods will be studied.

[A] Suppose $X$ is a compact Hausdorff space and $\mathcal{K}$ the Borel sets. Then $\Omega$ is a compact Hausdorff space in the product topology. A straightforward application of the Stone-Weierstrass theorem and the Riesz-Markov theorem shows that there is a unique regular measure on the Borel subsets $\mathcal{B}$ of $\Omega$ which agrees with $P_0$ on $\mathcal{C}$, provided the finite-dimensional marginal measures are all regular. We call this measure $P$. This idea is due to S. Kakutani [3], and was discussed in detail by E. Nelson [8].

[B] By a condition is meant a set-valued function $k$ from $T$ to $\mathcal{E}$. For any condition $k$, we define

$$\Gamma(k) = \{\omega : \omega(t) \in k(t) \text{ for all } t \in T\},$$

and

$$\Gamma(S, k) = \{\omega : \omega(t) \in k(t) \text{ for all } t \in S\},$$

$S$ being a subset of $T$. It is possible to extend $P_0$ to a class of sets of the form $\Gamma(k)$, as follows.

The following lemma is a straightforward generalization of the separability lemma in [1], p. 56.

**Lemma 1.1.** For any condition $k$ there exists a countable set $S \subseteq T$ such that $P_0(\Gamma(S, k) - \Gamma(\{t\}, k)) = 0$ for all $t \in T$.

The proof is a simple exhaustion argument. Such a countable subset $S$ will be called determining for $k$.

Let $\mathcal{K}$ be a family of sets with the properties

(i) $X \in \mathcal{K}$

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(ii) any countable subfamily of $\mathcal{H}$ with the finite intersection property (F.I.P.) has nonnull intersection. Such a family will be called countably compact. If (ii) holds without the countability restriction, then $\mathcal{H}$ is called compact. If a condition $k$ has values in $\mathcal{H}$, it will be called a $\mathcal{H}$-condition.

The set of positive integers will be written $I$. Unions and intersections whose index set is $I$ will be written simply $\bigcup_j$, etc. rather than $\bigcup_{j\in I}$, etc. The following result can then be proven. It is stated in [7].

**Lemma 1.2.** Let $S_n$ be a determining set for the $\mathcal{H}$-condition $k_n$, $n \in I$. Let $\delta = \bigcup_n [\Gamma(S_n, k_n) - \Gamma(k_n)]$. Then $\delta$ has inner $P_0$-measure 0.

$\mathcal{C}_\mathcal{H}$ is now defined to be those subsets $\Gamma$ of $\Omega$ such that $\exists \Gamma''$ in $\mathcal{C}$ with $(\Gamma - \Gamma'') \cup (\Gamma'' - \Gamma)$ subset of a set of the form of $\delta$ in the above lemma. These sets $\Gamma$ form a $\sigma$-algebra, and the assignment to $\Gamma$ of the same measure as the $P_\sigma$-measure of $\Gamma''$ determines unambiguously a measure $P_\mathcal{H}$ on $\mathcal{C}_\mathcal{H}$, which is an extension of $P_\sigma$. This construction, based on ideas of Doob and Khintchine [4] is done by A. Mayer in [6], [7].

**Remark 1.1.** Notice that $\mathcal{C}_\mathcal{H}$ contains all sets of the form $\Gamma'(k)$, for any $\mathcal{H}$-condition $k$, assigning to such a set the measure $P_0(\Gamma(S, k))$, $S$ being any determining set for $k$.

**Remark 1.2.** If $X$ is compact Hausdorff, $\mathcal{H}$ the Borel sets, $\mathcal{K}$ the compact sets, and $P_0$ satisfies the regularity condition of [A], then $\mathcal{C}_\mathcal{H} \subset \mathcal{B}$, and $P|\mathcal{C}_\mathcal{H} = P_\mathcal{H}$. This is a consequence of the following (under the hypotheses of the last sentence):

**Lemma 1.3.** If $S$ is determining for the condition $k$, and $k(t)$ is compact for all $t$, then $P(\Gamma(k)) = P(\Gamma(S, k))$.

**Proof.** By Theorem 2.2 of [8] there is some countable subset $S_1$ of $T$ such that $P(\Gamma(S_1, k)) = P(\Gamma(k))$. Now, $\Gamma(S_1, k) \supset \Gamma(S \cup S_1, k) \supset \Gamma(k)$, so $P(\Gamma(S \cup S_1, k)) = P(\Gamma(k))$. But

$$\Gamma(S, k) = \Gamma(S \cup S_1, k) \cap \bigcap_{s \in S} \{\Gamma(S_1, k) = \Gamma([s], k)\}.$$  

Thus $P(\Gamma(S, k)) = P(\Gamma(S \cup S_1, k))$.

We will deal mainly with the situation where $T$ is a topological space, and with a certain $\sigma$-subalgebra $\mathcal{D}_\mathcal{H}$ of $\mathcal{C}_\mathcal{H}$, where $\mathcal{D}_\mathcal{H}$ is defined like $\mathcal{C}_\mathcal{H}$, except that the only conditions $k$ used for $\mathcal{D}_\mathcal{H}$ will be those of the form
For such a $k$, we write $I(k)$ as $\mathcal{A}(U, K)$. The restriction of $P\mathcal{X}$ to $D\mathcal{X}$ will be called $Q\mathcal{X}$.

If $\mathcal{K}$ consists of closed sets in a metric space, $T$ is locally compact, and $\tau$ is a regular measure on $T$, then $(D\mathcal{X}, Q\mathcal{X})$ has the convenient property that whenever the map $t \to x_t$ (where $x_t(\omega) = \omega(t)$) is measurable in probability, i.e. is continuous in probability outside of some $\tau$-null set, then the map $(\omega, t) \to \omega(t)$ can be made measurable the $\mu \times \tau$-completion of $\mathcal{K} \times T$, where $T$ is the Borel sets of $T$ and $($,$\mu)$ some extension of $(D,\mathcal{X} Q\mathcal{X})$. (See [7], Theorem 2.) This says, in a sense, that $D\mathcal{X}$ is “not too large.” On the other hand, it is “not too small,” in the sense that it contains many natural subsets which are not in $\mathcal{E}$; this will be shown.

In § 2 are given some examples and general remarks concerning compact and countably compact families.

In [8], with $X$ and $T$ compact metrizable spaces, various natural subsets of $\Omega$ and $\Omega \times T$ were shown to be in $\mathcal{B}$, $\mathcal{D}$, or product $\sigma$-algebras derived from them (the bar over a $\sigma$-algebra signifies completion with respect to the measure being considered on it). In § 3 and 4 we show (in a somewhat more general context) that these subsets are in $D\mathcal{X}$, $D\mathcal{X}$, or the corresponding product $\sigma$-algebras, where $\mathcal{K}$ is a countably compact family of closed subsets of $X$ which contains a complete system of neighborhoods for each point of $X$ (or, briefly, generates the topology of $X$).

2. Some topological considerations.

**Lemma 2.1.** Let $X$ be a 1-st countable Hausdorff space. Then any countable compact family $\mathcal{K}$ of subsets of $X$ which generates the topology of $X$ consists of closed sets only.

**Proof.** Suppose $K \in \mathcal{K}$, and $x \notin K$. Choose a countable family $\{K_n | n \in I\}$ of neighborhoods of $x$ in $\mathcal{K}$, with $\bigcap_n K_n = \{x\}$. If $x \in \bar{K}$, then $K \cap K_1 \cap \cdots \cap K_n$ is never empty. Thus, $K \cap \bigcap_n K_n$ is nonempty, so $x \in K$.

**Remark 2.1.** If we assume that $X$ actually has a countable base for its open sets, then clearly any intersection of sets of $\mathcal{K}$ can be reduced to a countable intersection. In particular, it follows that $\mathcal{K}$ is actually a compact family, not just countably compact.

**Lemma 2.2.** (Alexander). Let $\mathcal{K}$ be a compact family of subsets
of a set $X$. Let $\mathcal{K}$ be the family of arbitrary intersections of finite unions of sets of then $\mathcal{K}$ is closed under arbitrary intersections and finite unions, and is again a compact family.

**Proof.** See [9], p. 139.

**Corollary 2.1.** The most general compact family of sets on a set $X$ arises by choosing a subfamily of the closed sets, for some compact topology on $X$.

**Proof.** Given a compact family $\mathcal{K}$ on a set $X$, use $\mathcal{K}$ as the family of closed sets for $X$; this gives a compact space.

**Remark 2.2.** The property of countable compactness does not persist from $\mathcal{K}$ to $\mathcal{K}$. For example, let $A$ be all ordinals up to and including the first uncountable ordinal $\alpha_0$. Let $B$ be the rational numbers $\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots \}$. Let $X = A \times B - \{(\alpha_0, 0)\}$. Let $\mathcal{K}$ consist of all sets of the form $K_{\alpha, n} = \{(\alpha^i, x) | \alpha^i > \alpha, x < 1/n\}$, where $\alpha$ is a countable ordinal and $n \in I$. Then no countable intersection of sets $K_{\alpha, n}$ is empty, so $\mathcal{K}$ is countably compact. But let $L_n = \bigcap_{\alpha < \alpha_0} K_{\alpha, n} = \{(\alpha_0, x) | x < 1/n\}$. Then the $L_n$ have the F.I.P., but $\bigcap_n L_n = \emptyset$.

In § 3 we shall be considering countably compact families $\mathcal{K}$ on separable metrizable spaces $X$, $\mathcal{K}$ generating the topology of $X$. Some examples follow.

(a) $X$ a Banach space which is separable and a dual, $\mathcal{K}$ the set of all closed spheres. This is mentioned in [6].

In this connection, however, notice that the separable Banach space $C$ of all continuous functions on, say, the closed interval $[-1, 1]$, is not a dual; and, in fact, the family of all closed spheres in this Banach space is not a countably compact family. To see this, let

$$f_n(\lambda) = \begin{cases} 1 & \text{if } -1 \leq \lambda \leq 0 \\ 1 - n\lambda & \text{if } 0 < \lambda < \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} \leq \lambda \leq 1, \end{cases}$$

and let $f'_n(\lambda) = -f_n(-\lambda)$. Let $K_n$ be the closed sphere of radius 2 about $f_n - 2$, and $K'_n$ the closed sphere of radius 2 about $f'_n + 2$. Then

$$K_n \cap K'_n = \{g | f'_n \leq g \leq f_n\} = \emptyset.$$

Since $f_1 \geq f_2 \geq \cdots$ and $f'_1 \leq f'_2 \leq \cdots$, we have $K_1 \cap K'_1 \supset K_2 \cap K'_2 \supset \cdots$. Thus, the spheres $\{K_n, K'_n | n = 1, 2, \cdots\}$ have the F.I.P., but there is
no continuous function in their intersection. The author does not know, however, whether some $\mathcal{H}$ does not exist for $C$.

(b) An example where the metric space is not complete: let $X$ be the nondyadic numbers in the unit interval. $\mathcal{H}$ will be defined as follows. Let $S_n$ be the set of dyadics of the form $k/2^n$, $k = 0, \ldots, 2^n$. Then $X = [0, 1] - \bigcup_n S_n$. Let $\mathcal{H}_n$ be the intersection with $X$ of intervals $[a, b]$, where $a = (k + 1/8)1/2^n$, $b = (k + 7/8)1/2^n$, $k = 0, 1, \ldots, 2^n - 1$. Let $\mathcal{H} = \bigcup_n \mathcal{H}_n$.

To see that $\mathcal{H}$ generates the topology of $X$, we must show that any $x \in X$ is an interior point of some interval in $\mathcal{H}_n$, for arbitrarily large $n$. But a nondyadic number $x$ is characterized by the property that a zero followed by a one occurs arbitrarily far out in its dyadic expansion. Thus, for arbitrarily large $n$, we can get $k/2^n + 1/2^{n+2} < x < k/2^n + 1/2^{n+1}$, so that $x$ is interior to an interval of $\mathcal{H}_n$.

To see that $\mathcal{H}$ is countably compact, suppose we have a sequence $K_1, K_2, \ldots$ with the F.I.P. Assume repetitions have been eliminated. Then no two can come from the same $\mathcal{H}_n$, since two members of $\mathcal{H}_n$ are either identical or disjoint. Consider now the closed intervals $\overline{K}_n$ in $[0, 1]$. These have the F.I.P., and are closed in $[0, 1]$. Thus their intersection is nonempty. Further, let $K_n \in \mathcal{H}_n$. Then $\overline{K}_n \cap S_{i_n} = \phi$, so $(\bigcap_n \overline{K}_n) \cap (\bigcup_n S_{i_n}) = \phi$. Since $i_n$ does not repeat itself, and since $S_1 \subset S_2 \subset \cdots$, we have $\bigcup_m S_{i_m} = \bigcup_n S_n$. Thus, $(\bigcap_n \overline{K}_n) \cap X \neq \phi$. But this is the same as $\bigcap_n \overline{K}_n$.

(c) A metric space for which no countably compact family can generate the topology: let $X$ be the dyadic numbers in $[0, 1]$. Suppose, in fact, we had such a family $\mathcal{H}$. Let $x_1, x_2, \ldots$ be an enumeration of $X$. Then one could choose a sequence $K^n_j$ of neighborhoods of $x_j, K^n_j \in \mathcal{H}$, and with the length of $K^n_j$ less than $1/n + j$. Let $U^n_j$ be the interior of $\overline{K}_j^n$. Then $x_j \in U^n_j$. Consider now the set $\bigcap_n \bigcup_j U^n_j$. This is a $G_\delta$ in the reals, and contains all the dyadics. Then it must contain some non-dyadics, since the dyadics are not a $G_\delta$. On the other hand, if $\xi$ is a non-dyadic in $\bigcap_n \bigcup_j U^n_j$, then $\xi$ is in some $\bigcap_n U^n_j$. Thus $\{K^n_j \mid n \in I\}$ has the F.I.P. But $\bigcap_n \overline{K}_j^n = \{\xi\}$, since the lengths of the $K^n_j$ go to zero as $n \to \infty$. Thus $\bigcap_n K^n_j = \bigcap_n (\overline{K}_j^n \cap X) = \phi$.

The question remains open whether, for example, every complete separable metric space has a countably compact family which generates its topology.

3. Measurability of various classes of functions. Throughout this section, let $X$ be a separable metric space; $\mathcal{B}$ the Borel sets. Let $\mathcal{H}$ be a collection of sets in $\mathcal{B}$ such that
Let $T$ be a compact metric space, and consider $\mathcal{D}_T, Q_T$, as defined in § 1. For brevity, we write simply $\mathcal{D}, Q$. We remark that the results of this section extend immediately to the case where $T$ is locally compact metrizable, and separable, since the classes of functions discussed are defined by their local properties in $T$.

Let $\mathcal{K}_0$ be a countable subset of $\mathcal{K}$ which still contains a complete system of neighborhoods at each point. Also, let $K_{\epsilon, n}$ be an enumeration of the sets of $\mathcal{K}_0$ of diameter $\leq \epsilon$. Let $A(\epsilon, S) = \bigcap_{s \in S} \{ \omega \mid \exists$ some open neighborhood $U$ of $s$ and some $n$ such that $\omega$ sends $U$ into $K_{\epsilon, n}\}$. Finally, let $\Phi(\epsilon, S) = \{ \omega \mid \exists$ some open $U \subseteq S$ and $n$ such that $\omega$ sends $U$ into $K_{\epsilon, n}\}$.

**Lemma 3.1.** $A(\epsilon, S)$ and $\Phi(\epsilon, S)$ are in $\mathcal{D}$ for any closed set $S$ and any $\epsilon > 0$.

**Proof.** Let $\mathcal{U}$ be a countable base for the open sets of $T$. Let $\mathcal{U}_1, \mathcal{U}_2, \cdots$ be an enumeration of the finite coverings of $S$ by sets in $\mathcal{U}$. Then $A(\epsilon, S) = \bigcup_n \bigcup_m \bigcap_{U_n \in \mathcal{U}_n} A(U, K_{\epsilon, m})$, and $\Phi(\epsilon, S) = \bigcup_n \bigcup_m \bigcap_{U_n \in \mathcal{U}_n} \Phi(U, K_{\epsilon, m})$.

**Theorem 3.1.** The set of all functions which are continuous at all points of the closed set $S \subseteq T$ is in $\mathcal{D}$.

**Proof.** This set is precisely $\bigcap_n A(1/n, S)$.

**Theorem 3.2.** For any regular measure $\nu$ on $T$, the set of $\nu$-almost everywhere continuous functions is in $\mathcal{D}$.

**Proof.** Let $V_{n, m}, n, m \in I$, be an enumeration of those finite unions of sets $\mathcal{U}$ such that $\nu(V_{n, m}) < 1/n$. A function $\omega$ is $\nu$-almost everywhere continuous if and only if for arbitrary small $\epsilon > 0$ there is a closed set $S$ whose complement has arbitrarily small measure, such that $\omega \in A(\epsilon, S)$. But $\omega \in A(\epsilon, S) \Rightarrow \omega \in A(\epsilon, U)$ for some open set $U \supset S$. New, $S^\perp$ is a union of sets in $\mathcal{U}$. Since $S^\perp \supset U^\perp$, and $U^\perp$ is compact, $U^\perp$ is covered by a finite union of sets of $\mathcal{U}$ which does not intersect $S$, and thus has $\nu$-measure no greater than that of $S$. Hence, the set of $\nu$-almost everywhere continuous functions is contained in $\bigcap_j \bigcap_n \bigcup_m A(1/j, V_{n, m})$. The converse inclusion is obvious.

**Theorem 3.3.** The set of functions whose points of discontinuity form a first category set, is in $\mathcal{D}$.
Proof. Let \( O_\varepsilon(\omega) = \{ s \mid \text{for every open } U \ni s \exists r, t \in U \text{ with } d(\omega(r), \omega(t)) > \varepsilon \} \). \( O_\varepsilon(\omega) \) is a closed set, and increases as \( \varepsilon \) decreases. Thus, the set \( \bigcup_{\varepsilon} O_\varepsilon(\omega) \) is of first category if and only if each \( O_\varepsilon(\omega) \) is nowhere dense. Let \( D \) be a countable dense subset of \( T \), and let \( D_{n,m} \) be an enumeration of the finite \( 1/m \)-dense subsets of \( D \) (i.e., every point of \( T \) is within \( 1/m \) of some point of \( D_{n,m} \), for every \( n, m \)). Then following Nelson in Theorem 3.3 of [8], \( O_\varepsilon(\omega) \) is nowhere dense if and only if, for every \( m \in I \), \( O_\varepsilon(\omega) \subset \text{some } D_{n,m} \). Thus, \( \omega \) has a first category set of discontinuities if and only if

\[
\omega \in \bigcap_{\varepsilon} \bigcap_{m} \bigcup_{s} O_\varepsilon(s, D_{n,m}).
\]

**Theorem 3.4.** Let \( T \) be a compact interval. Then the set of all \( \omega \) with discontinuities of the first kind only, is in \( \mathcal{B} \).

Proof. If \( \omega \) has only discontinuities of the first kind, then for any \( \varepsilon > 0 \) one can choose, for each \( t \in T \), an open interval \( R_t \) such that there are some fixed integers \( n_+ \) and \( n_- \) for which \( \omega(s) \in K_{n_+} \) for all \( s \in (R_t - \{t\})_+ \cap T \) and \( \omega(s) \in K_{n_-} \) for all \( s \in (R_t - \{t\})_- \cap T \). (Note: \( (R_t - \{t\})_\pm \) denotes the \( \pm \) of the two intervals into which \( R_t - \{t\} \) splits.)

Let \( S_t \) be a rational open interval with \( t \in S_t \subseteq \bar{S}_t \subset R_t \), and, for given \( \delta > 0 \), let \( U_t \) be another rational interval, of length \( < \delta \), with \( t \in U_t \cup S_t \). Then \( \omega \in \Theta(\varepsilon, (\bar{S}_t - U_t)_+ \cap T) \), and \( \omega \in \Theta(\varepsilon, (\bar{S}_t - U_t)_- \cap T) \). Since \( T \) can be covered by finitely many of the \( S_t \), we finally get the following: let \( \mathcal{F}_1, \mathcal{F}_2, \ldots \) be an enumeration of the finite coverings of \( T \) by rational open intervals. For any rational open interval \( S \), let \( \mathcal{F}(S) \) be the set of all open rational subintervals of \( S \) having length \( < 1/k \). Then if \( \omega \) has only discontinuities of the first kind, we have

\[
\omega \in \bigcap_{S} \bigcup_{r \in \mathcal{F}(S)} \bigcap_{s} \cap \bigcup_{m} \bigcup_{U \in \mathcal{B}} \bigcap_{\varepsilon} \{ \Theta(1/m, (\bar{S} - U)_+ \cap T) \cap \Theta(1/n, (\bar{S} - U)_- \cap T) \}.
\]

And conversely, if \( \omega \) has a discontinuity of the second kind at \( t_0 \), then there is some integer \( n \) such that no matter what open rational interval \( S \) one chooses about \( t_0 \), \( \omega \) will oscillate by more than \( 1/n \) either in \( (\bar{S} - U)_+ \cap T \) or \( (\bar{S} - U)_- \cap T \), provided \( U \) is a sufficiently short interval. Thus, the inclusion is an equality.

**Theorem 3.5.** The set \( \theta \) of pairs \((\omega, t)\) in \( \Omega \times T \) such that \( \omega \) is discontinuous at \( t \), is in \( \mathcal{B} \times \mathcal{B} \) (\( \mathcal{B} \) being the Borel sets in \( T \)). The function \((\omega, t) \mapsto \omega(t) \in \mathcal{B} \) is \( \Theta^1 \)-measurable, and a fortiori \( \mathcal{B} \times \mathcal{B} \)-measurable.

(Note: for a \( \sigma \)-algebra \( \mathcal{A} \) on a set \( Z \), and a set \( Z_0 \subseteq Z \), we denote by \( \mathcal{A} \mid Z_0 \) the \( \sigma \)-algebra \( \{ A \cap Z_0 \mid A \in \mathcal{A} \} \). In case \( Z_0 \in \mathcal{A} \), we get

\[
\mathcal{A} \mid Z_0 = \{ A \in \mathcal{A} \mid A \subset Z_0 \}.
\]
Proof of Theorem 3.5. \( \mathcal{U} \) is again a countable basis for the open sets of \( T \). Then we have \( \Theta^\perp = \bigcap_n \bigcup_{U \in \mathcal{U}} \bigcup_{m} [A(U, K_{1/n,m}) \times U] \). As for measurability of the function \( (\omega, t) \rightarrow \omega(t) \): let \( T_0 \) be a countable dense subset of \( T \). Let \( \mathcal{Y}_k \) be a finite covering of \( T \) by sets of diameter \( < 1/k \). Let \( \{g_{k,r} \mid V \in \mathcal{Y}_k\} \) be a partition of unity for \( \mathcal{Y}_k \). Let \( f \) be a continuous function on \( X \). Let \( f_k = \sum_{r} g_{k,r}(t) \sup_{s \in x \cap V} f(\omega(s)) \). Then \( f_k \) is \( \mathcal{C} \times \mathcal{B}_\tau \)-measurable, and, for fixed \( \omega \), \( f_k(t, \omega) \) is continuous in \( t \). Furthermore, at all points \( (\omega, t) \) in \( \Theta^\perp \), we have \( f_k(\omega, t) \rightarrow f(\omega(t)) \). Thus, \( f(\omega(t)) \) is \( \mathcal{C} \times \mathcal{B}_\tau \mid \Theta^\perp \)-measurable for each continuous \( f \). Now: for any closed set \( K \) in \( X \) there is a continuous function \( f_K \) which is 1 only on that set. Then \( \{(\omega, t) \mid \omega(t) \in K \} \cap \Theta^\perp = \{(\omega, t) \mid f_K(\omega(t)) = 1 \} \cap \Theta^\perp \), which is in \( \mathcal{C} \times \mathcal{B}_\tau \mid \Theta^\perp \). This completes the proof.

The generalization of Theorem 4.1 of [8] now goes through exactly as done there, by applying Fubini’s theorem. Namely, if \( \nu \) is a regular measure on \( T \), then \( \{\omega \mid \omega \text{ continuous at } t\} \) has \( Q \)-measure 1 for \( \nu \)-almost every \( t \). \( \{t \mid \omega \text{ continuous at } \omega \} \) has \( \nu \)-measure 1 for \( Q \)-almost every \( t \). \( \Theta \) has \( Q \times \nu \)-measure 0. Similarly, Theorem 4.2 of [8] generalizes to the present context: if \( \{\omega \mid \omega \text{ continuous at } t\} \) has \( Q \)-measure 0 for each \( t \in T \), then \( \{\omega \mid \text{the discontinuities of } \omega \text{ form a cat I set in } T\} \) has \( Q \)-measure 1. The proof is gotten in the same way, but substituting \( f^\perp \) of Theorem 3.5 above for Nelson’s \( f^+ \). The details will be omitted.

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