ON CONFORMAL MAPPING OF NEARLY CIRCULAR REGIONS

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Introduction. A Jordan curve $C$ in the $w$-plane, starshaped with respect to $w = 0$ and represented in polar coordinates by $\rho(\theta)e^{i\theta}$, will be said to satisfy an $\varepsilon$-condition ($\varepsilon \geq 0$) if

1. $\rho(\theta)$ is absolutely continuous in $(-\pi, +\pi)$

2. \[
\left| \frac{\rho'}{\rho} (\theta) \right| \leq \varepsilon \text{ for almost all } \theta \text{ in } (-\pi, +\pi).
\]

Sometimes the condition

\[
1 \leq \rho(\theta) \leq 1 + \varepsilon \text{ for all } \theta \text{ in } (-\pi, +\pi)
\]

will be added.

Let $w = f(z)$ be the conformal mapping of $|z| < 1$ to the interior of $C$ such that $f(0) = 0$, $f'(0) > 0$. Then one can ask: How "close" is $f(z)$ to the identity mapping $z$? This question has been studied by many authors, notably Marchenko [3] and, more recently, by Warschawski [9—14] and Specht [7]. For example, Marchenko stated:

**Theorem A.** If $C$ satisfies an $\varepsilon$-condition and also (0.2), then

\[
|f(z) - z| \leq K \cdot \varepsilon \quad (|z| \leq 1)
\]

for a universal constant $K$.

Furthermore, estimates for $M_p[f(z) - z]$ and $M_p[f'(z) - 1]$ have been given [9] where we write, for example,

\[
\|f(z) - z\|_p \equiv M_p[f(z) - z] = \left\{\frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\varphi}) - re^{i\varphi}|_{p,q} \right\}^{1/p}
\]

($p > 0; |z| = r < 1$).

In this connection, the theorem of M. Riesz [6] on conjugate harmonic functions is of importance.

**Theorem B.** Let $f(z) = u(z) + iv(z)$ be regular in $|z| < 1$ and $v(0) = 0$, so that $v(z)$ is a "normed conjugate" of $u(z)$. Then for every $p > 1$

\[
M_p[v(z)] \leq A_p M_p[u(z)] \quad (|z| = r < 1),
\]

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where $A_p$ is a constant that depends on $p$ only; one can take $A_2 = 1$, $A_p \leq 2p (p \geq 2)$ and $A_p = A_p$ for $p^{-1} + p'^{-1} = 1$. If the right-hand side of (0.4) is bounded in $0 \leq r < 1$, then $f(re^{i\varphi})$ has radial boundary values of class $L_p$ almost everywhere and (0.4) holds for $r = 1$.

In this paper we would like to make a few remarks about Marchenko's theorem and about estimates for $M_p[f'(z) - 1]$. First, we give a new proof of Theorem A which we hope is slightly simpler than Specht's [7] while giving only a slightly larger constant $K$. Next we ask whether we could replace the condition (0.1.ii) by the assumption of convexity of $C$ and still get (0.3). A counter example is constructed in I.2. Then Specht's method of proof is used to give a localized version of Theorem A, in which the $\varepsilon$-condition is fulfilled only for a part of $C$.

In the second part of the paper we obtain new estimates for $M_p[f'(z) - 1]$. Their source is a sharp and best possible estimate for $\int_0^{2\pi} [\vartheta'(\varphi)]^{1 + p} d\varphi$ where $\vartheta(\varphi) = \arg f(e^{i\varphi})$. It avoids the restriction $\varepsilon < 1$ of Warschawski [9] and gives all values of $p$ for which $M_p[\vartheta'(\varphi)] < \infty$ or $M_p[f'(z)]$ remains bounded for all $r = |z| < 1$.

**PART I**

I.1. New proof of Marchenko's theorem. While Specht's approach to Theorem A depends on a suitable integral representation of $\vartheta(\varphi) - \varphi$ ([7], p. 187), and Warschawski's on an estimate of $M_2[f'(e^{i\varphi}) - 1]$ ([9], p. 566), our proof will depend on a sharp estimate of $M_1[\vartheta'(\varphi) - 1]$. We shall prove:

**THEOREM 1.** If the Jordan curve $C$ satisfies an $\varepsilon$-condition and also (0.2) for some $\varepsilon \geq 0$, then $|f(z) - z| \leq K(\varepsilon) \cdot \varepsilon$ in $|z| \leq 1$, where $K(\varepsilon) \leq 3.7$ for all $\varepsilon \geq 0$ and $\lim_{\varepsilon \to 0} K(\varepsilon) = \sqrt{1 + \pi^2}3 \sim 2.1$.

Specht's proof yields another function $\bar{K}(\varepsilon)$ with $\bar{K}(\varepsilon) \leq 3.3$ and $\lim_{\varepsilon \to 0} \bar{K}(\varepsilon) = \sqrt{1 + (2 \log 2)^2} \sim 1.7$. The best possible bounds are not known.

In order to prove the theorem, we need the following

**LEMMA.** Let $F(x)$ be absolutely continuous in $<0, 2\pi>$, periodic with $2\pi$ and $\int_0^{2\pi} F(x) \ dx = 0$, and assume $F'(x) \in L_2(0, 2\pi)$. Then for all $x$ in $<0, 2\pi>$

$$|F(x)| \leq \frac{\pi}{\sqrt{3}} \|F'(x)\|_2. \tag{1.1}$$
This lemma is also used in Friberg’s thesis ([2], p. 14 ff). The constant \( \frac{\pi}{\sqrt{3}} \) cannot be improved as \( F(x) = \frac{x^2}{4} - \frac{\pi}{2} x + \frac{\pi^2}{6} (0 \leq x \leq 2\pi) \) shows.

**Proof.** It suffices to estimate \( F(0) \). For that we expand \( F(x) \) in its Fourier series \( F(x) = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \) and get

\[
|F(0)| = \left| \sum_{n=1}^{\infty} a_n \right| = \sum_{n=1}^{\infty} |a_n| \cdot n \cdot \frac{1}{n} \leq \left[ \sum_{n=1}^{\infty} n^2 a_n^2 \right]^{1/2} \left[ \sum_{n=1}^{\infty} n^{-2} \right]^{1/2}.
\]

The first factor is at most \( \left[ \sum_{n=1}^{\infty} n^2 a_n^2 \right]^{1/2} = \sqrt{2} \| F'(x) \|_2 \), by Parseval’s equality, the second is \( \pi/\sqrt{6} \).

**Proof of the theorem.** Putting \( f(\rho) = \rho(\theta) e^{i\phi} \), \( \theta = \theta(\phi) \), we first estimate \( |\theta(\phi) - \phi| \) if \( \varepsilon \) is assumed to be < 1. By the lemma, it is sufficient to estimate \( ||\theta'(\phi) - 1||_2 \). To do this, we note that \( \log(f(z)/z) = u(z) + iv(z) \) is regular in \( |z| < 1 \), continuous in \( |z| \leq 1 \), and \( v(0) = 0 \) since \( f'(0) > 0 \), so that \( v \) is a normed conjugate of \( u \): \( v = K[u] \). On \( |z| = 1 \) this gives

\[
\theta(\phi) - \phi = K[\log \rho(\theta(\phi))], \quad \theta(\phi + h) - (\phi + h) = K[\log \rho(\theta(\phi + h))]
\]

and hence

\[
\frac{\theta(\phi + h) - \theta(\phi)}{h} - 1 = K\left[ \frac{\log \rho(\theta(\phi + h)) - \log \rho(\theta(\phi))}{h} \right].
\]

By (0.1), for all \( \phi \) and \( h > 0 \)

\[
|\log \rho(\theta(\phi + h)) - \log \rho(\theta(\phi))| = \left| \int_{\theta(\phi)}^{\theta(\phi + h)} \frac{\rho'}{\rho}(t)dt \right| \leq \varepsilon |\theta(\phi + h) - \theta(\phi)|,
\]

and therefore

\[
\left| \frac{\theta(\phi + h) - \theta(\phi)}{h} - 1 \right|_2 = \left| \frac{\theta(\phi + h) - \theta(\phi)}{h} \right|_2 \leq \varepsilon |\theta(\phi + h) - \theta(\phi)|.
\]

Now we claim that

\[
\left| \frac{\theta(\phi + h) - \theta(\phi)}{h} - 1 \right|_2^2 = \left| \frac{\theta(\phi + h) - \theta(\phi)}{h} \right|_2^2 - 1.
\]

To show this, we write the left-hand side as

\[
\frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{\theta(\phi + h) - \theta(\phi)}{h} - 1 \right]^2 d\phi = \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{\theta(\phi + h) - \theta(\phi)}{h} \right]^2 d\phi + 1
\]

\[- 2 \frac{1}{2\pi h} \int_0^{2\pi} \{[\theta(\phi + h) - (\phi + h)] - [\theta(\phi) - \phi] + h\} d\phi.
\]
Since $\theta(\varphi) - \varphi$ is periodic with $2\pi$, the last term is $-2$, and (1.4) follows. Together with (1.3) we get $\|[[\theta(\varphi + h) - \theta(\varphi)]/h - 1\|^2 \leq \varepsilon^2/(1 - \varepsilon^2)$ for all $h > 0$. But since $C$ is rectifiable, $\theta(\varphi)$ is absolutely continuous [5] and hence $\theta'(\varphi)$ exists almost everywhere, and Fatou’s lemma yields now for $h \to 0$

$$\| \theta'(\varphi) - 1 \|_2 \leq \frac{\varepsilon}{\sqrt{1 - \varepsilon^2}}.$$  

This, incidentally, is a best possible estimate; see Theorem 6.

Now we apply our lemma to $F(\varphi) = \theta(\varphi) - \varphi$, the condition $\int_0^{2\pi} F(\varphi) d\varphi = 0$ following from (1.2), and we get for all $\varphi$

$$|\theta(\varphi) - \varphi| \leq \frac{\pi}{\sqrt{3}} \frac{\varepsilon}{\sqrt{1 - \varepsilon^2}}.$$  

From this we obtain an estimate of $|f(z) - z|$. An elementary consideration gives

$$|f(z) - z|^2 \leq \varepsilon^2 + (1 + \varepsilon)|\theta(\varphi) - \varphi|^2 \text{ on } |z| = 1;$$  

note that $1 \leq |f(e^{i\varphi})| \leq 1 + \varepsilon$. Together with (1.6) we obtain

$$|f(z) - z| \leq \varepsilon \left(1 + \frac{\pi^2}{3(1 - \varepsilon^2)}\right)^{1/2}$$  

for $|z| = 1$ and hence, by the maximum principle, for $|z| \leq 1$; this is valid whenever $\varepsilon < 1$. For all $\varepsilon \leq 20/27$ the factor of $\varepsilon$ is $\leq 3.7$; for $\varepsilon > 20/27$ we have

$$|f(z) - z| \leq 1 + \varepsilon + 1 = 2 + \varepsilon < \frac{54}{20} \varepsilon + \varepsilon = 3.7 \varepsilon .$$  

This proves $K(\varepsilon) \leq 3.7$ for all $\varepsilon \geq 0$, and (1.8) gives $\lim_{\varepsilon \to 0} K(\varepsilon) = \sqrt{1 + \pi^2}/3$.

Specht ([7], p. 188) obtains $|\theta(\varphi) - \varphi| \leq \varepsilon(2 \log 2 + \varepsilon)$. Combining this for $\varepsilon \leq 0.9$ with $|f(z) - z| \leq \varepsilon + |\theta(\varphi) - \varphi| (|z| = 1)$ and taking $|f(z) - z| \leq 2 + \varepsilon$ for $\varepsilon > 0.9$, one obtains $K(\varepsilon) \leq 3.3$ for all $\varepsilon > 0$; for $\varepsilon \to 0$ use (1.7).

1.2. Convex regions. Our next problem is to decide whether Marchenko’s theorem remains valid if the condition $|\rho'/\rho| \leq \varepsilon$ is replaced by the convexity of $C$. To study a suitable counter example, it will be convenient to use the following localization theorem.

\[ \frac{1}{1 - \varepsilon^2} \leq (1 - \varepsilon^2)^{-1/2} \] 

\[ \frac{3}{2} \leq 1 - \varepsilon^2. \]

This also follows directly from $||\theta'||_2 \leq (1 - \varepsilon^2)^{-1/2}$ ([8], p. 26) and $||\theta' - 1||_2^2 = ||\theta'||_2^2 - 1$, but we wanted to give an independent proof of (1.5).

\[ \frac{1}{1 - \varepsilon^2} \leq (1 - \varepsilon^2)^{-1/2} \] 

\[ \frac{3}{2} \leq 1 - \varepsilon^2. \]

The application of Warschawski’s inequality ([8], p. 18) would have given a slightly larger bound for $K$ in Theorem 1.
THEOREM 2. Let \( C: \rho(\theta)e^{i\theta} \) be a Jordan curve, starshaped with respect to \( w = 0 \) and contained in \( 1 \leq |w| \leq 1 + \varepsilon \), and let \( w = f(z) \) with \( f(0) = 0 \), \( f'(0) > 0 \) map \(|z| < 1\) conformally to the interior of \( C \); put \( \theta(\varphi) = \arg f(e^{i\varphi}) \).

Then to every \( \delta, 0 < \delta < \pi \), corresponds a constant \( D = D(\delta) \) such that

\[
|\theta(\varphi) - \varphi| - \frac{1}{2\pi} \int_{\varphi-\delta}^{\varphi+\delta} \log \rho(\theta(t)) \frac{t}{2} \, dt \leq D \cdot \varepsilon
\]

for all \( \varphi \), the integral being a Cauchy principal value.

Proof. Since \( \theta(\varphi) - \varphi \) is a normed conjugate of \( \log \rho(\theta(\varphi)) \) (see (1.2)), we have

\[
\theta(\varphi) - \varphi = \frac{1}{2\pi} \int_{\varphi-\pi}^{\varphi+\pi} \log \rho(\theta(t)) \frac{t}{2} \, dt = \frac{1}{2\pi} \int_{\varphi-\delta}^{\varphi+\delta} \frac{t}{2} \, dt + \frac{1}{2\pi} \int_{\varphi-\pi}^{\varphi-\delta} \log \rho(\theta(t)) \frac{t}{2} \, dt + \frac{1}{2\pi} \int_{\varphi+\delta}^{\varphi+\pi} \frac{t}{2} \, dt.
\]

In the last term \(|\frac{t}{2}|\) is bounded by \( \frac{\varepsilon}{2} \) while \( 0 \leq \log \rho(\theta(t)) \leq \varepsilon \). This proves (1.9) with \( D(\delta) = \text{ctg} [\frac{\delta}{2}] \).

Furthermore, we shall use another theorem of Marchenko ([3], p. 289) which, in the generalization by Warschawski ([10], p. 343), reads as follows. Let \( R \) be a simply connected region containing \( w = 0 \) whose boundary is contained in \( 1 \leq |w| \leq 1 + \varepsilon \). Let \( \lambda \) be such that any two points in \( R \) with distance \( < \varepsilon \) may be connected in \( R \) by an arc of diameter \( < \lambda \). If \( f(z) \) is the normalized mapping of \(|z| < 1\) to \( R \), then

\[
|f(z) - z| \leq M \varepsilon |\log \varepsilon| + M_1 \lambda
\]

for two absolute constants \( M \) and \( M_1 \). Ferrand ([1], p. 133) states without proof that one can take \( M = 1/\pi \) as the best possible constant; note that in her paper the boundary is assumed to be in \( 1 - \varepsilon \leq |w| \leq 1 + \varepsilon \). Obviously \( \lambda \leq 3\varepsilon \) if \( R \) is starshaped with respect to \( w = 0 \).

Now we shall study the following family of conformal maps. Let the Jordan curve \( C = C(\varepsilon)(0 < \varepsilon < 1/2) \) be defined as follows:

\[
|w| = 1 \quad \text{if} \quad -\pi \leq \arg w \leq 0,
\]

\[
|w| = 1 + \varepsilon \quad \text{if} \quad 0 < \theta_2 \leq \arg w \leq \frac{\pi}{2} + \kappa, \text{ where } 0 < \kappa < \frac{\pi}{2} \text{ and } \sin \kappa = 1/(1 + \varepsilon),
\]

and where these two circular arcs are connected by straight line segments. The angle \( \theta_2 \) will also depend on \( \varepsilon \) and is subject to
(1.11) \[ \theta_2 \to 0 \text{ and } \frac{\theta_2}{\varepsilon |\log \varepsilon|} \to +\infty \quad (\varepsilon \to 0). \]

Let \( w = f(z) \) map \( |z| < 1 \) to the interior of \( C \) with \( f(0) = 0, f(0) > 0 \) and let

\[ f(e^{i\varphi_1}) = 1 = e^{i\theta_1}, \quad f(e^{i\varphi_2}) = (1 + \varepsilon)e^{i\theta_2}. \]

By (1.10) we have for all \( \varphi \) and \( \varepsilon \)

(1.12) \[ |\theta(\varphi) - \varphi| \leq M \varepsilon |\log \varepsilon| + O(\varepsilon), \]

in particular \( \varphi_1 \to 0, \varphi_2 \to 0(\varepsilon \to 0) \). We therefore get from Theorem 2

\[ \theta(\varphi_1) - \varphi_1 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \log \rho(\theta(t)) \cot \frac{\varphi_1 - t}{2} dt + O(\varepsilon), \]

\[ |\theta(\varphi_1) - \varphi_1| = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \log \rho(\theta(t)) \cot \frac{t - \varphi_1}{2} dt + O(\varepsilon) > \frac{1}{2\pi} \int_{\varphi_1}^{\pi/2} + O(\varepsilon); \]

note that \( \rho(\theta(t)) = 1 \) for \( t \) in \( < \pi/2, \varphi_1 > \). The last integral is equal to

\[ \log(1 + \varepsilon) \int_{\varphi_1}^{\pi/2} \cot \frac{t - \varphi_1}{2} dt = 2 |\log(\varphi_2 - \varphi_1)| \varepsilon + O(\varepsilon). \]

Here we have by (1.11) and (1.12)

\[ \varphi_2 - \varphi_1 = \theta_2 - \theta_1 + O(\varepsilon |\log \varepsilon|) = (\theta_2 - \theta_1)(1 + o(1)) = \theta_2(1 + o(1)) , \]

so that altogether we obtain

(1.13) \[ |\theta(\varphi_1) - \varphi_1| > \frac{\log \theta_2}{\pi} \varepsilon + O(\varepsilon) \quad (\varepsilon \to 0). \]

Before we specialize (1.13), we remark that for the regions considered here

(1.14) \[ |f(z) - z| = |\theta(\varphi) - \varphi| + O(\varepsilon) \quad (z = e^{i\varphi}). \]

We have namely on \( |z| = 1 \)

\[ 2 \sin \frac{\theta(\varphi) - \varphi}{2} \leq |f(z) - z| \leq 2 \sin \frac{\theta(\varphi) - \varphi}{2} + (|f(z)| - 1). \]

By (1.12), \( |\theta(\varphi) - \varphi| = O(\varepsilon |\log \varepsilon|) \) and (1.14) follows.

We now make two special choices of \( \theta_2 = \Theta_2(\varepsilon) \), always subject to (1.11). For our first choice \( \Theta_3(\varepsilon) = \varepsilon |\log \varepsilon|^2 \) we obtain from (1.13) and (1.14)
Thus we proved that the best constant $M$ in (1.10) must satisfy $M \geq 1/\pi$, in agreement with Ferrand.

Next we choose $\theta_2$ such that $1 = (1 + \varepsilon) \cos \theta_2$, which makes $C(\varepsilon)$ convex. If we insert $\theta_2 = \sqrt{2\varepsilon} + O(\varepsilon)$ in (1.13), we obtain

$$|f(z) - z| \geq \frac{\varepsilon |\log \varepsilon|}{2\pi} (1 + o(1))$$

$$(z = e^{i\varphi}; \varepsilon \to 0).$$

**THEOREM 3.** If $C(\varepsilon)$ is the family of convex curves defined by $\cos \theta_2(\varepsilon) = [1/1 + \varepsilon]$, we have

$$\max_{|z| = 1} |f(z) - z| \geq \frac{\varepsilon |\log \varepsilon|}{2\pi} (1 + o(1))$$

$$(\varepsilon \to 0).$$

In particular, Theorem A does not hold if the condition $|\rho'(\theta)||\rho(\theta)| \leq \varepsilon$ is replaced by the convexity of $C$.

**I.3. Localization of the theorem of Marchenko.** In I.1 we have seen that Theorem A can be proved with a quite satisfactory constant $K$ by a "global" method, a method involving means rather than the function itself. Nevertheless, Specht’s proof of Theorem A, directly aiming at $|\theta(\varphi) - \varphi|$, has besides giving a slightly better constant the advantage of being useful to obtain a localization of Theorem A, where $|[\theta'/\rho]| \leq \varepsilon$ is known only for a part of $C$.

We begin with the following localization of Specht’s representation theorem ([7], p. 187).

**THEOREM 4.** Let $C : \rho(\theta)e^{i\theta}$ be a Jordan curve, starshaped with respect to $w = 0$ which satisfies:

(i) $1 \leq \rho(\theta) \leq 1 + \varepsilon$ for all $\theta$ and some $\varepsilon > 0$;

(ii) $\rho(\theta)$ has bounded difference quotients for $\theta$ in $(a, b)$.

Then to every $\delta > 0$ corresponds an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ and a constant $N(\delta)$ such that for $\varepsilon < \varepsilon_0$ we have

$$(1.15) \quad |[\theta(\varphi) - \varphi] - \frac{1}{\pi} \int_a^b \log \sin \frac{t(\theta) - \varphi}{2} \left| \frac{\rho'(\theta)}{\rho(\theta)} \right| d\theta| \leq N(\delta) \cdot \varepsilon,$$

for all $\varphi$ in $(a + \delta, b - \delta)$ for which $\theta'(\varphi)$ and $\rho'(\theta(\varphi))$ exist and $\theta'(\varphi) \neq 0$, i.e. for almost all $\varphi$ in $(a + \delta, b - \delta)$.

Here $t = t(\theta)$ is the inverse function of $\theta(t)$, and the integral exists as a Lebesgue integral.
Proof. For our fixed $\delta > 0$, choose $\varepsilon_0(\delta)$ such that $\alpha = \theta^{-1}(a)$ and $\beta = \theta^{-1}(b)$ satisfy $|\alpha - a| < \delta/2$, $|\beta - b| < \delta/2$; this is asserted by (1.10) or (1.12) as soon as $\varepsilon < \varepsilon_0$. Then we can write

$$\theta(\varphi) - \varphi = \frac{1}{2\pi} \int_{\varphi-\pi}^{\varphi+\pi} \left[ \log \rho(\theta(t)) - \log \rho(\theta(\varphi)) \right] \text{ctg} \frac{\varphi - t}{2} dt$$

$$= \frac{1}{2\pi} \int_{a}^{\beta} \left| \text{ctg} \frac{\varphi - t}{2} dt + O(\varepsilon) \right| \Delta \;$$

compare Theorem 2. Now one applies partial integration to the integral as in Specht’s proof, and (1.15) follows.

Now we can prove the following localization of Theorem A.

**Theorem 5.** Let $C : \rho(\theta)e^{i\theta}$ be a rectifiable Jordan curve, starshaped with respect to $w = 0$ which satisfies

(i) $1 \leq \rho(\theta) \leq 1 + \varepsilon$ for all $\theta$ and some $\varepsilon \geq 0$,
(ii) $|\rho(\theta + \tau) - \rho(\theta)| \leq \rho(\theta)|\tau|\varepsilon$ for all $\theta$ in $\langle a, b \rangle$ and all real $\tau$.

Then to every $\delta > 0$ corresponds a constant $K_1(\delta)$ such that

$$|f(z) - z| \leq K_1(\delta) \cdot \varepsilon \quad \text{for all } z = e^{i\varphi}, \varphi \in \langle a + \delta, b - \delta \rangle.$$

Proof. It suffices to prove this for small $\varepsilon$. Condition (ii) implies that we can estimate the integral term in (1.15) by

$$\varepsilon \left| \frac{1}{\pi} \int_{\varphi-\pi}^{\varphi+\pi} \left| \sin \frac{t(\theta) - \varphi}{2} \right| d\theta \right| \leq - \varepsilon \frac{1}{\pi} \int_{\varphi-\pi}^{\varphi+\pi} \left| \sin \frac{t - \varphi}{2} \right| \theta'(t)dt \leq \varepsilon(2 \log 2 + \varepsilon)$$

(see [7], p. 188). Hence $|\theta(\varphi) - \varphi| \leq K_1(\delta) \cdot \varepsilon$ for almost all $\varphi$ in $\langle a + \delta, b - \delta \rangle$. By continuity, this holds for all $\varphi$ in $\langle a + \delta, b - \delta \rangle$, and (1.16) follows.

**Remark.** By a simple approximation argument it is seen that the rectifiability of $C$, needed for the last inequality in (1.17), is not necessary for the validity of Theorem 5.

**Part II**

II. 1. Sharp estimates for the means of $\theta'(\varphi)$. Our aim is now to give an estimate for $M_z[f'(z) - 1]$ if $C$ satisfies an $\varepsilon$-condition. As a first step we prove the following

**Theorem 6.** Let $C : \rho(\theta)e^{i\theta}$ be a Jordan curve, starshaped with respect to $w = 0$, which satisfies an $\varepsilon$-condition for some $\varepsilon \geq 0$, and
let \( w = f(z) \) with \( f(0) = 0 \) map \( |z| < 1 \) conformally to the interior of \( C \). Then \( \theta(\phi) = \arg f(e^{i\phi}) \) satisfies

\[
\int_0^{2\pi} |\theta'(\phi)|^{p+1} d\phi < \infty \text{ if } 0 \leq p < \frac{\pi}{2 \arctan \varepsilon}.
\]

More precisely, we have

\[
\frac{1}{2\pi} \int_0^{2\pi} |\theta'(\phi)|^p d\phi \leq \left( \frac{\cos \arctan \varepsilon}{\cos(\phi \arctan \varepsilon)} \right)^p \text{ if } 1 \leq p < \frac{\pi}{2 \arctan \varepsilon} \leq 1 \text{ if } 0 \leq p \leq 1
\]

and

\[
\frac{1}{2\pi} \int_0^{2\pi} |\theta'(\phi)|^{-p} d\phi \leq \left( \frac{1}{\cos \arctan \varepsilon \cos(\phi \arctan \varepsilon)} \right)^p \text{ if } 0 \leq p < \frac{\pi}{2 \arctan \varepsilon}.
\]

Moreover, the bounds in (2.2) and (2.3), as well as the upper bound for \( p \) in (2.1), are best possible.

**REMARKS.** a. It easily follows from F. Riesz’s theorem ([5], p. 95), that not only \( \theta = \theta(\phi) \) but also its inverse \( \phi = \phi(\theta) \) is an absolutely continuous and monotonically increasing function, whenever \( C \) satisfies an \( \varepsilon \)-condition for some \( \varepsilon \geq 0 \). The substitution \( \phi = \phi(\theta) \) in (2.1) is therefore permissible\(^4\) and gives

\[
\frac{1}{2\pi} \int_0^{2\pi} |\theta'(\phi)|^{p} d\phi = \frac{1}{2\pi} \int_0^{2\pi} |\phi'(\theta)|^{p} d\theta \text{ if } 0 \leq p < \frac{\pi}{2 \arctan \varepsilon},
\]

so that (2.2) and (2.3) contain also estimates for the means of \( \phi'(\theta) \). In particular, since \( \pi/(2 \arctan \varepsilon) > 1 \), (2.3) is always applicable for \( p = 1 \), and we obtain that \( \phi'(\theta) \in L_2 \) whenever \( C \) satisfies an \( \varepsilon \)-condition for some \( \varepsilon \geq 0 \). (For \( \theta'(\phi) \in L_2 \) we need \( \varepsilon < 1 \).)

b. For \( p = 2 \) the bounds in (2.2) and (2.3) become \( (1 - \varepsilon^2)^{-1} \) (see [8], p. 26) and \( (1 + \varepsilon^2)^3/(1 - \varepsilon^2) \).

**Proof.** We begin with three preliminary remarks. First, we have

\[
|\rho'(\theta)|/|\rho(\theta)| \leq \varepsilon \text{ for all } \theta, \text{ for which } \rho'(\theta) \text{ exists. For by (0.1)}
\]

\[
|\log \rho(\theta + h) - \log \rho(\theta)| = \left| \int_{\theta}^{\theta+h} \frac{\rho'(t)}{\rho(t)} dt \right| \leq \varepsilon |h|,
\]

for all \( \theta \) and \( h \neq 0 \); this implies our proposition.

Next, since $C$ is rectifiable, we know by F. Riesz’s theorem ([5], p. 95; see also [16], p. 157 ff.) that

(i) $f(e^{i\varphi})$ is absolutely continuous, so that $[df(e^{i\varphi})/de^{i\varphi}]$ exists almost everywhere and is integrable; furthermore

(ii) $f'(z) \in H_i$, i.e. $\int_0^{2\pi} |f'(re^{i\varphi})| d\varphi \leq A < \infty$ for all $r < 1$.

We claim that

\[(2.4) \quad f''(re^{i\varphi}) \to \frac{df(e^{i\varphi})}{de^{i\varphi}} \quad \text{as } r \to 1, \quad \text{for almost all } \varphi.\]

To prove this, let $f''(re^{i\varphi}) \to h(e^{i\varphi})(r \to 1, \text{almost all } \varphi)$, so that by (ii) $h(e^{i\varphi})$ is integrable and $\int_0^{2\pi} |f''(re^{i\varphi}) - h(e^{i\varphi})| d\varphi \to 0(r \to 1)$. Therefore, for any fixed $\varphi_0$,

\[
[f(re^{i\varphi_0}) - f(r)] - r \int_0^{\varphi_0} h(e^{i\varphi})i e^{i\varphi} d\varphi = r \left[\int_0^{\varphi_0} [f''(re^{i\varphi}) - h(e^{i\varphi})]i e^{i\varphi} d\varphi \to 0 \quad (r \to 1)\right]
\]

that is

\[
f(e^{i\varphi_0}) = f(1) + \int_0^{\varphi_0} h(e^{i\varphi})i e^{i\varphi} d\varphi.
\]

Differentiation yields $[df(e^{i\varphi})/de^{i\varphi}] = h(e^{i\varphi})$ almost everywhere, which is (2.4). From now on we shall put $[df(e^{i\varphi})/de^{i\varphi}] = f'(e^{i\varphi})$ whenever this exists.

Finally, since $f' \in H_1$ and $f' \not\equiv 0$, one knows (see, e.g., [4], p. 56) that $f'(e^{i\varphi})$ vanishes only on a null set.

To start the proof of theorem, let $M$ be the set of all $\varphi$ in $\langle 0, 2\pi \rangle$ for which (i) $f'(e^{i\varphi})$ exists and is $\neq 0$ and (ii) $\lim_{r \to 1} f'(re^{i\varphi}) = f'(e^{i\varphi})$; by our above remarks, $M$ is of measure $2\pi$.

We consider now the function $g(z) = zf'(z)/f(z)$, regular and $\neq 0$ in $|z| < 1$, $g(0) = 1$, and put

\[
F(z) = \log g(z) = \log |g(z)| + i \arg g(z) = u(z) + iv(z),
\]

which is regular in $|z| < 1$ and vanishes at $z = 0$. We study $u(z), v(z)$ for $|z| \to 1$.

(a) Since

\[
(2.5) \quad \frac{zf'(z)}{f(z)} = \left[1 - i \frac{\theta'(\theta(\varphi))}{\rho(\theta(\varphi))}\right] \theta'(\varphi) \quad (z = e^{i\varphi}, \varphi \in M)
\]

we have $\theta'(\varphi) \neq 0(\varphi \in M)$ and furthermore

\[
|g(re^{i\varphi})| \to \theta'(\varphi) \left|1 + \frac{\theta'}{\rho} i\right| = \frac{\theta'(\varphi)}{\cos \beta(\theta(\varphi))} \quad (r \to 1, \varphi \in M)
\]
where $\beta(\theta)$ denotes the angle between $\arg w = \theta$ and the normal to $C$ at $(\rho(\theta), \theta)$. Hence

$$u(re^{i\varphi}) \to \log \frac{\theta'(\varphi)}{\cos \beta(\theta(\varphi))} = u(e^{i\varphi}) \quad (r \to 1, \varphi \in M).$$

(b) On the other hand we have for $v(z)$

$$v(re^{i\varphi}) = \arg g(re^{i\varphi}) \to \arg f'(e^{i\varphi}) + \varphi - \arg f(e^{i\varphi}) = \beta(\theta(\varphi)) \quad (r \to 1, \varphi \in M).$$

In particular, $\beta(\theta)$ exists for $\theta = \theta(\varphi)$, $\varphi \in M$, and hence by our first remark $|\beta(\theta(\varphi))| \leq \arctg \varepsilon(\varphi \in M)$.

(c) This implies that $|v(re^{i\varphi})| \leq \arctg \varepsilon$ for $r < 1$. For $v(re^{i\varphi})$ is harmonic in $r < 1$ and clearly represents the angle between $\arg w = \theta$ and the normal to the level curve corresponding to $|z| = r$, which is again starshaped. Thus $|v(re^{i\varphi})| < \pi/2$, and $v(re^{i\varphi})$ can therefore be represented by its Poisson integral in $r < 1$. Since the boundary values are $\leq \arctg \varepsilon$, also $|v(re^{i\varphi})| \leq \arctg \varepsilon(r < 1)$.

For the main part of the proof, we apply a method of Zygmund ([15], p. 286). Let $p > 0$ and consider

$$(2.6) \quad 1 = e^{\pm p F^{(0)}} = \frac{1}{2\pi i} \int_{|z| = r < 1} \frac{e^{\pm p F(z)}}{z} \, dz = \frac{1}{2\pi} \int_{|z| = r < 1} e^{\pm pu(z)} \cos[pv(z)] \, d\varphi.$$ 

By (c) and our assumption on $p$, we have $|pv(z)| \leq p \arctg \varepsilon < \pi/2$, so that the integrand in the last integral is positive for all $r < 1$ and $\varphi$. Recalling (a) and (b), an application of Fatou’s lemma yields

$$\frac{1}{2\pi} \int_M e^{\pm pu(e^{i\varphi})} \cos[pv(e^{i\varphi})] \, d\varphi \leq 1$$

that is

$$(2.7) \quad \frac{1}{2\pi} \int_M \left[\theta'(\varphi)\right]^{\pm p} \frac{\cos[p\beta(\theta(\varphi))]}{[\cos \beta(\theta(\varphi))]^{\pm p}} \, d\varphi \leq 1.$$ 

Now we note that $|\beta(\theta(\varphi))| \leq \arctg \varepsilon(\varphi \in M)$, and the fact that

$$\frac{\cos px}{(\cos x)^p}$$

is monotonically decreasing in $0 \leq x < \pi/2p$ if $p > 1$

increasing in $0 \leq x < \pi/2$ if $0 < p < 1$.

This proves our estimates (2.2) and (2.3).

We now show that our bounds are best possible. More precisely:

For every $\varepsilon \geq 0$ and for every $p$ with $0 \leq p < \pi/(2 \arctg \varepsilon)$, there exists a curve $C$ such that Theorem 6 holds with equality in (2.2) and (2.3), respectively.

For $\varepsilon = 0$, and for $\varepsilon > 0$, $0 \leq p \leq 1$ in (2.2), we simply let $C$ be
For $\varepsilon > 0$ and the other two cases in (2.2) and (2.3) we consider the curve $C: \rho(\theta) = e^{\varepsilon \theta}(|\theta|) \leq \pi$, which is composed of two pieces of logarithmic spirals that meet in $w = 1$ and $w = -e^{\pi\varepsilon}$. Let $f(z)$ be such that $f(1) = 1$ and $f(-1) = -e^{\varepsilon\pi}$.

We claim that for this mapping we have equality in (2.7) whenever $0 \leq p < \pi/(2 \arctg \varepsilon)$. Since $tg \beta(\theta(\phi)) = \pm \varepsilon$ for all $\phi \neq 0, \pi$, this would immediately give equality in (2.2) and (2.3).

To prove equality in (2.7), we study the behaviour of $f'(z)$ in $|z| < 1$. The curve $C$ is composed of two analytic arcs meeting at angles $\alpha_1 \pi$ and $\alpha_2 \pi$ with $\alpha_1 = 1 + [2/\pi] \arctg \varepsilon$ and $\alpha_2 = 1 - [2/\pi] \arctg \varepsilon$. By a theorem of Warschawski ([13], p. 835), we have therefore

\begin{align*}
\text{(2.8)} \\
&f'(z)(z - 1)^{-\frac{2\pi}{\arctg \varepsilon}} \to C_1 \neq 0(z \to 1) \\
\text{and } &f'(z)(z + 1)^{\frac{2\pi}{\arctg \varepsilon}} \to C_2 \neq 0(z \to -1),
\end{align*}

for unrestricted approach within $|z| < 1$. Thus,

\[ f'(z)(z + 1)^{\frac{2\pi}{\arctg \varepsilon}} \text{ and } |f'(z)|^{-1}(z - 1)^{\frac{2\pi}{\arctg \varepsilon}} \text{ are continuous in } |z| \leq 1, \text{ and we have for } re^{i\phi}, 0 \leq r < 1, 0 < |\phi| < \pi, \]

\[ |f'(re^{i\phi})| \leq \text{const} \frac{1}{(\pi - |\phi|)^{\frac{2\pi}{\arctg \varepsilon}}} \text{ and } |f'(re^{i\phi})|^{-1} \leq \text{const} \frac{1}{|\phi|^{\frac{2\pi}{\arctg \varepsilon}}}. \]

Therefore, if $2p\arctg \varepsilon < \pi$, $\exp \{\pm pu(re^{i\phi})\} = |g(re^{i\phi})|^{\pm p}$ is bounded by an integrable function, uniformly for all $r$ in $0 \leq r < 1$, so that Lebesgue's convergence theorem can be applied to (2.6) for $r \to 1$, giving equality in (2.7).

Finally, also the bound on $p$ is best possible. For this we simply note that by (2.5) and (2.8)

\[ |\theta'(\phi)|^{-1}|\phi|^{\frac{2\pi}{\arctg \varepsilon}} \geq D_1 > 0 \text{ near } \phi = 0 \]

\[ \text{and } \theta'(\phi) \cdot (\pi - |\phi|)^{\frac{2\pi}{\arctg \varepsilon}} \geq D_2 > 0 \]

near $\phi = \pi$, so that for $p = \pi/(2 \arctg \varepsilon)$ the functions $[\theta'(\phi)]^p$ and $[\theta'(\phi)]^{-p}$ are not integrable.

**Corollary.** Under the assumptions of Theorem 6, we have for $0 \leq r < 1$

\begin{align*}
\text{(2.9)} \\
\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\phi})|^{\pm p} d\phi \leq \frac{\max |\rho(\theta)|^{\pm p}}{\cos (p \arctg \varepsilon)} \text{ if } 0 \leq p < \frac{\pi}{2 \arctg \varepsilon}. \end{align*}

For $p = \pi/(2 \arctg \varepsilon)$, the left-hand side need not be uniformly bounded in $0 \leq r < 1$.

\textsuperscript{4} See also a similar estimate for smooth curves ([11], p. 254).
For the proof we note that by (2.6)
\[ 1 = \frac{1}{2\pi} \int_{|z| = r < 1} |g(z)|^p \cos (p \nu(z)) \, d\phi . \]

Recalling \( |p \nu(z)| \leq p \arctg \varepsilon \) and that \( |z|f(z)|^{\pm p} \) assumes its minimum for \( |z| = 1 \), we arrive at (2.9).

II.2. An estimate for \( M_p[f'(z) - 1] \). Theorem 6 enables us to derive an estimate for the mean of \( f'(z) - 1 \), which is small for small \( \varepsilon \).

**Theorem 7.** Let \( C : \rho(\theta)e^{i\theta} \) be a Jordan curve, starshaped with respect to \( w = 0 \), which satisfies an \( \varepsilon \)-condition and which lies in the ring \( 1 \leq |w| \leq 1 + \varepsilon \) for some \( \varepsilon \geq 0 \). Let \( w = f(z) \) with \( f(0) = 0 \), \( f'(0) > 0 \) map \( |z| < 1 \) conformally to the interior of \( C \). Then we have for all \( r \) with \( 0 \leq r < 1 \)

\begin{equation}
(2.10) \quad M_p[f'(re^{i\phi}) - 1] \leq \left\{ (1 + \varepsilon) \frac{\cos \arctg \varepsilon}{[\cos (p \arctg \varepsilon)]^{1/p}} + \varepsilon \right\} (1 + A_p) \cdot \varepsilon
\end{equation}

if \( 1 < p < \frac{\pi}{2 \arctg \varepsilon} \),

where \( A_p \) denotes the constant in Riesz’s Theorem B. The upper bound for \( p \) is best possible.\(^5\)

This improves a theorem of Warschawski ([9], p. 566) with respect to the restrictions on \( \varepsilon \) and \( p \).

**Proof.** We first estimate \( M_p[|z f'(z)|f(z)] - 1 \) (see [9], p. 565) and write by (2.5)

\[ \frac{z f'(z)}{f(z)} - 1 = (\theta'(\phi) - 1) - i \frac{\rho'(\theta)}{\rho(\theta)} \theta'(\phi) \quad (z = e^{i\phi}) . \]

Since the left-hand side vanishes at \( z = 0 \), Riesz’s theorem gives

\[ M_p[\theta'(\phi) - 1] \leq A_p M_p \left[ \frac{\theta'(\theta(\phi))}{\rho(\theta(\phi))} \theta'(\phi) \right] \leq A_p M_p[\theta'(\phi)] \cdot \varepsilon . \]

With (2.2) and Minkowski’s inequality we obtain

\begin{equation}
(2.11) \quad M_p \left[ \frac{e^{i\phi} f'(e^{i\phi})}{f(e^{i\phi})} - 1 \right] \leq (1 + A_p) M_p[\theta'(\phi)] \cdot \varepsilon
\end{equation}

\[ \leq (1 + A_p) \frac{\cos \arctg \varepsilon}{[\cos (p \arctg \varepsilon)]^{1/p}} \cdot \varepsilon . \]

\(^5\) For \( 0 < p \leq 1 \) an estimate can be obtained by an application of Hölder’s inequality.
Next, we use the estimate
\[ M_p[f'(z) - 1] \leq (1 + \varepsilon)M_p\left[\frac{zf''(z)}{f(z)} - 1\right] + M_p\left[\frac{f(z)}{z} - 1\right] \quad (|z| = r < 1), \]
where the last term is \( \leq (1 + A_r)e \varepsilon \); see [9], p. 564–566. Combining this with (2.11) and using the monotonicity of \( M_p[|zf''(z)/f(z)| - 1] \) with respect to \( r \), we arrive at (2.10).

For \( p = \pi/(2 \arctg \varepsilon) \), \( M_p[f'(re^{i\delta}) - 1] \) need not be uniformly bounded in \( 0 \leq r < 1 \). To see this, one has to modify our example in II.1 slightly in an obvious way so that it satisfies also \( 1 \leq \rho(\theta) \leq 1 + \varepsilon \); note that only the angle \( \pi \alpha \) is of importance.

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