

# Pacific Journal of Mathematics

**ON THE NUMBER OF PURE SUBGROUPS**

PAUL DANIEL HILL

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A problem due to Fuchs [3] is to determine the cardinality of the set  $\mathcal{P}$  of all pure subgroups of an abelian group. Boyer has already given a solution for nondenumerable groups  $G$  [1]; he showed that  $|\mathcal{P}| = 2^{|G|}$  if  $|G| > \aleph_0$ , where  $|A|$  denotes the cardinality of a set  $A$ . Our purpose is to complement the results of [1] by determining those groups for which  $|\mathcal{P}|$  is finite,  $\aleph_0$ , and  $c = 2^{\aleph_0}$ . In the following, group will mean abelian group.

**LEMMA 1.** *If  $G$  is a torsion group with  $|G| \leq \aleph_0$ , then  $|\mathcal{P}| = c$  unless*

$$(1) \quad G = p_1^\infty \oplus p_2^\infty \oplus \cdots \oplus p_n^\infty \oplus B,$$

*a direct sum of (at most) a finite number of groups of type  $p^\infty$  and a finite group, where  $p_i \neq p_j$  if  $i \neq j$ . If  $G$  is of the form (1), then  $|\mathcal{P}|$  is finite.*

*Proof.* The latter statements is clear, and if none of the following hold

- (i)  $G$  decomposes into an infinite number of summands
- (ii)  $G$  contains  $p^\infty \oplus p^\infty$  for some prime  $p$
- (iii)  $|B| = \aleph_0$ , where  $B$  is the reduced part of  $G$ ,

then  $G$  is of the form (1). Moreover, if (i) holds, then obviously  $|\mathcal{P}| = c$ . Every automorphism of  $p^\infty$  determines a pure subgroup of  $p^\infty \oplus p^\infty$ , and distinct automorphisms correspond to distinct subgroups. Since  $|A(p^\infty) = \text{automorphism group}| = c$ , it follows that  $p^\infty \oplus p^\infty$  has  $c$  pure subgroups. Thus if (ii) holds,  $|\mathcal{P}| = c$  since  $p^\infty \oplus p^\infty$  is a direct summand of  $G$ . Finally, if (iii) holds and if (i) does not, then the following argument shows that  $|\mathcal{P}| = c$ . We may write<sup>1</sup>  $B = C_1 \oplus B_1 = C_1 \oplus C_2 \oplus B_2$ , and continuing in this way define an infinite sequence  $C_n$  of cyclic groups such that no  $C_i$  is contained in the direct sum of any of the others. The direct sum of any subcollection of these cyclic groups is a pure subgroup of  $B$  and, therefore, of  $G$ .

An interesting corollary is noted: there is no torsion group with exactly  $\aleph_0$  pure subgroups.

**LEMMA 2.** *If  $G = F \oplus B$  is the direct sum of a torsion free group*

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<sup>1</sup> This is precisely the proof of Boyer that such a group has  $c$  subgroups [2].

$F$  of rank  $r$  and a finite group  $B$  with  $|G| \leq \aleph_0$ , then  $|\mathcal{S}|$  is finite,  $\aleph_0$ , or  $c$ , depending on whether  $r = 1$ ,  $1 < r < \infty$ , or  $r = \infty$ .

*Proof.* First, assume that  $B = 0$ . Let  $H$  be the minimal divisible group containing  $G$ . The correspondence  $D \rightarrow D \cap G$  is one-to-one between pure (divisible) subgroups  $D$  of  $H$  and pure subgroups of  $G$ . Thus only divisible groups  $G$  need be considered, and the proof is already clear except, possibly, the relation  $|\mathcal{S}| \leq \aleph_0$  for the case  $1 < r < \infty$ . However, let  $R^*$  denote the direct sum of  $r - 1$  copies of  $R$ , the additive rationals. Since  $G = R^* \oplus R$ , any pure subgroup  $P$  of  $G$  is a subdirect sum of a subgroup  $S^*$  of  $R^*$  and a subgroup  $S$  of  $R$ . Moreover,  $S^*$  and  $S^* \cap P$  are pure in  $R^*$ ;  $S$  and  $S \cap P$  are pure in  $R$ . Since  $|A(R)| = \aleph_0$ , it follows by induction that  $|\mathcal{S}| \leq \aleph_0$ .

Now consider the case  $B \neq 0$ . The lemma has already been proved if  $r = \infty$ , so assume that  $r$  is finite. Any pure subgroup  $P$  of  $G = F \oplus B$  is a subdirect sum of a pure subgroup  $E$  of  $F$  and a subgroup  $A$  of  $B$ . Since  $E \cap P$  has index in  $E$  which divides the order of  $B$ , there are only a finite number of choices of  $E \cap P$  for a given  $E$  (and consequently only a finite number of choice of  $P$ ). Thus the lemma is proved.

The theorem follows almost immediately from the lemmas.

**THEOREM.** *For any group  $G$ ,  $|\mathcal{S}| \leq \aleph_0$  if and only if:  $G = F \oplus T$  where  $T$  is torsion of the form (1) and  $F$  is torsion free of finite rank  $r \geq 0$ ; further if the prime  $p$  is in the collection  $\pi = \{p_1, p_2, \dots, p_n\}$  of the decomposition (1) of  $T$ , then  $F$  has no pure subgroup which can be mapped homomorphically onto  $p^\infty$ . In all other cases,  $|\mathcal{S}| = 2^{|\pi|}$ . Moreover,  $|\mathcal{S}|$  is finite if and only if either  $r = 0$  or  $r = 1$  and  $T$  is finite.*

*Proof.* Suppose that  $|\mathcal{S}| \neq 2^{|\pi|}$ . Then  $|G| \leq \aleph_0$  and the torsion part  $T$  of  $G$  is of the form (1). Hence  $G$  splits into its torsion and torsion free components,  $G = F \oplus T$ . Also,  $F$  is of finite rank  $r \geq 0$ . And there exists no homomorphism of a pure subgroup of  $F$  onto  $p^\infty$  where  $p \in \pi$  (since there would be  $c$  such homomorphisms, each determining a pure subgroup of  $G$ ). But suppose that  $G = F \oplus T$ , where  $F$  and  $T$  satisfy the given conditions. Let  $T'$  denote the divisible part of  $T$  and set  $F' = F \oplus B$ , where  $T = T' \oplus B$ . Since  $B$  is finite,  $|\mathcal{S}(F')| \leq \aleph_0$  is given by Lemma 2. Evidently, a pure subgroup  $P$  of  $G$  is the direct sum of a divisible subgroup of  $T'$  and a subdirect sum of a pure subgroup of  $F'$  and a finite subgroup of  $T'$ . Thus  $|\mathcal{S}| \leq \aleph_0$ .

If  $r = 1$ , then  $|\mathcal{S}(F \oplus p^\infty)| \geq \aleph_0$ , for there are at least  $\aleph_0$  homomorphisms of  $F$  into  $p^\infty$ , each determining a pure subgroup. In view of Lemmas 1 and 2, this completes the proof of the theorem.

## REFERENCES

1. D. L. Boyer, *A note on a problem of Fuchs*, Pacific J. Math., **10** (1960), 1147.
2. ———, *Enumeration theorems in infinite abelian groups*, Proc. Amer. Math. Soc., **7** (1956), 565-570.
3. L. Fuchs, *Abelian groups*, Hungarian Academy of Sciences (1958), Budapest.

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# Pacific Journal of Mathematics

Vol. 12, No. 1

January, 1962

Jonathan L. Alperin, <i>Groups with finitely many automorphisms</i> .....	1
Martin Arthur Arkowitz, <i>The generalized Whitehead product</i> .....	7
John D. Baum, <i>Instability and asymptoticity in topological dynamics</i> .....	25
William Aaron Beyer, <i>Hausdorff dimension of level sets of some Rademacher series</i> .....	35
Frank Herbert Brownell, III, <i>A note on Cook's wave-matrix theorem</i> .....	47
Gulbank D. Chakerian, <i>An inequality for closed space curves</i> .....	53
Inge Futtrup Christensen, <i>Some further extensions of a theorem of Marcinkiewicz</i> .....	59
Charles Vernon Coffman, <i>Linear differential equations on cones in Banach spaces</i> .....	69
Eckford Cohen, <i>Arithmetical notes. III. Certain equally distributed sets of integers</i> .....	77
John Irving Derr and Angus E. Taylor, <i>Operators of meromorphic type with multiple poles of the resolvent</i> .....	85
Jacob Feldman, <i>On measurability of stochastic processes in products space</i> .....	113
Robert S. Freeman, <i>Closed extensions of the Laplace operator determined by a general class of boundary conditions, for unbounded regions</i> .....	121
Robert E. Fullerton, <i>Geometric structure of absolute basis systems in a linear topological space</i> .....	137
Dieter Gaier, <i>On conformal mapping of nearly circular regions</i> .....	149
Andrew Mattei Gleason and Hassler Whitney, <i>The extension of linear functionals defined on <math>H^\infty</math></i> .....	163
Seymour Goldberg, <i>Closed linear operators and associated continuous linear operators</i> .....	183
Basil Gordon, Aviezri Siegmund Fraenkel and Ernst Gabor Straus, <i>On the determination of sets by the sets of sums of a certain order</i> .....	187
Branko Grünbaum, <i>The dimension of intersections of convex sets</i> .....	197
Paul Daniel Hill, <i>On the number of pure subgroups</i> .....	203
Robert Peter Holten, <i>Generalized Goursat problem</i> .....	207
Alfred Horn, <i>Eigenvalues of sums of Hermitian matrices</i> .....	225
Henry C. Howard, <i>Oscillation and nonoscillation criteria for <math>y''(x) + f(y(x))p(x) = 0</math></i> .....	243
Taqdir Husain, <i>S-spaces and the open mapping theorem</i> .....	253
Richard Eugene Isaac, <i>Markov processes and unique stationary probability measures</i> .....	273
John Rolfe Isbell, <i>Supercomplete spaces</i> .....	287
John Rolfe Isbell, <i>On finite-dimensional uniform spaces. II</i> .....	291
N. Jacobson, <i>A note on automorphisms of Lie algebras</i> .....	303
Antoni A. Kosinski, <i>A theorem on families of acyclic sets and its applications</i> .....	317
Marvin David Marcus and H. Minc, <i>The invariance of symmetric functions of singular values</i> .....	327
Ralph David McWilliams, <i>A note on weak sequential convergence</i> .....	333
John W. Milnor, <i>On axiomatic homology theory</i> .....	337
Victor Julius Mizel and Malempati Madhusudana Rao, <i>Nonsymmetric projections in Hilbert space</i> .....	343
Calvin Cooper Moore, <i>On the Frobenius reciprocity theorem for locally compact groups</i> .....	359
Donald J. Newman, <i>The Gibbs phenomenon for Hausdorff means</i> .....	367
Jack Segal, <i>Convergence of inverse systems</i> .....	371
Józef Siciak, <i>On function families with boundary</i> .....	375
Hyman Joseph Zimmerberg, <i>Two-point boundary conditions linear in a parameter</i> .....	385