GENERALIZED GOURSAT PROBLEM

ROBERT PETER HOLTEN
1. Introduction. The linear first order system of partial differential equations in two independent variables

\[(1-1) \quad V_i = \sum_{j=1}^{N} b_{ij}(x, y)V_j^i + \sum_{j=1}^{N} e_{ij}(x, y)V_j + f_i(x, y), \quad i = 1, \ldots, N\]

with coefficients that are continuous functions of the independent variables is \textit{hyperbolic} at \((0, 0)\) if there is a real matrix \(T = (t_{ij})\) non-singular with continuously differentiable components in some neighborhood of \((0, 0)\) such that \(T^{-1}BT\) is a diagonal matrix and \(B = (b_{ij})\). We consider problems of the following kind:

(1-2) To find such conditions that the hyperbolic system \((1-1)\) has a unique solution which satisfies a number of linear equations along several arcs issuing from the origin.

Picard was probably the first to consider a non-analytic problem of this type [7]. Two types of hypotheses are needed for (1-2). The first is geometrical i.e. we require certain curves determined by the functions \(b_{ij}\) (the characteristic curves) to intersect the arcs issuing from \((0, 0)\) (the data arcs) in a manner described in § 2 as Conditions (2.1). The second group of assumptions concern certain matrices made up from \(b_{ij}, t_{ij}\) and the coefficients of the linear equations mentioned in (1-2) and the slopes of the data arcs at \((0, 0)\). Some of these matrices are required to be non-singular and others to have eigenvalues with modulus less than one. In § 3 we consider the case that all the data arcs lie between two consecutive characteristic curves through \((0, 0)\). In this case we generalize the theorem proved in § 2 by giving conditions for there to be a unique solution which is \(C^n\). In § 5 we state conditions under which the hypotheses of Theorem 3.1 can always be satisfied for sufficiently large \(n\). We show at the end of § 5 that if some of the hypotheses of Theorem 3.1 are omitted the solution (if it exists) is no longer unique. In § 6 we solve a mixed problem for the general second order hyperbolic equation.

The equations (1-1) are simplified by the linear transformation

\[U^i = \sum_{j=1}^{N} t_{ij} V^j.\]

Without loss of generality we consider the problem (1-2) in the reduced form.

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Data arcs and characteristic curves. Under what conditions does a system of $N$ linear first order equations of the sort

$$
\begin{align*}
U^i_x + A^i(x, y)U^i_y &= \sum_{j=1}^{N} E^{ij}(x, y)U^j + G^i(x, y) & i = 1, \ldots, N \\
\end{align*}
$$

(S)

and some linear combinations of $U^1, \ldots, U^N$ given along arcs issuing from a point determine $U^1, \ldots, U^N$ uniquely?

We are concerned with real valued functions of real variables. Suppose the functions $A^i$ are $C^1$ (actually all the conditions we will impose need only hold in some neighborhood of $(0, 0)$). Let $\gamma_i(x, y)$ be the curve passing through the point $(x, y)$ and which has slope $A^i(\xi, \eta)$ at every point $(\xi, \eta)$ on it. These are called the characteristic curves of (S). The equation for $\gamma_i(x, y)$ is $\gamma = y(\xi; x, y)$ where

$$
\gamma_i(\xi; x, y) = A_i(\xi, y(\xi; x, y))
$$

$$
y_i(\xi; x, y) = \gamma_i(\xi; x, y).
$$

We come to the arcs along which we specify linear combinations of $U^1, \ldots, U^N$. Let $N_0$ be any positive integer less than $N + 1$ and let $C_1, \ldots, C_N, \tilde{C}_1, \ldots, \tilde{C}_{N_0-1}$ be curves issuing from $(0, 0)$ which have continuously turning tangents. Let these curves be given non-parametrically by

$$
C_i: \quad y = \varphi_i(x) \quad \varphi_i(0) = 0 \quad i = 1, \ldots, N
$$

$$
\tilde{C}_k: \quad y = \tilde{\varphi}_k(x) \quad \tilde{\varphi}_k(0) = 0 \quad k = 1, \ldots, N_0 - 1.
$$

The conditions (2-1) below help determine whether the range for $x$ is either $x \geq 0$ or $x \leq 0$. Our problem (S) may be started more explicitly in terms of data arcs:

$$
\begin{align*}
U^i_x + A^i(x, y)U^i_y &= \sum_{j=1}^{N} E^{ij}(x, y)U^j + G^i(x, y) & i = 1, \ldots, N \\
\sum_{j=1}^{N} a_{ij}(x)U^j(x, \varphi_i(x)) &= \hat{H}_i(x), & i = 1, \ldots, N \\
\sum_{j=1}^{N} \tilde{a}_{kj}(x)U^j(x, \tilde{\varphi}_k(x)) &= \tilde{H}_k(x), & k = 1, \ldots, N_0 - 1.
\end{align*}
$$

We seek solutions of (S) on closed domains, $R_{N_0}$, satisfying the following:

1. The boundary of $R_{N_0}$ is a piecewise smooth simply closed curve.
2. The origin is on the boundary of $R_{N_0}$ and $R_{N_0}$ contains a nonzero length segment of each data arc issuing from $(0, 0)$.

(2-1) 3. For every $(x, y)$ in $R_{N_0}$ and $i < N_0$, $\gamma_i(x, y)$ intersects $C_i$ or $\tilde{C}_i$ just once at a point we denote by $P_i(x, y)$. If $\gamma_i(x, y)$ intersects both
C_i and \( \hat{C}_{\hat{i}} \), then the point of intersection is \((0, 0)\). For \( i \geq N_o \) and every \((x, y)\) in \( R_{N_o} \), \( \gamma_i(x, y) \) intersects \( C_i \) just once at \( P_i(x, y) \).

4. For each \((x, y)\) in \( R_{N_o} \), the entire segment of \( \gamma_i(x, y) \) from \((x, y)\) to \( P_i(x, y) \) lies in \( R_{N_o} \).

We assume temporarily that there are domains \( R_{N_o} \) which satisfy these conditions and which have small as we please diameter. Later in this paper we discuss the existence of these domains. Loosely speaking the subscript \( N_o \) has the significance that \( N_o - 1 \) characteristic curves issue from \((0, 0)\) into the interior of \( R_{N_o} \) and as we will see consequently linearly combinations must be given along \( N + N_o - 1 \) arcs.

Notice that if \( N_o > 1 \), \((S)\) over determines the values of a solution at \((0, 0)\). We suppose \((S)\) is consistent at \((0, 0)\). That is, there are numbers \( b_i, c_i, d_i \) (to be interpreted as \( U^i(0, 0), U_2^i(0, 0), U^i_3(0, 0) \)) which satisfy the equations:

\[
\sum_{j=1}^{N} a_{ij}(0)b_j = H_i(0)
\]

\[
\sum_{j=1}^{N} \hat{a}_{kj}(0)b_j = \hat{H}_k(0)
\]

and

\[
\sum_{j=1}^{N} a_{ij}(0)[\varphi_i^j(0) - A^i(0, 0)]d_j = -\sum_{j=1}^{N} a_{ij}(0)\left[ \sum_{m=1}^{N} E^{jm}(0, 0)b_m + G^i(0, 0) \right]
\]

\[
- \sum_{j=1}^{N} a_{ij}(0)b_j + H_i(0)
\]

and same equation with \( i, a, \varphi, H \) replaced respectively by \( k, \hat{a}, \varphi, \hat{H} \) and

\[
c_i + A^i(0, 0)d_i = \sum_{j=1}^{N} E^{ij}(0, 0)b_j + G^i(0, 0) .
\]

Certain matrices play an important role in what follows. Let \( Q(n) \) be the square \( N \times N \) matrix such that

\[
Q(n)_{ii} = 0
\]

\[
Q(n)_{ij} = \max \left\{ \left| \frac{a_{ij}(0)}{a_{ii}(0)} \right| \cdot \left| \frac{\varphi_i^j(0) - A^i(0, 0)}{\varphi_i^j(0) - A^i(0, 0)} \right|^* \left| \frac{\hat{a}_{ij}(0)}{\hat{a}_{ii}(0)} \right| \cdot \left| \frac{\hat{\varphi}_i^j(0) - A^i(0, 0)}{\hat{\varphi}_i^j(0) - A^i(0, 0)} \right|^* \right\}
\]

\[
i = 1, \ldots, N_o - 1 \text{ and } j = 1, \ldots, N
\]

\[
Q(n)_{ij} = \left| \frac{a_{ij}(0)}{a_{ii}(0)} \right| \cdot \left| \frac{\varphi_i^j(0) - A^i(0, 0)}{\varphi_i^j(0) - A^i(0, 0)} \right|^*
\]

\[
i = N_o, \ldots, N \text{ and } j = 1, \ldots, N .
\]
We assume that the slopes of $C_i$ and $\hat{C}_i$ differ from the slope of $\gamma_i(0, 0)$ at $(0, 0)$. That is, $\varphi_i'(0) \neq A^i(0, 0)$ and $\varphi_i(0) \neq A^k(0, 0)$. We also assume that $a_i(0) \neq 0$.

Let $M(n)$ be the $N \times N$ matrix such that
\[
M(n)_{ij} = a_{ij}(0) \cdot (\varphi_i'(0) - A^i(0, 0))^n.
\]

Let $\hat{M}(n)$ be the $N \times (N_0 - 1)$ matrix such that
\[
\hat{M}(n)_{ij} = \hat{a}_{ij}(0)(\varphi_i'(0) - A^i(0, 0))^n, \quad i = 1, \ldots, N_0 - 1.
\]

Let $\bar{M}(n)$ be the compound $N \times (N_0 - 1)$ matrix
\[
\bar{M}(n) = \begin{pmatrix} M(n) \\ \hat{M}(n) \end{pmatrix}.
\]

For any matrix $P$ let $\bar{\lambda}(P)$ be the maximum modulus of all the eigenvalues of $P$.

**Lemma 2.1.** If $\bar{\lambda}(Q(n)) < 1$, then $M(n)$ is nonsingular.

**Proof.** Suppose $M(n)$ is singular. There is then a nonzero vector $x$ such that $M(n)x = 0$. That is
\[
\sum_{j=1}^{N} a_{ij}(0)[\varphi_i'(0) - A^i(0, 0)]^n x_j = 0.
\]

Dividing by $|a_{ii}(0)| \cdot |\varphi_i'(0) - A^i(0, 0)|^n$ we get
\[
|x_i| \leq \sum_{j \neq i}^{N} |a_{ij}(0)| |\varphi_i'(0) - A^i(0, 0)|^n |x_j|.
\]

Let $|x|$ be the vector whose $i$th component is $|x_i|$ then $|x| \leq Q(n) |x|$ the inequality is understood to hold for each pair of corresponding components. Hence
\[
(p + 1) |x| \leq \sum_{k=0}^{n} Q^k(n) |x|
\]
so that $\sum_{k=0}^{n} Q^k(n) |x|$ diverges as $p \to \infty$. Therefore $\bar{\lambda}(Q(n)) \geq 1$.

**Theorem 2.1.** If $(S)$ is consistent at $(0, 0)$ and all its given functions are $C^1$ and $\bar{\lambda}(Q(0)) < 1$ and $\bar{\lambda}(Q(1)) < 1$, then on some $R_{N_0}$ there is a unique $C^1$ solution of $(S)$.

**Proof.** We construct the solution by iteration. Let $^0U^i(x, y) = b_i + c_i x + d_i y$ and obtain $^sU^{i_1}, \ldots, ^sU^{i_s}$ from $^sU^1, \ldots, ^sU^N$ using
(2-2) \[ \frac{s+1}{s+1} U^i_x + A^i(x, y) \frac{s}{s+1} U^i_y = \sum_{j=1}^N E^{ij}(x, y) U^j + G^i(x, y) \]
and
\[ \frac{s+1}{s+1} U^i(x, \varphi_i(x)) = -\sum_{j=1}^N \frac{a_{ij}(x)}{a_i(x)} U^j(x, \varphi_i(x)) + \frac{1}{a_i(x)} H_i(x) \]
and
\[ \frac{s+1}{s+1} U^k(x, \hat{\varphi}_k(x)) = -\sum_{j \neq k} \frac{\hat{a}_{kj}(x)}{\hat{a}_{kk}(x)} U^j(x, \hat{\varphi}_k(x)) + \frac{1}{\hat{a}_{kk}(x)} \hat{H}_k(x) . \]

Equivalently,
\[ (2-3) \frac{s+1}{s+1} U^i(x, y) = \frac{s+1}{s+1} U^i(P_i(x, y)) + \int_{P_i(x, y)} \left[ \sum_{j=1}^N E^{ij}(\xi, \eta)^j U^j(\xi, \eta) \right] \cdot G^i(\xi, \eta) d\xi . \]

[Where the integral is taken along \( \gamma_i(x, y) \) with \( C_i \cup \hat{C}_i \) and for \( P_i(x, y) \) on \( C \),

(2-4) \[ \frac{s+1}{s+1} U^i(P_i(x, y)) = -\sum_{j \neq i} \frac{a_{ij}(\alpha^i(x, y))}{a_i(\alpha^i(x, y))} U^j(P_i(x, y)) + \frac{1}{a_i(\alpha^i(x, y))} H_i(\alpha^i) \]
and for \( P_i(x, y) \) an \( C_i \)

(2-5) \[ \frac{s+1}{s+1} U^i(P_i(x, y)) = -\sum_{j \neq i} \frac{\hat{a}_{ij}(\alpha^i)}{\hat{a}_{ii}(\alpha^i)} U^j(P_i) + \frac{1}{\hat{a}_{ii}(\alpha^i)} \hat{H}_i(\alpha^i) \]
where \( \alpha^i(x, y) \) is the abscissa of \( P_i(x, y) \).

From the assumption that (S) is consistent at \( (0, 0) \) we can conclude that \( \frac{s}{s+1} U^i(P_i(x, y)) \) is properly defined when \( P_i(x, y) \) is the origin [i.e. when \( (x, y) \) lies on \( \gamma_i(0, 0) \)]. It is easy to see that

\[ \left\| \frac{s+2}{s+1} U^i - \frac{s+1}{s+1} U^i \right\| \leq \sum_{j=1}^N T_{ij} \left\| \frac{s+1}{s+1} U^j - s U^j \right\| \]

where \( |T_{ij} - Q_{ij}| \) can be made as small as we please by taking the diameter of \( R_{N_0} \) small enough. Since \( \lambda(Q(0)) < 1 \) we conclude that \( 0 U^i, \frac{1}{s+1} U^i, \ldots, \frac{s}{s+1} U^i, \cdots \) converges uniformly for each \( i = 1, \ldots, N \). That there is at most one solution follows also immediately. The proof that the first partial derivatives also converge uniformly depends in the following way on the fact that \( \lambda(Q(0)) < 1 \):

By taking the \( y \)-partial derivative of (2-3) and (2-4) and using (2-2) to eliminate \( \frac{s+1}{s+1} U^i_z \) we have

\[ [\varphi^i(x) - A^i(x, \varphi_i(x))] \frac{s+1}{s+1} U^i(x, \varphi_i(x)) \]
\[ = \sum_{\substack{j=1 \atop j \neq i}}^{N} \frac{a_{ij}(x)}{a_{ii}(x)} \left[ \varphi_i^j(x) - A_i^j(x, \varphi_i(x)) \right] U^j_i(x, \varphi_i(x)) \]

+ terms not involving derivatives of \( U \)

and

\[ s^{+1} U^j(y, x) = [\varphi_i^j(\alpha^i) - A_i^j(x, y)] \cdot \alpha_i^j(x, y) \cdot s^{+1} U^j_i(\alpha^i, \varphi_i(\alpha^i)) \]

\[ + \int_{a_i}^{x} \sum_{j=1}^{N} E^{i,j}(\xi, \eta) \cdot s U^j_\eta(\xi, \eta) \cdot y^j_\eta(\xi; x, y) \cdot d\xi \]

+ lower order terms.

It is not hard to show that

\[ \alpha_i^j(0, 0) = \frac{1}{\varphi_i^j(0) - A_i^j(0, 0)} \]

therefore

\[ [\varphi_i^j(\alpha^i) - A_i^j(x, y)]\alpha_i^j \]

has the limit 1 as \((x, y)\) approaches \((0, 0)\). Consequently considering both (2-4) and (2-5) we have

\[ ||s^{+1} U^i_y - s U^i_y|| \leq \sum_{j=1}^{N} (Q_{ij}(1) + \varepsilon_1)(1 + \varepsilon_2) ||s U^j_y - s^{-1} U^j_y|| \]

\[ + \beta \sum_{j=1}^{N} ||s U^j - s^{-1} U^j|| \]

where \( \varepsilon_1 \) and \( \varepsilon_2 \) approach zero as the diameter of \( R_{N_0} \) approaches zero. \( \beta \) is some fixed constant. Since \( \lambda(Q(1)) < 1 \) by selecting \( R_{N_0} \) with small enough diameter the eigenvalues of the matrix \( L \) where

\[ L_{ij} = (Q_{ij}(1) + \varepsilon_1)(1 + \varepsilon_2) \]

also have modulus less than one. Let

\[ v_* = ||s U^i_y - s^{-1} U^i_y|| \text{ and } u^i = ||U^i - 0 U^i|| \]

them

\[ v_{s+1} \leq Lv_* + \beta T^s u \]

where the inequality must hold between pairs of corresponding components. It is easily seen that

\[ \sum_{s=1}^{\infty} v_* \leq (1 - L)^{-1} v_1 + \beta(1 - L)^{-1}(1 - T)^{-1} u \]

and our convergence is assured.

In [4] Meltzer assumed that there are only two data arcs. The
method used in [5] by Mihailow permits the characteristic curves and data arcs only to be straight lines. In [3] the author obtained results in the large by making more assumptions relating the slopes of the data arcs and the characteristic curves. In [6] Peyser in effect requires that \( N - 1 \) of the data arcs be identical and consequently the matrices \( Q(0) \) and \( Q(1) \) are nilpotent. Finally in [9] Yosida assumes that the matrix \( M(0) \) is diagonal and consequently \( Q(0) \) is the zero matrix.

3. Higher order solutions. In this section we prove a generalization of Theorem 2.1 for \( N_0 = 1 \). This is the case that all the data arcs lie between two consecutive characteristic curves through the origin. With the addition of a consistency hypothesis for higher order derivatives at \((0, 0)\) the generalization when \( N_0 > 1 \) is also true. We begin by proving a lemma about

\[
(S_0) \quad \left\{ \begin{array}{l}
U_x^i + A^i(x, y)U_y^i = F^i(x, y) \\
N \sum_{j=1} U^j(x, \varphi_i(x)) = H_i(x), \quad i = 1, \cdots, N
\end{array} \right.
\]

[this is \((S)\) with \( N_0 = 1 \) and \( E^{ij} = 0 \)]

**Lemma 3.1.** If \( n \) is any nonnegative integer and \( A^i, F^i, H^i, a_{ij}, \varphi_i \) are \( C^{n+1} \) and \( \lambda(\gamma(n)) < 1 \) and \( \lambda(Q(n + 1)) < 1 \) and \( M(0), \cdots, M(n - 1) \) each have rank \( N \) (i.e., are nonsingular), then on some \( R_i \) (we assume that such domains exist) there is exactly one \( C^{n+1} \) solution of \((S_0)\). Moreover \( R_i \) depends on neither \( F^i \) nor \( H^i \).

**Proof.** As we did before we perform the iteration:

\[
s^{+1} U_x^i + A^i(x, y)s^{+1} U_y^i = F^i(x, y)
\]

\[
s^{+1} U^i(x, \varphi_i(x)) = -\sum_{j=1}^N a_{ij}(x)s U^j(x, \varphi_i(x)) + \frac{1}{a_{ii}(x)} H_i(x).
\]

Taking the \( n \)th derivative of the second equation we get

\[
\sum_{p=0}^n s^{+1} U^i_{p, n-p}(x, \varphi_i(x)) \binom{n}{n-p} [\varphi_i(x)]^{n-p} = -\sum_{j=1}^N a_{ij} \sum_{p=0}^n U^j_{p, n-p}(x, \varphi_i(x)) \binom{n}{n-p} [\varphi_i(x)]^{n-p}
\]

+ terms involving derivatives of order less than \( n \) of \( s^{+1} U \) and \( s^i U \), where

\[
U^i_{p, n-p} = \frac{\partial^n U^i}{\partial x^p \partial y^{n-p}} \quad \text{and} \quad \binom{n}{n-p} = \frac{n!}{p!(n-p)!}.
\]

Using the first equation we have
\[ s^i U_{p,n-p} = [-A^i(x, y)]^p s^i U_{0,n} \]

+ terms involving derivatives of \( s^i U \) of order less than \( n \).

Consequently,

\[
\phi(x) - A^i(x, \phi(x)) = \sum_{j=1}^{n} \frac{a_{ij}(x)}{a_{ii}(x)} [\phi(x) - A^i(x, \phi(x)) ]^n s^i U_{0,n}(x, \phi(x))
\]

+ terms of order less than \( n \).

Since \( M(0), \cdots, M(n-1) \) and \( M(n), M(n+1) \) are nonsingular (see Lemma 2.1) the values of any solution of \( (S_0) \) and all their derivatives up to and including order \( (n+1) \) are uniquely determined at \((0, 0)\). Let \( c_i(p_1, p_2) \) be the value determined for \( U_{p_1,p_2}^i(0,0) \). We begin our iteration with

\[
0^i U(x, y) = \sum_{0 \leq p_1 + p_2 \leq n+1} c_i(p_1, p_2) \cdot \frac{x^{p_1} y^{p_2}}{p_1! p_2!}.
\]

It follows that

\[ s^i U_{p_1,p_2}(0, 0) = c_i(p_1, p_2) \quad \text{for} \quad 0 \leq p_1 + p_2 \leq n+1, \quad s > 0. \]

Now since \( \lambda(Q(n)) < 1 \) we see that all the \( n \)th order derivatives of the sequence \( 0^i U, \cdots, s^i U, \cdots \) converge uniformly on some \( R \) of sufficiently small diameter. Also it is clear that the functions \( F^i \) and \( G^i \) are not involved in how small the diameter must be chosen. That there is at most one solution which is \( C^\infty \) follows in the customary way. It remains to see that we have a \( C^{n+1} \) solution to \( (S_0) \).

Since

\[ s^i U_{p,n+1-p} = [-A^i(x, y)]^p s^i U_{0,n+1} \]

+ terms involving derivatives of \( s^i U \) of order less than \( n+1 \), we need consider only the convergence of

\[ 0^i U_{0,n+1}, \cdots, s^i U_{0,n+1}, \cdots \]

Now

\[
(s+1) U_i(x, y) = (s+1) U_i(P_i(x, y)) + \int_{P_i(x,y)} F_i(\xi, \eta) d\xi
\]

\[ [P_i(x, y) = \text{the point } (\alpha^i(x, y), \phi_i(\alpha^i(x, y)))] \]

\[
(s+1) U_{0,n+1}^i = [\alpha_{i}^n+1 \sum_{p=0}^{n+1} [\phi_i]^{n+1-p} (s+1) U_{p,n+p}^i] \frac{n+1}{p}
\]

+ terms of order less than \( n+1 \).
can be made as near 1 as we please by taking the diameter of \( R \lambda \) small enough. From this observation and the assumption \( \lambda(Q(n+1)) < 1 \) the series

\[\sum_{i=0}^{\infty} U_{i,n+1}, \ldots, U_{0,n+1}, \ldots\]

converge uniformly. Consequently we have indeed the unique \( C^{n+1} \) solution of \((S_0)\).

For the system

\[
\begin{cases}
U_i + A^i(x, y) U_i = \sum_{j=1}^{N} E^{ij}(x, y) U_j + G^i(x, y) \\
\sum_{j=1}^{N} a_{ij}(x) U_j(x, \varphi_i(x)) = H_i(x), \quad i = 1, \ldots, N
\end{cases}
\]

we have

**Theorem 3.1.** If \( n \) is a nonnegative integer and \( A^i, E^{ij}, G^i, a_{ij}, \varphi_i, H_i \) are \( C^{n+1} \) and \( \lambda(Q(n)) < 1 \) and \( \lambda(Q(n+1)) < 1 \) and \( M(0), \ldots, M(n-1) \) each are nonsingular, then on some \( R_i \) there is exactly one \( C^{n+1} \) solution of \((S_i)\).

**Proof.** Using Lemma 3.1 we can define a sequence of functions which are \( C^{n+1} \) on some \( R_i \) as follows:

\[s_{i+1} U_i + A^i(x, y) s_{i+1} U_i = \sum_{j=1}^{N} E^{ij}(x, y) s_{i+1} U_j + G^i(x, y)\]

\[\sum_{j=1}^{N} a_{ij}(x) s_{i+1} U_j(x, \varphi_i(x)) = H_i(x) .\]

We can show that all the \((n+1)\)th order derivatives converge on some possibly smaller \( R_i \). Using the same kind of calculations as before it is easy to see that

\[
\| s_{i+1} U_{0,n+1} - s_{i+1} U_{0,n+1} \| \leq \sum_{j=1}^{N} T_{ij} \| s_{i+1} U_{0,n+1} - s_{i+1} U_{0,n+1} \|
\]

\[+ \sum_{j=1}^{N} S_{ij} \| s_{i+1} U_{0,n+1} - s_{i+1} U_{0,n+1} \| \]

where \( \lambda(T) < 1, T_{ij} \geq 0 \) and by taking the diameter of \( R_i \) small enough each \( S_{ij} \) can be made arbitrarily close to zero. We have in vectors

\[V_{s+1} \leq T \cdot V_{s+1} + S \cdot V, \text{ (the inequality must hold for each component).} \]

\[(I - T) V_{s+1} \leq S \cdot V_s .\]
Since $(I - T)^{-1} = I + T + T^2 + \cdots$, $(I - T)^{-1} \geq 0$, $V_{s+1} \leq (I - T)^{-1}SV_s$. By choosing $R_1$ small enough
\[ \ln((I - T)^{-1}S) < 1. \]
Except for this the proof of Theorem 3.1 is like Lemma 3.1.

4. Constructing the domains of dependence. We discuss this topic only for the case that the data arcs and characteristic curves are straight lines. The subject has been treated more completely in [3].

Suppose $\varphi_i(x) = m_i x$, $\hat{\varphi}_k(x) = \hat{m}_k x$ and $A^i$ are constant. By possibly renaming the variables we can assume
\[ A^1 \leq A^2 \leq \cdots \leq A^N. \]
Let $\bar{R}_{N_0}$ be the region lying below both the line $\gamma_N(0, 0)$ and the line $\gamma_{N_0}(0, 0)$. $R_{N_0}$ will be a part of $\bar{R}_{N_0}$.

Assume that all the data arcs lie in $\bar{R}_{N_0}$. Let the first $l$ data arcs lie to the left of the $y$-axis (i.e. $\varphi_1, \cdots, \varphi_i$ defined only for $x \leq 0$, of course $l$ may be zero) and suppose
\[ A^N \geq m_1 \geq \cdots \geq m_i. \]
We suppose the remaining data arcs are ordered so that
\[ m_{i+1} \leq m_{i+2} \leq \cdots \leq m_N \leq \hat{m}_1 \leq \cdots \leq \hat{m}_{N_0-1} \leq A^{N_0}. \]
We further assume that
\[
\begin{align*}
m_i &> A^i, \quad i = 1, \cdots, l \\
m_i &< A^i, \quad i = l + 1, \cdots, N \\
\hat{m}_k &> A^k, \quad k = 1, \cdots, N_0 - 1.
\end{align*}
\]
These last assumptions assure us that $C_i$ lies below $\gamma_i(0, 0)$ and $\hat{C}_k$ lies above $\gamma_i(0, 0)$.

Our final assumption excludes the possibility that by omitting some data arcs a domain $R_{N_0}$ satisfying conditions (2.1) can be constructed with $M_0 < N_0$. This final assumption is: For $N_0 \neq 1$ assume
\[ m_N > A^1 \text{ and } \hat{m}_k > A^{k+1}, \quad k = 1, \cdots, N_0 - 2. \]
We construct the domains $R_{N_0}$ for $N_0 \neq 1$. The case $N_0 = 1$ offers no new difficulties. We begin by choosing a point $P_0$ with negative abscissa on $\gamma_N(0, 0)$ (we exclude the special case that $C_{N-1}$ lies along $\gamma_1(0, 0)$). Define points $p_1, \cdots, p_N$ as follows:
\[ p_i = \text{the intersection of } \gamma_i(p_{i-1}) \text{ with } C_i, \quad i = 1, \cdots, N. \]
Since $C_a$ lies above $\gamma_1(0, 0)$ and $\hat{C}_k$ above $\gamma_{k+1}(0, 0)$, we can continue
with points \( \hat{p}_1, \ldots, \hat{p}_{N_0} \):

\[
\begin{align*}
\hat{p}_i &= \text{the intersection of } \gamma_i(p_{N_0}) \text{ with } \hat{c}_1, \\
\hat{p}_k &= \text{the intersection of } \gamma_k(\hat{p}_{k-1}) \text{ with } \hat{c}_k, \ k = 2, \ldots, N_0 - 1, \\
\hat{p}_{N_0} &= \text{the intersection of } \gamma_{N_0}(\hat{p}_{N_0} - 1) \text{ with } \gamma_{N_0}(0, 0)
\end{align*}
\]

where \( A < A^< < A^{N_0} \) and no \( A^< \) exists such that \( A^< < A^< < A^{N_0} \).

The boundary of \( R_{N_0} \) consists of \( \gamma_{N_0}(0, 0) \) from \((0, 0)\) to \( p_0 \), \( \gamma_1(p_0) \) from \( p_0 \) to \( p_1 \), \ldots, \( \gamma_{N_0}(p_{N-1}) \) from \( p_{N-1} \) to \( p_N \), \( \gamma_i(p_N) \) from \( p_N \) to \( p_1 \), \( \gamma_3(p_1) \) from \( \hat{p}_1 \) to \( \hat{p}_2 \), \ldots, \( \gamma_{N_0-1}(\hat{p}_{N_0-2}) \) from \( \hat{p}_{N_0-2} \) to \( \hat{p}_{N_0-1} \), \( \gamma_{N}(\hat{p}_{N_0-1}) \) from \( \hat{p}_{N_0-1} \) to \( \hat{p}_{N_0} \), \( \gamma_{N_0}(0, 0) \) from \( \hat{p}_{N_0} \) to \((0, 0)\). The domains \( R_{N_0} \) constructed in this way satisfy conditions (2.1). Also, if we let \( p_0 \) approach \((0, 0)\) along \( \gamma_N(0, 0) \) the diameter of \( R_{N_0} \) approaches zero.

5. Certain special systems. We turn our attention to the hypotheses of Theorem 3.1. We have \( N_0 = 1 \), that is, all the data arcs lie between two consecutive characteristic curves through \((0, 0)\). In this case we call a system \((S_j)\) regular if

![Diagram](image_url)
\[ A^i(0, 0) \neq A^j(0, 0) \text{ and } \varphi_i'(0) \neq \varphi_j'(0) \text{ for } i \neq j. \]

For a regular system we can suppose without loss of generality that
\[
A^i(0, 0) \leq \varphi_i'(0) < \varphi_i'(0) < \cdots < \varphi_N'(0) < \varphi_i'(0) \\
\leq A^i(0, 0) < A^2(0, 0) < \cdots < A^N(0, 0).
\]

We can show, in case \((S_j)\) is regular, that there is always an \(\bar{n}\) such that \(\bar{\lambda}(Q(n)) < 1\) whenever \(n > \bar{n}\).

Recall that for \(i \neq j\)
\[
Q(n)_{ij} = \left| \frac{a_{ij}(0)}{a_{ii}(0)} \right| \frac{\varphi_i'(0) - A^i(0, 0)}{\varphi_j'(0) - A^j(0, 0)}^n \\
Q(n)_{ii} = 0.
\]

The eigenvalues of \(Q(n)\) satisfy the equation \(\text{Det}(Q(n) - \lambda I) = 0\).

**Lemma 5.1.** If \(d_1 < d_2 < \cdots < d_N \leq e_1 < e_2 < \cdots < e_N\), then each term (except the diagonal which is one) in the expansion of determinant, \(D\), of the matrix \((c_{ij}/c_{ii})\), \(c_{ij} = |d_i - e_j|\), has absolute value less than one.

**Proof.** We proceed by induction. The lemma is vacuously true in case \(N=1\). Suppose \(N>1\) and let us look at a typical nonzero term of \(D\). Let this term, \(\pi\), contain as a factor \(c_{qN}/c_{qq}\) from the \(N\)th column and \(c_{NP}/c_{NN}\) from the \(N\)th row. Suppose that \(p \neq N\), then we have \(q \neq N\) and except possibly for sign \([\pi(c_{pq}/c_{qq})]/[(c_{qN}/c_{qq})(c_{NP}/c_{NN})]\) is a term in the expansion of the \((N-1) \times (N-1)\) determinant and hence its absolute value is no larger than one. To show \(|\pi| < 1\) we need only show that
\[
\frac{c_{qN} \cdot c_{NP}}{c_{qp} \cdot c_{NN}} < 1 \\
\frac{c_{qN} \cdot c_{NP}}{c_{qp} \cdot c_{NN}} = \frac{(e_N - d_q)(e_p - d_N)}{(e_p - d_q)(e_N - d_N)} < 1
\]
if and only if
\[-d_qe_p - d_Ne_N < -d_qe_N - d_Ne_p \\
(d_N - d_q)e_p < (d_N - d_q)e_N \]
eq p < e_N which is one of our assumptions. If \(p = N\), then \(\pi\) is a term of the \((N-1) \times (N-1)\) determinant. This completes the induction.

When we have established the following lemma we can immediately conclude that \(\lim_{n \to \infty} \text{Det}(Q(n) - \lambda I) = (-\lambda)^N\).

**Lemma 5.2.** For a regular system \((S_j)\) each term (except the diagonal term which is one) in the expansion of the determinant of \((b_{ij}/b_{ii})\) where
\[ b_{ij} = |\varphi_i(0) - A^i(0, 0)| \text{ has absolute value less than one.} \]

**Proof.** We delete the first column and row of \((b_{ij}/b_{ii})\) and use Lemma 5.1. Suppose \(\pi\) is any term of \(\text{Det}(b_{ij}/b_{ii})\). Let \(\pi\) contain \(b_{ip}/b_{11}\) from the 1st row and \(b_{iq}/b_{qq}\) from the 1st column. Suppose \(p \neq 1\), then \(q \neq 1\). Use Lemma 5.1 to see that

\[
\frac{|\pi| b_{qp}}{b_{qq}} \leq 1
\]

\[
\frac{b_{ip} \cdot b_{q1}}{b_{11} \cdot b_{qq}} = \frac{A^p(0, 0) - \varphi_i(0)}{A^p(0, 0) - \varphi_q(0)} \cdot \frac{\varphi_i(0) - A^i(0, 0)}{\varphi_q(0) - A^i(0, 0)} < 1
\]

if and only if

\[
A^p(0, 0)\varphi_i(0) + A^i(0, 0)\varphi_q(0) < A^p(0, 0)\varphi_q(0) + A^i(0, 0)\varphi_i(0)
\]

\[
[A^p(0, 0) - A^i(0, 0)]\varphi_q(0) < [A^p(0, 0) - A^i(0, 0)]\varphi_i(0)
\]

\[
\varphi_q(0) < \varphi_i(0).
\]

In case \(p = 1\) Lemma 5.1 yields our result immediately.

**THEOREM 5.1.** If \((S_0)\) is regular, then for any \(\varepsilon > 0\) there is an \(\bar{n}\) such that

\[ \lambda(Q(n)) < \varepsilon \text{ for all } n > \bar{n}. \]

Let us consider systems with constant coefficients of the form

\[
(S_0)
\]

\[
\begin{cases}
U^i_x + A^iU^i_y = F^i(x, y) \\
\sum_{j=1}^{N} a_{ij}U^i(x, m_ix) = H_i(x), i = 1, \cdots, N.
\end{cases}
\]

We suppose that the constants \(m_1, \cdots, m_N, A^1, \cdots, A^N\) are ordered so that

\[ A^1 \leq m_2 \leq m_3 \leq \cdots \leq m_N \leq m_1 \leq A^2 \leq \cdots \leq A^N. \]

We have shown that \((S_0)\) has at most one \(C^{n+1}\) solution on \(R\), if \(M(0), \cdots, M(n-1)\) are nonsingular and \(F^i, H^i\) are \(C^{n+1}\) and \(\lambda(Q(n)) < 1\). We will now investigate to what extent these conditions for uniqueness are necessary.

In \((S_0)\) suppose \(M(p)\) is singular for some integer \(p \geq 0\). Let \(e\) be a nonzero vector such that \(M(p)e = 0\). Then

\[ U^i(x, y) = (y - A^i x)p^i e_i, \]

\[ i = 1, \cdots, N \]

is a nontrivial polynomial solution of
We express this in

**Theorem 5.2.** If $M(p)$ is singular for some integer $p \geq 0$, then $(\bar{S}_0)$ has a nontrivial polynomial solution.

It is harder to show that the condition $\lambda(Q(n)) < 1$ is needed. Without loss of generality we can suppose that $a_{i1} = a_{22} = \cdots = a_{NN} = 1$, then Lemma 5.2 shows us that

$$\lim_{n \to \infty} \frac{\det M(n)}{[\varphi_1'(0) - A^1(0, 0)]^n \cdots [\varphi_N'(0) - A^N(0, 0)]^n} = 1 .$$

We define for all real numbers $r \geq 0$:

$$\tilde{M}(r)_{ii} = a_{ii}(m_i - A^i)^r, \quad i = 1, \ldots, N$$

$$\tilde{M}(r)_{ij} = a_{ij}(A^i - m_j)^r, \quad i = 1, \ldots, N, \quad j = 2, \ldots, N .$$

Then

$$\lim_{r \to \infty} \frac{\det \tilde{M}(r)}{|m_1 - A^1|^r \cdots |m_N - A^N|^r} = +1 .$$

Let

$$\tilde{Q}(n)_{ii} = 1, \quad \tilde{Q}(n)_{ij} = -Q(n)_{ij}, \quad i \neq j .$$

Since $\tilde{Q}(n)_{ii} > 0, \tilde{Q}(n)_{ij} \leq 0, \quad i \neq j$, if each principal minor of $\tilde{Q}(n)$ is positive, then each component of $\tilde{Q}(n)^{-1}$ is nonnegative. Using this fact we prove

**Lemma 5.3.** If each principal minor of $\tilde{Q}(n)$ is positive, then $\lambda(Q(n)) < 1$.

**Proof.** We suppose $\lambda(Q(n)) \geq 1$ and deduce that $\tilde{Q}(n)^{-1}$ has at least one negative component. Let $e$ be a nonzero vector and $|\lambda| \geq 1$ and $Q(n)e = \lambda e$. Then

$$|\lambda||e_i| \leq \sum_{j=1}^N Q(n)_{ij} |e_j|$$

$$|e_i| - \sum_{j=1}^N Q(n)_{ij} |e_j| \leq -(|\lambda| - 1) |e_i|$$

$$\sum_{j=1}^N \tilde{Q}(n)_{ij} |e_j| \leq -(|\lambda| - 1) |e_i| \leq 0 .$$

Let
\[ f_i = \sum_{j=1}^{N} \tilde{Q}(n)_{ij} |e_j| \]

Since \(-f_i \geq 0\), if \(\tilde{Q}(n)^{-1}\) had no negative components each component of \(\tilde{Q}(n)^{-1}(-f)\) would be nonnegative. But then \(\tilde{Q}(n)^{-1}f = |e| \leq 0\) implies that \(e = 0\). This is contrary to our assumption that \(e \neq 0\).

We call a system \((S_o)\) uniform if each term (except the diagonal which is one) in the expansion of \(\text{Det}(M(0))\) is either negative or zero. We have assumed that \(a_{ii} = 1\). Any \((S_o)\) with \(N = 3\) satisfying \(a_{13} \leq 0, a_{21} \leq 0, a_{12} \geq 0, a_{23} \geq 0, a_{22} \geq 0\) is uniform.

**Lemma 5.4.** If \((S_o)\) is uniform and \(\text{Det}(\tilde{M}(n)) > 0\), then each principal minor of \(\tilde{Q}(n)\) is positive.

Combining the last two lemmas we have

**Lemma 5.5.** If \((S_o)\) is uniform and \(\text{Det}(\tilde{M}(n)) \geq 1\), then \(\text{Det}(\tilde{M}(n)) \leq 0\). Now \(\text{Det}(\tilde{M}(n))\) is eventually positive and \(\text{Det}(\tilde{M}(r))\) is a continuous function of \(r\).

If \(\text{Det}(\tilde{M}(r)) = 0\), then \(\tilde{M}(r)e = 0\) for some \(e \neq 0\) and

\[
\begin{align*}
U^i(x, y) &= (y - A^i x)^r e_i, \\
U^i(x, y) &= (A^i x - y)^r e_i, \quad i = 2, \ldots, N
\end{align*}
\]

is a \(C^{[r]}\) solution of \((S_{oo})\). We have proved

**Theorem 5.3.** If \((S_o)\) is uniform and \(\text{Det}(\tilde{M}(n)) \geq 1\), then \((S_{oo})\) has a nontrivial solution which is \(C^n\).

We can give a more complete analysis of \((S_o)\) when \(N = 2: \)

\[
\begin{align*}
\begin{cases}
U^i_x + A^i U^i_y = 0, \quad i = 1, 2 \\
U^i(x, m_i x) + a U^i(x, m_i x) = 0 \\
U^i(x, m_2 x) + U^i(x, m_2 x) = 0
\end{cases}
\end{align*}
\]

where \(A^1 < m_2 < m_1 < A^2\).

The eigenvalues of \(M(n)\) satisfy the equation

\[
\lambda^2 = |a| \rho^n \quad \text{where} \quad \rho = \frac{A^2 - m_1}{m_1 - A^1} \frac{m_2 - A^1}{A^2 - m_2}.
\]

Since \((S_{oo})\) is regular, \(\rho < 1\).

If we had allowed \(m_2 = A^1\) or \(m_1 = A^2\) we would have had \(\rho = 0\). Let \(r\) be the real number such that \(|a| \rho^r = 1\). Suppose that \(r \geq 1\). As we know, if \(|a| < 1\), then \((S_{oo})\) has only the trivial solution. Let \([r]\) be the greatest integer less than or equal to \(r\). Two cases arise. First if \(a > 0\), then \(a \cdot \rho^r = +1\). In this case
is a nontrivial $C^{[r]}$ solution of $(S_{00})$. If $r$ is an integer these functions are polynomials. In this connection we notice that $M(n)$ is singular just in case $1 - a \rho^n = 0$ which can happen only if $r$ is an integer and $a > 0$. Now suppose $a$ is negative, then $a \rho = -1$ and

$$U \cdot (x, y) = (A^2 - m_x) (y - A^1 x)^r$$
$$U \cdot (x, y) = (m_2 - A^1)^r (A^2 x - y)^r$$

is a nontrivial $C^{[r]}$ solution of $(S_{00})$.

6. Application to second order equations. We apply our results to the 2nd order system $(S_2)$. Our method is however equally suited for the $n$th order case.

The system

$$\begin{cases}
Z_{xx} - Z_{yy} = A(x, y)Z_x + B(x, y)Z_y + C(x, y)Z + D(x, y) \\
b_{i1}(x)Z_x(x, \varphi_i(x)) + b_{i2}(x)Z_y(x, \varphi_i(x)) + b_{i3}(x)Z(x, \varphi_i(x)) = H_i(x), i = 1, 2 \\
Z(0, 0) = c
\end{cases}$$

is transformed into the equivalent system

$$\begin{cases}
U_x - U_y = 1/2(A + B) U + 1/2(-A + B) V + CZ + D \\
V_x + V_y = -1/2(A + B) U - 1/2(-A + B) V - CZ - D \\
(b_{i1} + b_{i2}) U(x, \varphi_i(x)) + (-b_{i1} + b_{i2}) V(x, \varphi_i(x)) \\
= -2b_{i3}Z(x, \varphi_i(x)) + 2H_i(x) \\
Z_y = 1/2(U + V), Z_x = 1/2(U - V) \\
Z(0, 0) = c
\end{cases}$$

by the substitution $U = Z_x + Z_y$ and $V = -Z_x + Z_y$.

If we iterate as follows:

$$\begin{align*}
U^{s+1} - U^{s} &= 1/2(A + B) U^s + 1/2(-A + B) V^s + CZ^s + D \\
V^{s+1} + V^{s} &= -1/2(A + B) U^s - 1/2(-A + B) V^s - CZ^s - D \\
(b_{i1} + b_{i2}) U^{s+1}(x, \varphi_i(x)) + (-b_{i1} + b_{i2}) V^{s+1}(x, \varphi_i(x)) \\
&= -2b_{i3}Z^{s}(x, \varphi_i(x)) + 2H_i(x) \\
Z^{s+1}_y = 1/2(U^{s+1} + V^{s+1}), Z^{s+1}_x = 1/2(U^{s+1} - V^{s+1})
\end{align*}$$
and let \( a_{i1} = b_{i1} + b_{i2}, a_{i2} = -b_{i1} + b_{i2}, a_{21} = b_{21} + b_{22}, a_{22} = -b_{21} + b_{22}, A^1 = -1, A^2 = +1 \) we have using the same methods as in § 3.

**Lemma 6.1.** If \( n \) is any nonnegative integer and \( A, B, C, D, b_{ij}, H_i, \varphi_i \) are \( C^{n+1} \) and \( \lambda(Q(n)) < 1 \) and \( \lambda(Q(n+1)) < 1 \) and \( M(0), \cdots, M(n-1) \) are nonsingular, then on some \( R \) there is exactly one \( C^{n+1} \) solution of (\( S_2 \)).

If we assume that \(-1 \leq \varphi'_i(0) < \varphi_i(0) \leq 1 \) and let
\[
 a = \frac{-b_{11}(0) + b_{12}(0)}{b_{11}(0) + b_{12}(0)}, \quad b = \frac{b_{21}(0) + b_{22}(0)}{-b_{21}(0) + b_{22}(0)},
\]
\[
 r = \frac{1 - \varphi'_i(0)}{1 + \varphi'_i(0)} \cdot \frac{1 + \varphi'_i(0)}{1 - \varphi'_i(0)}
\]
[Notice that \( 0 \leq r < 1 \)]
we have immediately

**Theorem 6.1.** If \( n \) is a nonnegative integer and \( |ab| r^n < 1 \) and \( abr^k \neq 1 \) for \( k = 0, \cdots, n-1 \) and \( A, B, C, D, b_{ij}, H_i, \varphi_i \) are \( C^{n+1} \), then on some \( R \) there is exactly one \( C^{n+1} \) solution of (\( S_2 \)).

Since \( 0 \leq r < 1 \) there always is a nonnegative integer such that
\[
 |ab| \cdot r^n < 1.
\]
It is interesting to notice that if
\[
 ab \cdot r^p = 1 \text{ for some } p \geq 1 \text{ which need not be an integer, then}
\]
\[
 Z(x, y) = (1 - m_z)^p \frac{(x + y)^{p+1}}{p + 1} + b(1 + m_z)^p \frac{(x - y)^{p+1}}{p + 1}
\]
is a non-trivial solution of
\[
 Z_{xx} - Z_{yy} = 0
\]
\[
 b_{12}Z_x(x, m_i x) + b_{i2}Z_y(x, m_i x) = 0, \ i = 1, 2
\]
\[
 Z(0, 0) = 0.
\]
This \( Z \) is a polynomial in case \( p \) is an integer.

We finish by applying our theorem to a problem solved by Goursat [2]:
\[
 Z_{xx} - Z_{yy} = AZ_x + BZ_y + D
\]
\[
 Z(x, m_i x) = H_i(x), \ i = 1, 2 \quad \text{where}
\]
\[
 H_i(0) = H_i(0), \ -1 \leq m_z < m_i \leq 1.
\]
An equivalent problem is
\[ Z_{xx} - Z_{yy} = AZ_x + BZ_y + CZ + D \]
\[ Z_x(x, m, x) + m_i Z_y(x, m, x) = H_i(x), \quad i = 1, 2 \]
\[ Z(0, 0) = H_i(0). \]

We have in this case
\[ |ab| = \frac{1 - m_1}{1 + m_1} \cdot \frac{1 + m_2}{1 - m_2} = r < 1. \]

Consequently according to Theorem 6.1 this problem has exactly one \( C^1 \) solution.

In [1] the authors treat a somewhat more general system the functions of which satisfy certain Lipschitz conditions. They make in our notation the hypotheses \( |ab| < 1 \) and \( r < 1 \). In [8] Szmydt solves the same problem with the hypothesis that some of the Lipschitz constants are small. The result is essentially the same.

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